ANNALS
OF
DISCRETE
MATHEMATICS

32

MONOGRAPH

Planar Graphs: Theory and Algorithms

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NORTH-HOLLAND

PLANAR GRAPHS: THEORY AND ALGORITHMS

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1988

NORTH-HOLLAND — AMSTERDAM • NEW YORK • OXFORD • TOKYO

ISBN: 0 444 70212 1

Publishers:
ELSEVIER SCIENCE PUBLISHERS B.V.
P.O. BOX 1991
1000 BZ AMSTERDAM
THE NETHERLANDS

Sole distributors for the U.S.A. and Canada:
ELSEVIER SCIENCE PUBLISHING COMPANY, INC.
52 VANDERBILT AVENUE
NEW YORK, N.Y. 10017
U.S.A.

Library of Congress Cataloging in Publication Data

Nishizeki, T. (Takao), 1947–
Planar graphs : theory and algorithms / T. Nishizeki and N. Chiba.
p. cm. -- (North-Holland mathematics studies : 140) (Annals of discrete mathematics : 32).
Bibliography: p.
Includes index.
ISBN 0-444-70212-1
1. Graph theory. 2. Algorithms.
I. Chiba, N. (Norishige). II. Title. III. Series. IV. Series: Annals of discrete mathematics : 32.
QA166.N57 1988
511'.5--dc19
88-4980 CIP

PRINTED IN THE NETHERLANDS

NORTH-HOLLAND MATHEMATICS STUDIES 140
Annals of Discrete Mathematics (32)

NORTH-HOLLAND — AMSTERDAM • NEW YORK • OXFORD • TOKYO

Dedicated to our teacher
Professor Nobuji Saito

PREFACE

The theory of planar graphs was first discovered in 1736 by Euler when he found his important formula relating the numbers of vertices, edges and faces of polyhedrons, which can be represented by planar graphs. Since that time numerous results have been obtained on planar graphs. One of the most outstanding results is Kuratowski's theorem which gives a criterion for a graph to be planar. Another example is the famous four-color theorem: every planar graph can be colored with at most four colors so that no two adjacent vertices receive the same color. In recent years, planar graphs have attracted computer scientists' interest, and a lot of interesting algorithms and complexity results have been obtained for planar graphs. For example, Hopcroft and Tarjan have reported on a linear time algorithm which tests the planarity of a graph.

Recently it appeared to us that the time was ripe to collect and organize the many results on planar graphs, which have been our research topics for these ten years. In our opinion the theory and algorithms are complementary to each other in the research of planar graphs. For example, Hopcroft and Tarjan's algorithm was motivated by Kuratowski's theorem although it was not explicitly used in the algorithm. On the other hand many theoretic results have been obtained from the algorithmic view point. Thus we have tried to include most of the important theorems and algorithms that are currently known for planar graphs. Furthermore we have tried to provide constructive proofs for theorems, from which algorithms immediately follow. Most of the algorithms are written in Pidgin PASCAL in a manner that will make their adaptation to a practical programming language relatively easy. They are all efficient, and most of them are the best known ones; the complexities are linear or $O(n \log n)$.

A glance at the table of contents will provide an outline of the topics to be discussed. The first two chapters are introductory in the sense that they provide the foundations, respectively, of the graph theoretic notions and algorithmic techniques that will be used in the book. Experts in graph theory or algorithms may skip Chapters 1 or 2. The remaining chapters discuss the topics on planarity testing, embedding, drawing, vertex- or edge-coloring, maximum independent set, subgraph listing, planar separator theorem, Hamiltonian cycles, and single- or multicommodity flows. The topics reflect the authors' favor as graph theorists and computer scientists. The chapters are structured in such a way that the book will be suitable as a textbook in a course on algorithms, graph theory, or planar graphs. In addition, the book will be useful for computer scientists and graph theorists at the research level. An extensive reference section is also included.

Sendai Takao Nishizeki and Norishige Chiba

ACKNOWLEDGMENTS

It is a pleasure to record the authors' gratitude to those to whom we are indebted, directly or indirectly, in writing this book. First of all, our thanks are due to Professor Nobuji Saito of Tohoku University under whose guidance we have studied and worked since our student days; without his continual encouragement this book would not have been completed.

The present book is based primarily upon a series of the authors' own investigations. We wish to thank the coauthors of our joint papers: S. Abe, T. Asano, I. Baybars, H. Gabow, D. Hochbaum, O. Kariv, D. Leven, K. Matsumoto, K. Onoguchi, T. Ozawa, M. Sato, D. Shmoys, K. Suzuki, H. Suzuki, K. Takamizawa, O. Terada, T. Watanabe and T. Yamanouchi.

