Lecture Notes in Mathematics
1491

E. Lluis-Puebla J. L. Loday H. Gillet
C. Soulé V. Snaith

Higher Algebraic K-Theory: an overview

Springer-Verlag

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Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

Authors

Emilio Lluis-Puebla
Departamento de Matématicas
Facultad de Ciencias
Circuito Exterior
Ciudad Universitaria
México 04510 D. F., Mecixo

Jean-Louis Loday
Institut de Recherche Mathématique Avancée
ULP et CNRS, 7, rue René Descartes
67084 Strasbourg, France

Henri Gillet
Department of Mathematics
University of Illinois Chicago
Box 4348
Chicago, IL60680, USA

Christophe Soulé
Institut des Hautes Etudes Scientifiques
35, Route de Chartres
91440 Bures-sur-Yvette, France

Victor Snaith
Department of Mathematics and Statistics
Mc Master University
1280 Main St. W.,
Hamilton, Ontario, Canada L8S 4K1

Mathematics Subject Classification (1980): 19, 11, 13, 14, 55

ISBN 3-540-55007-0 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-55007-0 Springer-Verlag New York Berlin Heidelberg

Printed in Germany

Typesetting: Camera ready by author
Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
46/3140-543210 - Printed on acid-free paper

Preface

This volume consists of five different papers on Higher Algebraic K-theory. These are based on several series of lectures delivered during the "First seminar on Algebraic K-theory" at the "Universidad Nacional Autonoma de México" in 1985. Their purpose is to introduce students to this important field by explaining the basic concepts, surveying the main ideas and results, and describing some of the most recent developments.

Each series of lectures has its own perspective (algebra, algebraic geometry or number theory) and is somewhat independent of the other ones. For instance, a basic notion like the cohomology of groups is presented several times. On the other hand, their combination should give a fairly comprehensive overview of the field. Occasionally, results explained in one series of lectures are used (and then explicitly referred to) in another one.

Proofs are not always given, but we hope that the reader will find this volume enjoyable and useful as an introduction to the vast literature.

I would like to thank the Universidad Nacional Autonoma de México and the director of the Faculty of Sciences, Dr. Félix Recillas, for their support and encouragement without which this meeting would not have taken place. Also all my thanks go to Carolina Bello and Ingeborg Jebram for typing this manuscript.

Emilio Lluis-Puebla.

Contents

Introduction to Algebraic K-Theory

Emilio Lluis-Puebla

Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México

These notes contain a series of ten lectures delivered at the "Universidad Nacional Autónoma de México" on its "Primer Seminario de K-Teoría Algebraica". They provide an introduction to the subject as well as to the other expositions in this volume.

In Chapter I we review some important concepts from Homological Algebra starting from the elementary concepts and assuming a knowledge of the reader of Group and Ring Theory only. In Chapter II we present the (Co)Homology of Groups in a very elementary way underlying the relevant results used to establish its relation with Algebraic K-Theory.

In Chapter III we define the basic concepts of (classical) Algebraic K-Theory and establish its relation with the Homology of Groups.

Contents

I. Some Homological Algebra

I.1. Modules

Let Λ be a ring (not necessarily commutative) with $1 \neq 0$.

Definition 1.1. A left *Λ-module* or *module over Λ* is a pair (M, μ) where M is an additive abelian group and $\mu : \Lambda \times M \longrightarrow M$ is a function written $(\alpha, x) \mapsto \alpha x$ such that the following axioms are verified:

i) $\alpha(x + y) = \alpha x + \alpha y$
ii) $(\alpha + \beta)x = \alpha x + \beta x$
iii) $(\alpha\beta)x = \alpha(\beta x)$
iv) $1x = x \qquad (\alpha, \beta \in \Lambda; x, y \in M)$.

μ is called a *scalar multiplication of* M and the elements of Λ are called *scalars.*

For example, take $\Lambda = \mathbb{Z}$, hence any abelian group can be considered as a $\mathbb{Z}$-module. Also if Λ is a field k, a k-module is just a vector space over k. So the concept of Λ-module is a proper generalization of the concepts of abelian group and vector space.

