FOURIER SERIES

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Translated from the Russian by
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TRANSLATOR'S PREFACE

The present volume is the second in a new series of translations of outstanding Russian textbooks and monographs in the fields of mathematics, physics and engineering, under my editorship. It is hoped that Professor Tolstov's book will constitute a valuable addition to the English-language literature on Fourier series.

The following two changes, made with Professor Tolstov's consent, are worth mentioning:

1. To enhance the value of the English-language edition, a large number of extra problems have been added by myself and Professor Allen L. Shields of the University of Michigan. We have consulted a variety of sources, in particular, *A Collection of Problems in Mathematical Physics* by N. N. Lebedev, I. P. Skalskaya, and Y. S. Uflyand (Moscow, 1957), from which most of the problems appearing at the end of Chapter 9 have been taken.

2. To keep the number of cross references to a minimum, four chapters (8 and 9, 10 and 11) of the Russian original have been combined to make two chapters (8 and 9) of the present edition

I have also added a Bibliography, containing suggestions for collateral and supplementary reading. Finally, it should be noted that sections marked with asterisks contain material of a more advanced nature, which can be omitted without loss of continuity.

R. A. S.

FOURIER SERIES

CONTENTS

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TRIGONOMETRIC FOURIER SERIES

1. Periodic Functions

A function $f(x)$ is called ***periodic*** if there exists a constant $T > 0$ for which

$$f(x + T) = f(x), \tag{1.1}$$

for any x in the domain of definition of $f(x)$. (It is understood that both x and $x + T$ lie in this domain.) Such a constant T is called a *period* of the function $f(x)$. The most familiar periodic functions are $\sin x$, $\cos x$, $\tan x$, etc. Periodic functions arise in many applications of mathematics to problems of physics and engineering. It is clear that the sum, difference, product, or quotient of two functions of period T is again a function of period T.

If we plot a periodic function $y = f(x)$ on any closed interval $a \leqslant x \leqslant a + T$, we can obtain the entire graph of $f(x)$ by periodic repetition of the portion of the graph corresponding to $a \leqslant x \leqslant a + T$ (see Fig. 1).

If T is a period of the function $f(x)$, then the numbers $2T, 3T, 4T, \ldots$ are also periods. This follows immediately by inspecting the graph of a periodic function or from the series of equalities[1]

$$f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \cdots$$

[1] We suggest that the reader prove the validity not only of these equalities but also of the following equalities:

$$f(x) = f(x - T) = f(x - 2T) = f(x - 3T) = \cdots$$

which are obtained by repeated use of the condition (1.1). Thus, if T is a period, so is kT, where k is any positive integer, i.e., if a period exists, it is *not unique*.

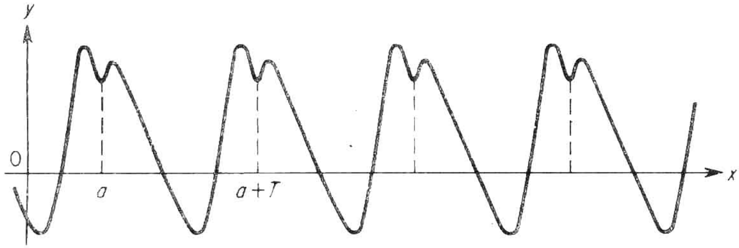

FIGURE 1

Next, we note the following property of any function $f(x)$ of period T:

If $f(x)$ is integrable on any interval of length T, then it is integrable on any other interval of the same length, and the value of the integral is the same, i.e.,

$$\int_a^{a+T} f(x)\,dx = \int_b^{b+T} f(x)\,dx, \tag{1.2}$$

for any a and b.

This property is an immediate consequence of the interpretation of an integral as an area. In fact, each integral (1.2) equals the area included between the curve $y = f(x)$, the x-axis and the ordinates drawn at the end points of the interval, where areas lying above the x-axis are regarded as positive and areas lying below the x-axis are regarded as negative. In the present case, the areas represented by the two integrals are the same, because of the periodicity of $f(x)$ (see Fig. 2).