A book such as this one owes a great deal, of course, to many previous workers and writers. Without trying to be complete, we would like to mention the very important succession of books and papers by Aho, Hopcroft and Ullman; Baker; Behzad, Chartrand and Lesniak-Foster; Bollobás; Booth and Lueker; Even; Frederickson; Golumbic; Hassin; Hopcroft and Tarjan; Lipton and Tarjan; Miller; Okamura; Okamura and Seymour; Ore; Papadimitriou and Steiglitz; Seymour; Tarjan; Thomassen; Tutte; and Wilson.

Finally, following a Western custom, we should thank our children's mothers, Kayoko Nishizeki and Yumiko Chiba for many things which have nothing at all to do with planar graphs.

CONTENTS

CHAPTER 1

GRAPH THEORETIC FOUNDATIONS

1.1. Introduction

A graph can be thought of as a diagram consisting of a collection of vertices together with edges joining certain pairs of vertices. A planar graph is a particular diagram which can be drawn on the plane so that no two edges intersect geometrically except at a vertex at which they are both incident.

First consider the example depicted in Fig. 1.1(a), which consists of six vertices (drawn by small black cycles) and 12 edges (drawn by straight lines there). Is the graph planar? That is, can you draw the graph on the plane by locating vertices and drawing edges appropriately in such a way that no two edges intersect except at a common endvertex? The drawing in Fig. 1.1(a), as it is, has two intersections in the circles drawn by dotted lines. However, one can avoid them if the vertex v_6 is located in the exterior of the square $v_1v_2v_3v_4\ v_1$ (drawn by a thick line), as shown in Fig. 1.1(b). Thus the graph is known to be planar. Next consider the graph depicted in Fig. 1.2(a), known as the complete graph K_5 on five vertices. Is K_5 planar? If we suppose so, then one may assume without loss of generality that $v_1v_2v_3v_4v_5$ is drawn on the plane as a regular pentagon. (Look on the plane as flexible rubber, and deform it as desired.) One may also assume that the edge (v_1, v_3) is drawn in the interior of the pentagon. Then both the edges (v_2, v_5) and (v_2, v_4) must be drawn in the exterior, and consequently edge (v_3, v_5) must be drawn in the interior, as shown in Fig. 1.3(a). Then an intersection must occur whether the edge (v_1, v_4) is drawn in the interior or exterior. Thus K_5 cannot be drawn on the plane without edge-crossing, so is nonplanar. Another example of nonplanar graphs is the "complete bipartite graph" $K_{3,3}$ depicted in Fig. 1.2(b). One may assume that edge (u_1, v_2) is drawn in the interior of the hexagon $u_1v_1u_2v_2u_3v_3$, and hence edge (v_1, u_3) in the exterior. Then (u_2, v_3) cannot be drawn without producing an intersection. Thus $K_{3,3}$ is also known to be nonplanar.

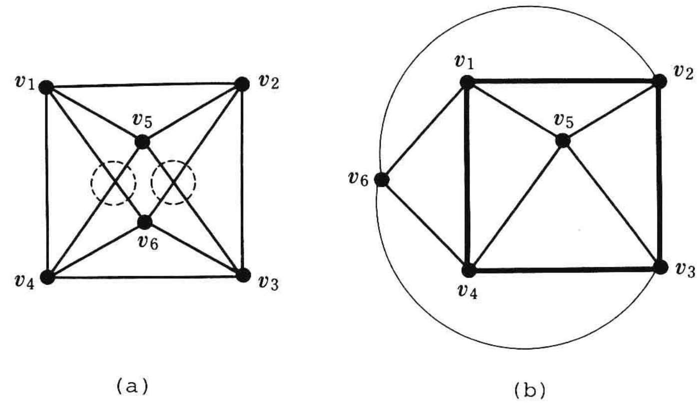

Fig. 1.1. (a) A graph G; and (b) A plane embedding of G.

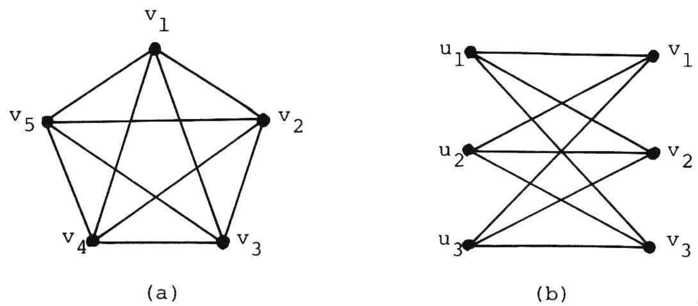

Fig. 1.2. Kuratowski's graphs: (a) Complete graph K_5; (b) Complete bipartite graph $K_{3,3}$.