Similarly we can define a right Λ-module using the scalar multiplication (on the right) $\mu : M \times \Lambda \longrightarrow M$ and writing appropriately the axioms. If Λ is commutative then every left Λ-module is a right Λ-module and vice versa.

This objects (the Λ-modules) are not so special in the sense that (in categorical language) every small abelian category can be considered inside a module category over and adequate ring.

How do we relate two Λ-modules? We relate two sets using functions; we relate two groups using functions that preserve the group structure. So we will relate two modules by means of functions that preserve the Λ-module structure called *homomorphisms.*

Definition 1.2. Let M and N be two Λ-modules. A function $f : M \longrightarrow N$ is called a *Λ-module homomorphism* if $f(x+y) = f(x)+f(y)$ and $f(\alpha x) = \alpha(f(x))$ for all $\alpha \in \Lambda$; $x, y \in M$.

In Module Theory we also talk about the *kernel* and the *image* of a homomorphism $f : M \longrightarrow N$ defined as follows:

$$\ker f = \{x \in M | f(x) = 0\}$$
$$\operatorname{im} f = \{f(x) \in N | x \in M\} \ .$$

Also, we define N to be a *submodule* of a Λ-module M if N is a subgroup of M and for all $\alpha \in \Lambda$, $\alpha N = \{\alpha x | x \in N\} \subset N$.

The composition of homomorphisms turns out to be a homomorphism; the image under a homomorphism of a submodule is a submodule; the inverse image of a submodule under a homomorphism is a submodule; and, in particular the kernel and the image of a homomorphism are submodules.

We also have the concept of quotient module whose elements are the distinct cosets of a submodule in a module. In particular we define the *cokernel* of a module homomorphism $f : M \to N$ as the quotient module: $\operatorname{coker} f = N/\operatorname{im} f$.

We will relate several Λ-modules by means of a collection of homomorphisms as follows:

Definition 1.3. We will say that a sequence of Λ-modules

$$\cdots \to M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \cdots$$

is *semiexact* in M_i if $\operatorname{im} f_{i-1} \subset \ker f_i$. If it is semiexact in each module we will call it a *semiexact sequence.*

This definition is equivalent to say that the composition $f \circ f_{i-1}$ is the *trivial homomorphism* denoted by 0, i.e. $f_i \circ f_{i-1} = 0$. We will say that the sequence of (1.3) is *exact* in M_i if it is semiexact and $\operatorname{im} f_{i-1} \supset \ker f_i$, i.e. if $\operatorname{im} f_{i-1} = \ker f_i$. If it is exact in each M_i it is called an *exact sequence.* Of course, every exact sequence is a semiexact sequence but the converse is not true.

An exact sequence of the form

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

where 0 denotes the trivial module is called a *short exact sequence.* It is immediate that f is an injective homomorphism called *monomorphism* and g is an onto homomorphism called *epimorphism.* The following notation is used to represent a short exact sequence:

$$M' \overset{f}{\rightarrowtail} M \overset{g}{\twoheadrightarrow} M'' \ .$$

It is just a mask of a submodule and a quotient module of a Λ-module M in the exact sequence

$$N \rightarrowtail M \twoheadrightarrow M/N \ .$$

There is a category Mod_Λ whose objects are the Λ-modules and whose morphisms are the homomorphisms of Λ-modules. Mod_Λ is studied by analyzing the behavior of certain functors defined on it. The most important are hom and $\otimes$ and certain functors derived from them.

Let $\hom_\Lambda(M, N)$ denote the set of all homomorphisms from the Λ-module M to the Λ-module N. If Λ is not commutative then $\hom_\Lambda(M, N)$ is always an abelian group and if Λ is commutative $\hom_\Lambda(M, N)$ is a Λ-module.

The following theorem can be considered as the beginning of Homological Algebra.