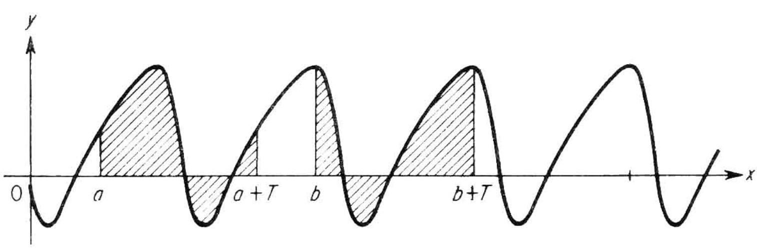

FIGURE 2

Hereafter, when we say that a function $f(x)$ of period T is *integrable*, we shall mean that it is integrable on an interval of length T. It follows from the property just proved that $f(x)$ is also integrable on any interval of finite length.

2. Harmonics

The simplest periodic function, and the one of greatest importance for the applications, is

$$y = A \sin (\omega x + \varphi),$$

where A, ω, and φ are constants. This function is called a *harmonic of amplitude* $|A|$, *(angular) frequency* ω, *and initial phase* φ. The period of such a harmonic is $T = 2\pi/\omega$, since for any x

$$A \sin \left[\left(x + \frac{2\pi}{\omega}\right)\omega + \varphi\right] = A \sin [(\omega x + \varphi) + 2\pi] = A \sin (\omega x + \varphi).$$

The terms "amplitude," "frequency," and "initial phase" stem from the following mechanical problem involving the simplest kind of oscillatory motion, i.e., *simple harmonic motion*: Suppose that a point mass M, of mass m, moves along a straight line under the action of a *restoring force* F which is proportional to the distance of M from a fixed origin O and which is directed towards O (see Fig. 3). Regarding s as positive if M lies to the right of O and

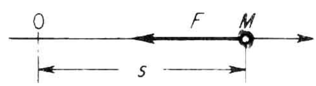

FIGURE 3

negative if M lies to the left of O, i.e., assigning the usual positive direction to the line, we find that $F = -ks$, where $k > 0$ is a constant of proportionality. Therefore

$$m \frac{d^2s}{dt^2} = -ks$$

or

$$\frac{d^2s}{dt^2} + \omega^2 s = 0,$$

where we have written $\omega^2 = k/m$, so that $\omega = \sqrt{k/m}$.

It is easily verified that the solution of this differential equation is the function $s = A \sin (\omega t + \varphi)$, where A and φ are constants, which can be calculated from a knowledge of the position and velocity of the point M at the initial time $t = 0$. This function s is a harmonic, and in fact is a periodic function of time with period $T = 2\pi/\omega$. Thus, under the action of the restoring force F, the point M undergoes oscillatory motion. The amplitude $|A|$ is the maximum deviation of the point M from O, and the quantity $1/T$ is the number of oscillations in an interval containing 2π units of time (e.g., seconds). This explains the term "frequency". The quantity φ is the initial phase and characterizes the initial position of the point, since for $t = 0$ we have $s_0 = \sin \varphi$.

We now examine the appearance of the curve $y = A \sin(\omega x + \varphi)$. We assume that $\omega > 0$, since otherwise $\sin(-\omega x + \varphi)$ is merely replaced by $-\sin(\omega x - \varphi)$. The simplest case is obtained when $A = 1$, $\omega = 1$, $\varphi = 0$; this gives the familiar *sine curve* $y = \sin x$ [see Fig. 4(a)]. For $A = 1$, $\omega = 1$, $\varphi = \pi/2$, we obtain the *cosine curve* $y = \cos x$, whose graph is the same as that of $y = \sin x$ shifted to the left by an amount $\pi/2$.