As above, not all graphs are planar. However planar graphs arise quite naturally in real-world applications, such as road or railway maps, electric printed circuits, chemical molecules, etc. Planar graphs play an important role in these problems, partly due to the fact that some practical problems can be efficiently solved for planar graphs even if they are intractable for general graphs. Moreover, a number of interesting and applicable results are known concerning the mathematical and algorithmic properties of planar graphs. Thus the theory of planar graphs has emerged as a worthwhile mathematical discipline in its own right.

1.2. Some basic definitions

Let us formally define the notion of a graph. A *graph* $G = (V, E)$ is a structure which consists of a finite set of *vertices* V and a finite set of *edges* E;

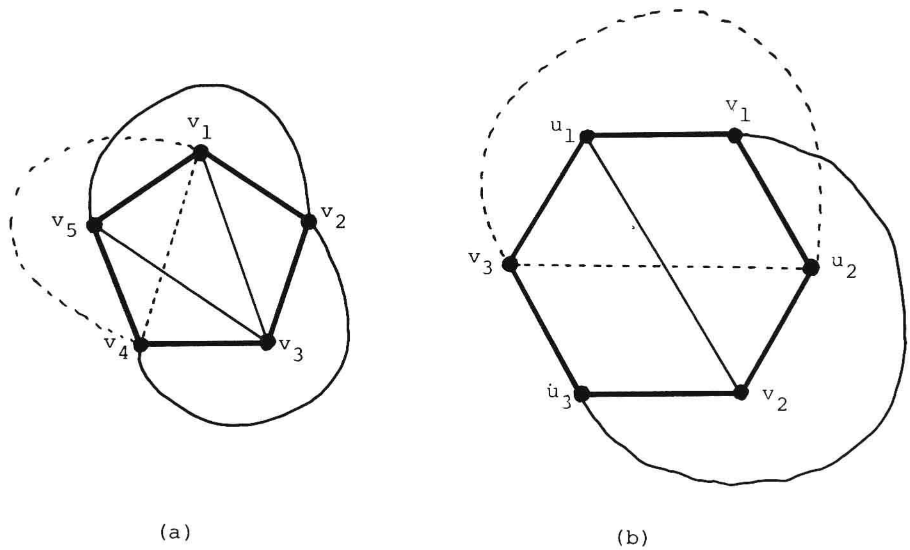

Fig. 1.3. Partial embedding of (a) K_5 and (b) $K_{3,3}$.

each edge is an unordered pair of distinct vertices. Throughout the book n denotes the number of vertices of G, that is, $n = |V|$, while m denotes the number of edges, that is, $m = |E|$. Any edge of the form (u, v) is said to *join* the vertices u and v. Our graph G is a so-called *simple finite graph*, that is, G has no "multiple edges" or "loops" and sets V and E are finite. *Multiple edges* join the same pair of vertices, while a *loop* joins a vertex to itself. The graph, in which loops and multiple edges are allowed, is called a *multigraph*. The graph, in which E is defined to be a set of ordered pairs of distinct vertices, is a *directed graph* (*digraph* for short).

If $(u, v) \in E$, then two vertices u and v of a graph G are said to be *adjacent*; u and v are then said to be *incident* to edge (u, v); u is a *neighbour* of v. The *neighbourhood* $N(v)$ is the set of all neighbours of v. Two distinct edges are *adjacent* if they have a vertex in common. The *degree* of a vertex v of G is the number of edges incident to v, and is written as $d(G, v)$ or simply $d(v)$. In the graph G depicted in Fig. 1.1(a) vertices v_1 and v_2 are adjacent; $N(v_1) = \{v_2, v_4, v_5, v_6\}$, and hence $d(v_1) = 4$.

We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. If $V' = V$ then G' is called a *spanning* subgraph of G. If G' contains all the edges of G that join two vertices in V' then G' is said to be *induced by V'*. If V' consists of exactly the vertices on which edges in E' are incident, then G' is said to be *induced by E'*. Fig. 1.4(a) depicts a spanning subgraph of G in Fig. 1.1(b); Fig. 1.4(b) depicts a subgraph induced by $V' = \{v_1, v_2, v_4, v_5\}$; Fig. 1.4(c) depicts a subgraph induced by $\{(v_1, v_2), (v_1, v_4), (v_1, v_5), (v_2, v_5)\}$.