Theorem 1.4. *a) If $N' \overset{\psi}{\rightarrowtail} N \xrightarrow{\psi'} N''$ is an exact sequence of Λ-modules then there is an exact induced sequence*

$$0 \longrightarrow \hom_\Lambda(M, N') \xrightarrow{\psi_*} \hom_\Lambda(M, N) \xrightarrow{\psi'_*} \hom_\Lambda(M, N'') \ .$$

b) If $M' \xrightarrow{\varphi} M \overset{\varphi'}{\twoheadrightarrow} M''$ *is an exact sequence of* Λ*-modules then there is an exact induced sequence*

$$\hom_\Lambda(M', N) \xleftarrow{\varphi^*} \hom_\Lambda(M, N) \xleftarrow{\varphi'^*} \hom_\Lambda(M'', N) \longleftarrow 0 \ .$$

We can expect that the case when ψ'_* is onto be of interest. It really is. The class of Λ-modules that satisfy ψ'_* to be onto whenever ψ' is onto are called *projective* Λ-modules. By imitating the case of k-modules (k a field) i.e. vector spaces, we say that a Λ-module is *free* if it has a *basis*. It turns out that every free module is projective and that every Λ-module is a quotient of a free module. Dually we can define the concept of an *injective* Λ-module as the one who makes φ^* an epimorphism whenever φ is a monomorphism. It turns out that every module is isomorphic to a submodule of an injective module, being this a very important fact.

For M a right Λ-module and N a left Λ-module we define an abelian group $M \otimes_\Lambda N$, called the *tensor product* of M and N over Λ, as the free abelian group generated by the symbols $x \otimes y$, $x \in M$, $y \in N$ modulo the subgroup generated by the expressions

$$\begin{aligned} &(x + x') \otimes y - (x \otimes y + x' \otimes y) \\ &x \otimes (y + y') - (x \otimes y + x \otimes y') \\ &x\lambda \otimes y - x\lambda \otimes y \qquad \lambda \in \Lambda,\ x, x' \in M;\ y, y' \in N \ . \end{aligned}$$

If Λ is commutative $M \otimes_\Lambda N$ is a Λ-module.

We have a theorem similar to 1.4:

Theorem 1.5. *a) If* $N' \xrightarrow{\psi} N \overset{\psi'}{\twoheadrightarrow} N''$ *is an exact sequence of* Λ*-modules then there is an exact induced sequence*

$$M \otimes N' \xrightarrow{1_M \otimes \psi} M \otimes_\Lambda N \xrightarrow{1_N \otimes \psi'} M \otimes_\Lambda N'' \longrightarrow 0 \ .$$

b) If $M' \xrightarrow{\varphi} M \overset{\varphi'}{\twoheadrightarrow} M''$ *is an exact sequence of* Λ*-modules then there is an exact induced sequence*

$$M' \otimes_\Lambda N \xrightarrow{\varphi \otimes 1_N} M \otimes_\Lambda N \xrightarrow{\varphi' \otimes 1_N} M'' \otimes_\Lambda N \longrightarrow 0 \ .$$

Observe that $(\hom_\Lambda -, -)$ is a functor of two variables from the category of Λ-modules Mod_Λ, to the category of abelian groups Ab. It is covariant on the second variable and contravariant on the first one.

Also $- \otimes_\Lambda -$ is a functor of two variables from Mod_Λ to Ab and is covariant on both variables.

One extremely important relation that ties the functors hom and $\otimes$ is given by the following isomorphism

$$\hom_{\Lambda'}(M \otimes_\Lambda N, U) \cong \hom_\Lambda(M, \hom_{\Lambda'}(N, U))$$

where M and N are Λ-modules (right and left respectively) and N and U are Λ'-modules (both right).

Exercise 1.6. Prove that if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence that splits then so are

$$0 \to M' \otimes_\Lambda N \to M \otimes_\Lambda N \to M'' \otimes_\Lambda N \to 0$$

and

$$0 \to \hom_\Lambda(M'', N) \to \hom_\Lambda(M, N) \to \hom_\Lambda(M', N) \to 0 \ .$$