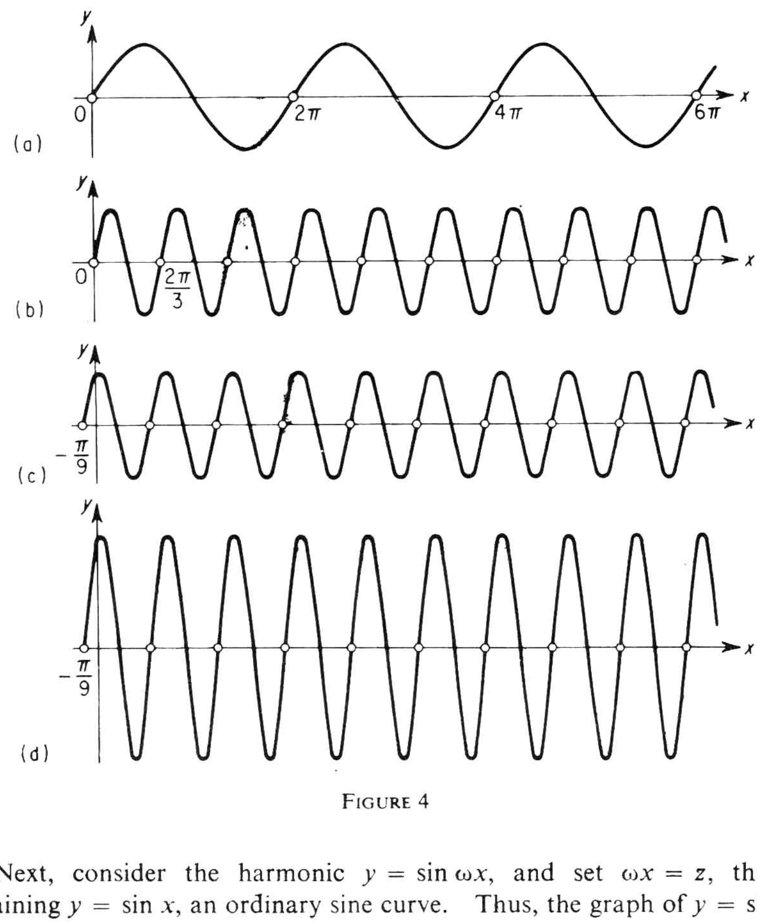

FIGURE 4

Next, consider the harmonic $y = \sin \omega x$, and set $\omega x = z$, thereby obtaining $y = \sin x$, an ordinary sine curve. Thus, the graph of $y = \sin \omega x$ is obtained by deforming the graph of a sine curve: This deformation reduces to a uniform compression along the x-axis by a factor ω if $\omega > 1$, and to a uniform expansion along the x-axis by a factor $1/\omega$ if $\omega < 1$. Figure 4(b) shows the harmonic $y = \sin 3x$, of period $T = 2\pi/3$.

Now, consider the harmonic $y = \sin(\omega x + \varphi)$, and set $\omega x + \varphi = \omega z$, so that $x = z - \varphi/\omega$. We already know the graph of $\sin \omega z$. Therefore, the graph of $y = \sin(\omega x + \varphi)$ is obtained by shifting the graph of $y = \sin \omega x$ along the x-axis by the amount $-\varphi/\omega$. Figure 4(c) represents the harmonic

$$y = \sin\left(3x + \frac{\pi}{3}\right)$$

with period $2\pi/3$ and initial phase $\pi/3$.

Finally, the graph of the harmonic $y = A \sin(\omega x + \varphi)$ is obtained from that of the harmonic $y = \sin(\omega x + \varphi)$ by multiplying all ordinates by the number A. Figure 4(d) shows the harmonic

$$y = 2 \sin\left(3x + \frac{\pi}{3}\right).$$

These results may be summarized as follows:

The graph of the harmonic $y = A \sin(\omega x + \varphi)$ is obtained from the graph of the familiar sine curve by uniform compression (or expansion) along the coordinate axes plus a shift along the x-axis.