We shall often construct new graphs from old ones by deleting some vertices or edges. If $V' \subset V$ then $G - V'$ is the subgraph of G obtained by deleting the vertices in V' and all edges incident on them, that is, $G - V'$ is a subgraph induced by $V - V'$. Similarly if $E' \subset E$ then $G - E' = (V, E - E')$. If $V' = \{v\}$ and $E' = \{(u, v)\}$ then this notation is simplified to $G - v$ and $G - (u, v)$.

We also denote by G/e the graph obtained by taking an edge e and *contracting* it, that is, removing e and identifying its ends u and v in such a way that the resulting vertex is adjacent to those vertices (other than u and v) which were originally adjacent to u or v. For $E' \subset E$ we denote by G/E' the graph which results from G after a succession of such contractions for the edges in E'. The graph G/E' is called a *contraction* of G.

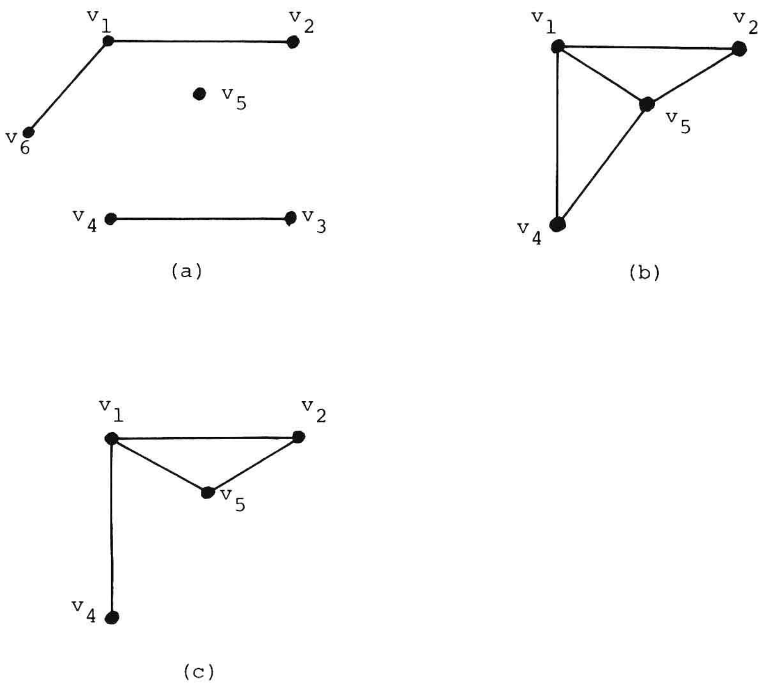

Fig. 1.4. Subgraphs of G in Fig. 1.1(b): (a) Spanning subgraph; (b) Vertex-induced subgraph; (c) Edge-induced subgraph.

A v_0-v_l *walk* in G is an alternating sequence of vertices and edges of G, $v_0, e_1, v_1, \ldots, v_{l-1}, e_l, v_l$, beginning and ending with a vertex, in which each edge is incident on two vertices immediately preceding and following it. The number l of edges is called its *length*. If the vertices $v_0, v_1, \ldots, v_l$ are distinct (except, possibly, $v_0 = v_l$), then the walk is called a *path* and is usually denoted by $v_0v_1 \cdots v_l$. A path or walk is *closed* if $v_0 = v_l$. A closed path containing at least one edge is called a *cycle*. A cycle of length 3, 4, 5, . . . , is called a triangle, quadrilateral, pentagon, etc. One example of walks in G depicted in Fig. 1.1(b) is

$$v_1, (v_1, v_2), v_2, (v_2, v_3), v_3, (v_3, v_5), v_5,$$

$$(v_5, v_2), v_2, (v_2, v_3), v_3, (v_3, v_4), v_4,$$

which is not closed, that is *open*. One example of cycles is $v_1v_2v_3v_4v_1$, a quadrilateral.

The dual concept of a cycle is a "cutset" which we now define. A *cut* of a graph G is a set of edges of G whose removal increases the number of components. A *cutset* is defined to be a cut no proper subset of which is a cut, that is, a cutset is a minimal cut. Fig. 1.5 illustrates these concepts; $\{a, b, c, d, e\}$ is a cut but not a cutset; both $\{a, b, c\}$ and $\{d, e\}$ are cutsets.