I.2. Resolutions and Homology

Let $\{C_n\}_{n\in\mathbb{Z}}$ be a family of Λ-modules and $\{\partial_n : C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$ a family of Λ-module homomorphisms such that $\partial_n \circ \partial_{n+1} = 0$. A *chain complex* or *chain* over Λ is the pair $C = \{C_n, \partial_n)$ and we write it as follows:

$$C : \ldots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \ .$$

This means that a chain is just a decreasing semiexact sequence of Λ-modules. A morphism $\varphi : C \to D$ between two chains is a family of Λ-module homomorphisms $\{\varphi_n : C_n \to D_n\}$ such that the following diagram commutes:

$$\begin{array}{ccccccccccc}
C: & \cdots & \longrightarrow & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & C_{n-2} & \longrightarrow & \cdots \\
\downarrow\varphi & & & \downarrow\varphi_n & & \downarrow\varphi_{n-1} & & \downarrow\varphi_{n-2} & & \\
D: & \cdots & \longrightarrow & D_n & \xrightarrow{\partial'_n} & D_{n-1} & \xrightarrow{\partial'_{n-1}} & D_{n-2} & \longrightarrow & \cdots \ .
\end{array}$$

The main concept in Homological Algebra is the following: Let C be a chain complex. Then the n-th homology module of C, denoted $H_n(C)$, is the quotient module $H_n(C) = \ker \partial_n / \operatorname{im} \partial_{n+1}$.

$H_n(C)$ measures the inexactness of the chain C. For example, if C is exact then $\operatorname{im} \partial_{n+1} = \ker \partial_n$, hence $H_n(C) = 0$. We have associated to a chain C a graded module $H_*(C) = \{H_n(C)\}$ which we call *the homology of the chain C*. A chain morphism induces a well defined morphism (of degree 0) $\varphi_* : H_*(C) \to H_*(D)$ between graded modules. Then $H_*(\_)$ is a covariant functor from the category of chain complexes to the category of graded Λ-modules.

If we consider semiexact families $\{C^n\}_{n\in\mathbb{Z}}$ with increasing index we obtain dual concepts; we have cochains, cochain morphisms, cohomology of a cochain, etc.

Given two chain complexes C, D and two morphisms between them $\varphi, \varphi' : C \to D$ when do they induce the same homomorphism between $H_*(C)$ and $H_*(D)$?

To answer this question we introduce the concept of *homotopy.*

Definition 2.1. Let $C = \{C_n, \partial_n\}$ and $D = \{D_n, \partial'_n\}$ be two chain complexes and $\varphi, \varphi' : C \to D$ two chain morphisms. We will say that φ is ***homotopic*** to φ' if there exists a family of Λ-module homomorphisms

$$h = \{h_n : C_n \to D_{n+1} \mid n \in \mathbb{Z}\}$$

such that $\partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n = \varphi_n - \varphi'_n$ for all $n \in \mathbb{Z}$ in the following diagram

$$\begin{array}{ccccccccc}
C: & \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots \\
& & & \varphi'_{n+1}\downarrow\downarrow\varphi_{n+1} & {}^{h_n}\swarrow & \varphi'_n\downarrow\downarrow\varphi_n & {}^{h_{n-1}}\swarrow & \varphi'_{n-1}\downarrow\downarrow\varphi_{n-1} & & \\
D: & \cdots & \longrightarrow & D_{n+1} & \xrightarrow[\partial'_{n+1}]{} & D_n & \xrightarrow[\partial'_n]{} & D_{n-1} & \xrightarrow[\partial'_{n-1}]{} & \cdots
\end{array}$$

The family $h = \{h_n\}$ is called a *chain homotopy* and we will say that φ *is homotopic to* φ'. In symbols

$$h : \varphi \sim \varphi' : C \to D \ .$$

It is easy to check that $\sim$ is an equivalence relation.

We will say that a chain morphism $\varphi : C \to D$ is a *homotopy equivalence* if there is a chain morphism $\varphi' : D \to C$ such that $\varphi' \circ \varphi \sim 1_C$ and $\varphi \circ \varphi' \sim 1_D$. In this case we will say that C and D are of the same *homotopy type.*

Theorem 2.2. *If* $\varphi \sim \varphi' : C \to D$ *then* $H_*(\varphi) = H_*(\varphi') : H_*(C) \to H_*(D)$.