Using a well-known formula from trigonometry, we write

$$A \sin(\omega x + \varphi) = A(\cos \omega x \sin \varphi + \sin \omega x \cos \varphi).$$

Then, setting

$$a = A \sin \varphi, \qquad b = A \cos \varphi, \tag{2.1}$$

we convince ourselves that every harmonic can be represented in the form

$$a \cos \omega x + b \sin \omega x. \tag{2.2}$$

Conversely, every function of the form (2.2) is a harmonic. To prove this, it is sufficient to solve (2.1) for A and B. The result is

$$A = \sqrt{a^2 + b^2}, \quad \sin \varphi = \frac{a}{A} = \frac{a}{\sqrt{a^2 + b^2}}, \quad \cos \varphi = \frac{b}{A} = \frac{b}{\sqrt{a^2 + b^2}},$$

from which φ is easily found.

From now on, we shall write harmonics in the form (2.2). For example, for the harmonic shown in Fig. 4(d), this form is

$$2 \sin\left(3x + \frac{\pi}{3}\right) = \sqrt{3} \cos 3x + \sin 3x$$

It will also be convenient to explicitly introduce the period T in (2.2). If we set $T = 2l$, then, since $T = 2\pi/\omega$, we have

$$\omega = \frac{2\pi}{T} = \frac{\pi}{l},$$

and therefore, the harmonic with period $T = 2l$ can be written as

$$a \cos \frac{\pi x}{l} + b \sin \frac{\pi x}{l}. \tag{2.3}$$

3. Trigonometric Polynomials and Series

Given the period $T = 2l$, consider the harmonics

$$a_k \cos \frac{\pi k x}{l} + b_k \sin \frac{\pi k x}{l} \qquad (k = 1, 2, \ldots) \tag{3.1}$$

with frequencies $\omega_k = \pi k/l$ and periods $T_k = 2\pi/\omega_k = 2l/k$. Since

$$T = 2l = kT_k,$$

the number $T = 2l$ is simultaneously a period of all the harmonics (3.1), for an integral multiple of a period is again a period (see Sec. 1). Therefore, every sum of the form

$$s_n(x) = A + \sum_{k=1}^{n} \left(a_k \cos \frac{\pi k x}{l} + b_k \sin \frac{\pi k x}{l}\right),$$

where A is a constant, is a function of period $2l$, since it is a sum of functions of period $2l$. (The addition of a constant obviously does not destroy periodicity; in fact, a constant can be regarded as a function for which *any* number is a period.) The function $s_n(x)$ is called a *trigonometric polynomial of order n* (and period $2l$).

Even though it is a sum of various harmonics, a trigonometric polynomial in general represents a function of a much more complicated nature than a simple harmonic. By suitably choosing the constants $A, a_1, b_1, a_2, b_2, \ldots$ we can form functions $y = s_n(x)$ with graphs quite unlike the smooth and symmetric graph of a simple harmonic. For example, Fig. 5 shows the trigonometric polynomial

$$y = \sin x + \tfrac{1}{2} \sin 2x + \tfrac{1}{4} \sin 3x.$$

The *infinite trigonometric series*

$$A + \sum_{k=1}^{\infty} \left(a_k \cos \frac{\pi k x}{l} + b_k \sin \frac{\pi k x}{l}\right)$$

(if it converges) also represents a function of period $2l$. The nature of functions which are sums of such infinite trigonometric series is even more diverse. Thus, the following question arises naturally: Can any given

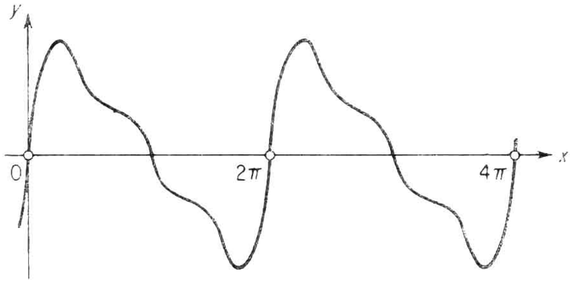

FIGURE 5

function of period $T = 2l$ be represented as the sum of a trigonometric series? We shall see later that such a representation is in fact possible for a very wide class of functions.