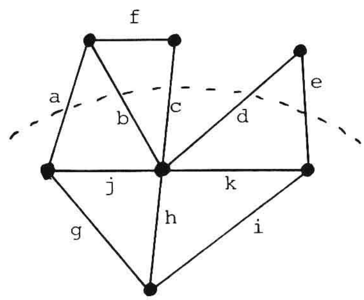

Fig. 1.5. Cut and cutset.

A graph G is *connected* if for every pair $\{u, v\}$ of distinct vertices there is a path between u and v. A (*connected*) *component* of a graph is a *maximal connected subgraph*. A *cutvertex* is a vertex whose deletion increases the number of components. Similarly an edge is a *bridge* if its deletion increases the number of components. G is 2-*connected* if G is connected and has no cutvertex. A *block* of G is a maximal 2-connected subgraph of G. A *separation pair* of a 2-connected graph G is two vertices whose deletion disconnects G. G is 3-*connected* if G has no cutvertex or separation pair. In general, a *separating set* of a connected graph G is a set of vertices of G whose deletion disconnects G. The graph in Fig. 1.4(a) is disconnected, and has three components; the graph in Fig. 1.4(c) which is not 2-connected has a cutvertex v_1, a bridge (v_1, v_4) and two blocks; the graph in Fig. 1.4(b) has no cutvertex but has a separation pair $\{v_1, v_5\}$, so is 2-connected but not 3-connected; G in Fig. 1.1(b) has no cutvertex or separation pair, so is 3-connected.

If G has a separation pair $\{x, y\}$, then we often split G into two graphs G_1

and G_2, called *split graphs*. Let $G_1' = (V_1, E_1')$ and $G_2' = (V_2, E_2')$ be two subgraphs satisfying the following conditions (a) and (b):

(a) $V = V_1 \cup V_2, V_1 \cap V_2 = \{x, y\}$;

(b) $E = E_1' \cup E_2', \quad E_1' \cap E_2' = \varnothing, \quad |E_1'| \geq 2, \quad |E_2'| \geq 2.$

Define G_1 to be the graph obtained from G_1' by adding a new edge (x, y) if it does not exist; similarly define G_2. (See Fig. 1.6.)

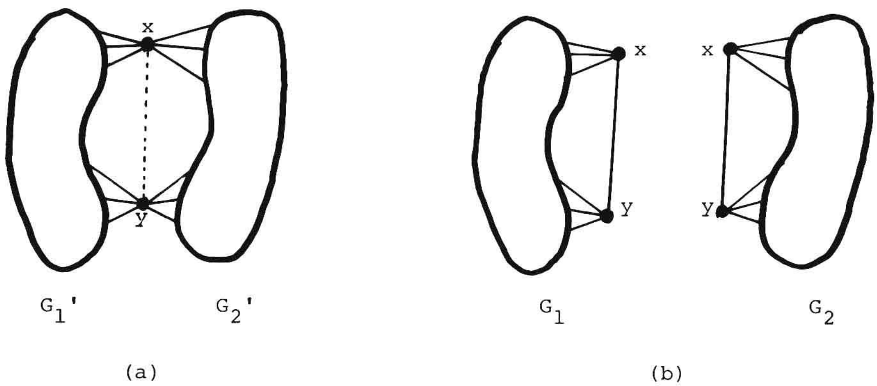

Fig. 1.6. (a) A graph G with a separation pair $\{x, y\}$, where edge (x, y) may not exist; (b) Split graphs G_1 and G_2.

Before ending this section, we will define some special graphs. A graph without any cycles is a *forest*; a *tree* is a connected forest. Fig. 1.4(a) is a forest having three components.

A graph in which every pair of distinct vertices are adjacent is called a *complete graph*. The complete graph on n vertices is denoted by K_n. K_5 has been depicted in Fig. 1.2(a).

Suppose that the vertex set V of a graph G can be partitioned into two disjoint sets V_1 and V_2, in such a way that every edge of G joins a vertex of V_1 to a vertex of V_2; G is then said to be a *bipartite graph*. If every vertex of V_1 is joined to every vertex of V_2, then G is called a *complete bipartite graph* and is denoted by $K_{s,r}$ where $s = |V_1|$ and $r = |V_2|$. Fig. 1.2(b) depicts a complete bipartite graph $K_{3,3}$ with partite sets $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$.

1.3. Planar graphs

Let us formally define a planar graph. Draw a graph G in the given space (e.g. plane) with points representing vertices of G and curves representing edges. G