Proof. Let $h : \varphi \sim \varphi'$ be the homotopy. Let $x \in H_n(C)$ be arbitrary, let $z \in Z_n(C)$ such that $p(z) = x$ where $p : Z_n(C) \to H_n(C)$ is the projection. Then

$$\varphi_n(z) - \varphi'_n(z) = \partial'_{n+1}h(z) + h_{n-1}\partial_n(z) = \partial'_{n+1}h_n(z)$$

because $\partial_n(z) = 0$. Since $\partial'_{n+1}h_n(z) \in B_n(D)$,

$$[H_*(\varphi)]\,(x) = [H_*(\varphi')]\,(x) \ .$$

Then $H_n(\varphi) = H_n(\varphi')$ for all $n \in \mathbb{Z}$, i.e. $\varphi(z)$ and $\varphi'(z)$ are homologous. □

The converse of this theorem is not true.

If $\varphi = 0 : C \to C$ is the trivial morphism and $\varphi' = 1_C : C \to C$ is the identity morphism then a homotopy $h : \varphi \sim \varphi'$ is called a *contraction* and we have $\partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n = 1$, $n \in \mathbb{Z}$. This implies by theorem 2.2 that $H_*(C) = 0$ and that C is exact.

Since the category of chain complexes is an abelian category we can form short exact sequences of chain complexes displayed vertically as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \xrightarrow{\varphi} & D & \xrightarrow{\varphi'} & E & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n+1} & \xrightarrow{\varphi_{n+1}} & D_{n+1} & \xrightarrow{\varphi'_{n+1}} & E_{n+1} & \longrightarrow & 0 \\
 & & \downarrow \partial_{n+1} & & \downarrow \partial'_{n+1} & & \downarrow \partial''_{n+1} & & \\
0 & \longrightarrow & C_n & \xrightarrow{\varphi_n} & D_n & \xrightarrow{\varphi'_n} & E_n & \longrightarrow & 0 \\
 & & \downarrow \partial_n & & \downarrow \partial'_n & & \downarrow \partial''_n & & \\
0 & \longrightarrow & C_{n-1} & \xrightarrow{\varphi_{n-1}} & D_{n-1} & \xrightarrow{\varphi'_{n-1}} & E_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & .
\end{array}$$

We have the following basic theorem:

Theorem 2.3. *Let $C \rightarrowtail D \twoheadrightarrow E$ be a short exact sequence of chain complexes. Then there is a homomorphism $\kappa_n : H_n(E) \to H_{n-1}(C)$ for each $n \in \mathbb{Z}$ such that the following sequence is exact:*

$$\ldots \longrightarrow H_n(C) \longrightarrow H_n(D) \longrightarrow H_n(E) \xrightarrow{\kappa_n}$$
$$\xrightarrow{\kappa_n} H_{n-1}(C) \longrightarrow H_{n-1}(D) \longrightarrow H_{n-1}(E) \xrightarrow{\kappa_{n-1}} \ldots .$$

Consider a positive exact chain complex of projective (free) Λ-modules $P = \{P_n, \partial_n\}$, that is, such that $H_n(P) = 0$ for $n \geq 1$ and let us assume it satisfies that $H_0(P) \cong M$. We will write it as follows

$$P : \ldots \longrightarrow P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \ldots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

and call it a *projective (free) resolution* of a Λ-module M.

Proposition 2.4. *Let M be a Λ-module. Then there exists a projective resolution of M.*

Proof. Since every Λ-module is quotient of a free Λ-module there is a short exact sequence

$$0 \longrightarrow M_0 \xrightarrow{\mu_0} F_0 \xrightarrow{\eta_0} M \longrightarrow 0$$

where F_0 is a free Λ-module. Since M_0 is a quotient of a free Λ-module F_1, there is a short exact sequence

$$0 \longrightarrow M_1 \xrightarrow{\mu_1} F_1 \xrightarrow{\eta_1} M_0 \longrightarrow 0$$

where F_1 is free. By induction we obtain a short exact sequence

$$0 \longrightarrow M_n \xrightarrow{\mu_n} F_n \xrightarrow{\eta_n} M_{n-1} \longrightarrow 0$$

with F_n free. Define a sequence

$$F : \ldots \longrightarrow F_{n+1} \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \longrightarrow \ldots$$

by

$$F_n = \begin{cases} M & \text{if } n = -1 \\ F_n & \text{if } n \geq 0 \\ 0 & \text{if } n < -1 \end{cases} \qquad \partial_n = \begin{cases} \eta_0 & \text{if } n = 0 \\ \mu_{n-1} \circ \eta_n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