For the time being, suppose that $f(x)$ belongs to this class. This means that $f(x)$ can be expanded as a sum of harmonics, i.e., as a sum of functions with a very simple structure. The graph of the function $y = f(x)$ is obtained as a "superposition" of the graphs of these harmonics. Thus, to give a mechanical interpretation, we can represent a complicated oscillatory motion $f(x)$ as a sum of individual oscillations which are particularly simple. However, one must not imagine that trigonometric series are applicable only to oscillation phenomena. This is far from being the case. In fact, the concept of a trigonometric series is also very useful in studying many phenomena of a quite different nature.

If

$$f(x) = A + \sum_{k=1}^{\infty} \left(a_k \cos \frac{\pi k x}{l} + b_k \sin \frac{\pi k x}{l} \right), \tag{3.2}$$

then, setting $\pi x / l = t$ or $x = tl/\pi$, we find that

$$\varphi(t) = f\left(\frac{tl}{\pi}\right) = A + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt), \tag{3.3}$$

where the harmonics in this series all have period 2π. This means that if a function $f(x)$ of period $2l$ has the expansion (3.2), then the function $\varphi(t) = f(tl/\pi)$ is of period 2π and has the expansion (3.3). Obviously, the converse is also true, i.e., if a function $\varphi(t)$ of period 2π has the expansion (3.3), then the function $f(x) = \varphi(\pi x / l)$ is of period $2l$ and has the expansion (3.2).

Thus, it is enough to know how to solve the problem of expansion in trigonometric series for functions of the "standard" period 2π. Moreover, in this case, the series has a simpler appearance. Therefore, we shall develop the theory for series of the form (3.3), and only the final results will be converted to the "language" of the general series (3.2).

4. A More Precise Terminology. Integrability. Series of Functions

We now introduce a more precise terminology and recall some facts from differential and integral calculus. When we say that $f(x)$ is integrable on the interval $[a, b]$, we mean that the integral

$$\int_a^b f(x)\,dx \tag{4.1}$$

(which may be improper) exists in the elementary sense. Thus, our integrable functions $f(x)$ will always be either continuous or have a finite number of points of discontinuity in the interval $[a, b]$, at which the function can be either bounded or unbounded.

In courses on integral calculus, it is proved that if a function has a finite number of discontinuities, then if the integral

$$\int_a^b |f(x)|\,dx$$

exists, so does the integral (4.1). (The converse is not always true.) In this case, the function $f(x)$ is said to be *absolutely integrable*. If $f(x)$ is absolutely integrable and $\varphi(x)$ is a bounded integrable function, then the product $f(x)\varphi(x)$ is absolutely integrable. The following rule for integration by parts holds:

Let $f(x)$ and $\varphi(x)$ be continuous on $[a, b]$, but perhaps non-differentiable at a finite number of points. Then, if $f'(x)$ and $\varphi'(x)$ are absolutely integrable,[2] *we have*

$$\int_a^b f(x)\varphi'(x)\,dx = \Big[f(x)\varphi(x)\Big]_{x=a}^{x=b} - \int_a^b f'(x)\varphi(x)\,dx. \tag{4.2}$$

Another familiar result is the fact that if the functions $f_1(x), f_2(x), \ldots, f_n(x)$ are integrable on $[a, b]$, then their sum is also integrable, and

$$\int_a^b \left[\sum_{k=1}^n f_k(x)\right] dx = \sum_{k=1}^n \int_a^b f_k(x)\,dx. \tag{4.3}$$

[2] Instead of absolute integrability of both derivatives, we can weaken this requirement to absolute integrability of just one of the derivatives. However, the stronger form of the requirement is sufficient for what follows.