Since μ_n is a monomorphism and η_n is an epimorphism we have that

$$\text{im } \partial_{m+1} = \text{im } \mu_n = \ker \eta_n = \ker \partial_n \quad .$$

Then the sequence is exact and clearly $H_0(F) \cong M$. Since every free module is projective we are done. □

Definition 2.5. A positive chain complex $C = \{C_n, \partial_n\}$ is called *acyclic* if $H_n(C) = 0$ for $n \geq 1$ (i.e. C is exact till $C_1, H_0(C)$ may be different of 0). Equivalently, the seqence

$$\ldots \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} H_0(C) \longrightarrow 0$$

is exact.

We can say, in view of 2.5, that a projective resolution of a Λ-module M is a projective and acyclic chain $P = \{P_n, \partial_n\}$ such that $H_0(P) \cong M$.

The following lemma is considered as the fundamental lemma in Homological Algebra. It tells us how we can construct chain morphisms and homotopies from a projective chain into an acyclic chain.

Lemma 2.6. *Let $C = \{C_n, \partial_n\}$ and $D = \{D_n, \partial'_n\}$ be two chain complexes. Let $\varphi = \{\varphi_i : C_i \to D_i\}_{i \leq n}$ be a family of Λ-module homomorphisms such that $\partial'_i \circ \varphi_i = \varphi_{i-1} \circ \partial_i$ for $i \leq n$. Suppose that C_i is projective for $i > n$ and that $H_i(D) = 0$ for $i \geq n$. Then $\{\varphi_i\}_{i \leq n}$ extends to a chain morphism $\varphi : C \to D$ and is unique up to homotopy.*

Definition 2.7. Let P be a projective resolution of a Λ-module M

$$P : \ldots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\varepsilon} M \longrightarrow 0 \ .$$

A *reduced projective resolution* P_M of M is a projective resolution of M in which M has been suppressed, i.e.

$$P_M : \ldots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \longrightarrow 0 \ .$$

Observe that we have not lost any information about P since $M = \operatorname{coker} \partial_1$. We consider the projective resolutions as a generalization of a Λ-module presentation, i.e. a generalization of the concepts of generators and relations. The advantage of P_M is that it consists only of projective Λ-modules.

Now, lets compare two projective resolutions of a Λ-module.

Theorem 2.8. *Let P and P' be projective resolutions of a Λ-module M. Then there exist a chain morphism $\varphi : P \to P'$ such that $\varepsilon'\varphi_0 = \varepsilon$. Furthermore, φ is unique up to homotopy and is a homotopy equivalence.*

Proof. By lemma 2.6 applied to $n = -1$ we obtain a chain morphism $\varphi : P \to P'$ such that $\varepsilon'\varphi_0 = \varepsilon$. Furthermore, φ is unique up to homotopy ($h_{-1} = 0$). Similarly there exists $\varphi' : P' \to P$. By 2.6, the composition $\varphi'\varphi : P \to P$ and the identity $1_P : P \to P$ are homotopic, i.e. $\varphi'\varphi \sim 1_P$. Analogously $\varphi\varphi' \sim 1_{P'}$. Then φ is a homotopy equivalence. □

By the previous theorem we can say that two projective resolutions of a Λ-module M are of the *same homotopy type* or that they are *unique up to homotopy equivalence.*

Example 2.9. Consider $\mathbb{Z}$-modules. The subgroups of a free group are free. Then, any abelian group G admits a free resolution of length ≤ 1:

$$0 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow G \longrightarrow 0 \ .$$

For example, the $\mathbb{Z}$-module $\mathbb{Z}/p$, p a prime, admits the following resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\mu} \mathbb{Z} \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

where μ is multiplication by p.

I.3. Torsion and Extension Functors

Let $P_M : \ldots \longrightarrow P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \ldots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \longrightarrow 0$ be a reduced projective resolution of a right Λ-module M. Let N be a left Λ-module and consider the tensor product $P_M \otimes_\Lambda N$ which is the sequence

$$P_M \otimes_\Lambda N : \ldots \longrightarrow P_n \otimes_\Lambda N \xrightarrow{\partial_n \otimes 1} P_{n-1} \otimes_\Lambda N \xrightarrow{\partial_{n-1} \otimes 1}$$
$$\ldots \longrightarrow P_1 \otimes_\Lambda N \xrightarrow{\partial_1 \otimes 1} P_0 \otimes_\Lambda N \longrightarrow 0 \ .$$

$P_M \otimes_\Lambda N$ is a semiexact sequence since, for all $n > 1$,

$$(\partial_{n-1} \otimes_\Lambda 1) \circ (\partial_n \otimes_\Lambda 1) = (\partial_{n-1} \circ \partial_n) \otimes_\Lambda 1 = 0 \otimes_\Lambda 1 = 0 \ .$$

Then we can form

$$H_*(P_M \otimes_\Lambda N) = \{H_n(P_M \otimes_\Lambda N)\}_{n\geq 0}$$

and have the following

Definition 3.1. For each $n \geq 0$, let $\mathrm{Tor}_n^\Lambda(M,N)$ denote $H_n(P_M \otimes_\Lambda N)$ and call it the n-th *Tor group over* Λ *of* M *and* N.

It is easy to see that $\mathrm{Tor}_n^\Lambda(M,N)$ is independent of the choice of the resolution. It only depends on n, M and N.

Let $f : M \longrightarrow M''$ and $g : N \longrightarrow N''$ be Λ-module homomorphisms. Let P_M and $P_{M''}$ be reduced projective resolutions of M and M'' respectively. By lemma 2.6 there is a chain morphism $\varphi : P_M \longrightarrow P_{M''}$ that extends f. Then $\varphi \otimes g$ is a chain morphism that induce

$$(\varphi \otimes g)_* : H_*(P_M \otimes_\Lambda N) \longrightarrow H_*(P_M \otimes_\Lambda N'')$$

i.e.

$$(\varphi \otimes g)_* : \mathrm{Tor}_*^\Lambda(M,N) \longrightarrow (M'',N'')\ .$$

It does not depend on φ but only on n, f and g. It is easy to prove the following

Proposition 3.2. $\mathrm{Tor}_n^\Lambda(\_\,,\_)$ *is a covariant bifunctor from the category of* Λ*-modules to the category of abelian groups.*

We leave to the reader the proof of the following

Exercise . Let $M' \rightarrowtail M \twoheadrightarrow M''$ be a short exact sequence of Λ-modules. Let $P'_{M'}$ and $P''_{M''}$ be reduced projective resolutions of M' and M'' respectively. Then there is a reduced projective resolution P_M of M such that

$$P'_{M'} \rightarrowtail P_M \twoheadrightarrow P''_{M''}$$

is an exact sequence of reduced projective resolutions that splits.

Let $N' \rightarrowtail N \twoheadrightarrow N''$ be a short exact sequence of left Λ-modules and P_M a reduced projective resolution of a right Λ-module M. Then

$$P_M \otimes_\Lambda N' \rightarrowtail P_M \otimes_\Lambda N \twoheadrightarrow P_M \otimes_\Lambda N''$$

is a short exact sequence of chains. By theorem 2.3 there is a homomorphism

$$\kappa_n : H_n(P_M \otimes_\Lambda N'') \longrightarrow H_{n-1}(P_M \otimes_\Lambda N')$$

such that the following sequence is exact:

$$\ldots \longrightarrow H_n(P_M \otimes_\Lambda N') \longrightarrow H_n(P_M \otimes_\Lambda N) \longrightarrow H_n(P_M \otimes_\Lambda N'')$$
$$\xrightarrow{\kappa_n} H_{n-1}(P_M \otimes_\Lambda N') \longrightarrow H_{n-1}(P_M \otimes_\Lambda N') \longrightarrow H_{n-1}(P_M \otimes_\Lambda N'') \longrightarrow \ldots\ .$$