Graduate Texts in Mathematics

Alan F. Beardon

The Geometry of Discrete Groups

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Preface

This text is intended to serve as an introduction to the geometry of the action of discrete groups of Möbius transformations. The subject matter has now been studied with changing points of emphasis for over a hundred years, the most recent developments being connected with the theory of 3-manifolds: see, for example, the papers of Poincaré [77] and Thurston [101]. About 1940, the now well-known (but virtually unobtainable) Fenchel–Nielsen manuscript appeared. Sadly, the manuscript never appeared in print, and this more modest text attempts to display at least some of the beautiful geometrical ideas to be found in that manuscript, as well as some more recent material.

The text has been written with the conviction that geometrical explanations are essential for a full understanding of the material and that however simple a matrix proof might seem, a geometric proof is almost certainly more profitable. Further, wherever possible, results should be stated in a form that is invariant under conjugation, thus making the intrinsic nature of the result more apparent. Despite the fact that the subject matter is concerned with groups of isometries of hyperbolic geometry, many publications rely on Euclidean estimates and geometry. However, the recent developments have again emphasized the need for hyperbolic geometry, and I have included a comprehensive chapter on analytical (not axiomatic) hyperbolic geometry. It is hoped that this chapter will serve as a "dictionary" of formulae in plane hyperbolic geometry and as such will be of interest and use in its own right. Because of this, the format is different from the other chapters: here, there is a larger number of shorter sections, each devoted to a particular result or theme.

The text is intended to be of an introductory nature, and I make no apologies for giving detailed (and sometimes elementary) proofs. Indeed, many geometric errors occur in the literature and this is perhaps due, to some extent, to an omission of the details. I have kept the prerequisites to a minimum and, where it seems worthwhile, I have considered the same topic from different points of view. In part, this is in recognition of the fact that readers do not always read the pages sequentially. The list of references is not comprehensive and I have not always given the original source of a result. For ease of reference, Theorems, Definitions, etc., are numbered collectively in each section (2.4.1, 2.4.2, . . .).

I owe much to many colleagues and friends with whom I have discussed the subject matter over the years. Special mention should be made, however, of P. J. Nicholls and P. Waterman who read an earlier version of the manuscript, Professor F. W. Gehring who encouraged me to write the text and conducted a series of seminars on parts of the manuscript, and the notes and lectures of L. V. Ahlfors. The errors that remain are mine.

Cambridge, 1982 ALAN F. BEARDON

Contents

CHAPTER 1
Preliminary Material

§1.1. Notation

We use the following notation. First, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the integers, the rationals, the real and complex numbers respectively: $\mathbb{H}$ denotes the set of quaternions (Section 2.4).

As usual, $\mathbb{R}^n$ denotes Euclidean n-space, a typical point in this being $x = (x_1, \ldots, x_n)$ with

$$|x| = (x_1^2 + \cdots + x_n^2)^{1/2}.$$

Note that if $y > 0$, then $y^{1/2}$ denotes the positive square root of y. The standard basis of $\mathbb{R}^n$ is $e_1, \ldots, e_n$ where, for example, $e_1 = (1, 0, \ldots, 0)$. Certain subsets of $\mathbb{R}^n$ warrant special mention, namely

$$B^n = \{x \in \mathbb{R}^n : |x| < 1\},$$

$$H^n = \{x \in \mathbb{R}^n : x_n > 0\},$$

and

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

In the case of $\mathbb{C}$ (identified with $\mathbb{R}^2$) we shall use Δ and $\partial\Delta$ for the unit disc and unit circle respectively.

The notation $x \mapsto x^2$ (for example) denotes the function mapping x to x^2: the domain will be clear from the context. Functions (maps or transformations) act on the *left*: for brevity, the image $f(x)$ is often written as fx (omitting brackets). The composition of functions is written as fg: this is the map $x \mapsto f(g(x))$.

Two sets A and B *meet* (or A meets B) if $A \cap B \neq \emptyset$. Finally, a property $P(n)$ holds *for almost all n* (or all sufficiently large n) if it fails to hold for only a finite set of n.

§1.2. Inequalities

All the inequalities that we need are derivable from Jensen's inequality: for a proof of this, see [90], Chapter 3.

Jensen's Inequality. *Let μ be a positive measure on a set X with $\mu(X) = 1$, let $f: X \to (a, b)$ be μ-integrable and let $\phi: (a, b) \to \mathbb{R}$ be any convex function. Then*

$$\phi\left(\int_X f \, d\mu\right) \leq \int_X (\phi f) \, d\mu. \tag{1.2.1}$$

Jensen's inequality includes Hölder's inequality

$$\int_X fg \, d\mu \leq \left(\int_X f^2 \, d\mu\right)^{1/2} \left(\int_X g^2 \, d\mu\right)^{1/2}$$

as a special case: the discrete form of this is the Cauchy–Schwarz inequality

$$|\sum a_i b_i| \leq (\sum |a_i|^2)^{1/2} (\sum |b_i|^2)^{1/2}$$

for real a_i and b_i. The complex case follows from the real case and this can, of course, be proved by elementary means.

Taking $X = \{x_1, \ldots, x_n\}$ and $\phi(x) = e^x$, we find that (1.2.1) yields the general Arithmetic–Geometric mean inequality

$$y_1^{\mu_1} \cdots y_n^{\mu_n} \leq \mu_1 y_1 + \cdots + \mu_n y_n,$$

where μ has mass μ_j at x_j and $y_j = \phi f(x_j)$.

In order to apply (1.2.1) we need a supply of convex functions: a sufficient condition for ϕ to be convex is that $\phi^{(2)} \geq 0$ on (a, b). Thus, for example, the functions cot, tan and $\cot^2$ are all convex on $(0, \pi/2)$. This shows, for instance, that if $\theta_1, \ldots, \theta_n$ are all in $(0, \pi/2)$ then

$$\cot\left(\frac{\theta_1 + \cdots + \theta_n}{n}\right) \leq \frac{\cot \theta_1 + \cdots + \cot \theta_n}{n}. \tag{1.2.2}$$

As another application, we prove that if x and y are in $(0, \pi/2)$ and $x + y < \pi/2$ then

$$\tan x \tan y \leq \tan^2\left(\frac{x + y}{2}\right). \tag{1.2.3}$$

Writing $w = (x + y)/2$, we have

$$\frac{\tan x + \tan y}{1 - \tan x \tan y} = \tan(x + y)$$

$$= \frac{2 \tan w}{1 - \tan^2 w}.$$

As tan is convex, (1.2.1) yields

$$\tan x + \tan y \geq 2 \tan w$$

and the desired inequality follows immediately (noting that $\tan^2 w < 1$ so both denominators are positive).

§1.3. Algebra

We shall assume familiarity with the basic ideas concerning groups and (to a lesser extent) vector spaces. For example, we shall use elementary facts about the group S_n of permutations of $\{1, 2, \ldots, n\}$: in particular, S_n is generated by transpositions. As another example, we mention that *if $\theta: G \to H$ is a homomorphism of the group G onto the group H, then the kernel K of θ is a normal subgroup of G and the quotient group G/K is isomorphic to H.*

Let g be an element in the group G. The elements *conjugate* to g are the elements hgh^{-1} in G ($h \in G$) and the conjugacy classes $\{hgh^{-1}: h \in G\}$ partition G. In passing, we mention that the maps $x \mapsto xgx^{-1}$ and $x \mapsto gxg^{-1}$ (both of G onto itself) play a special role in the later work. The *commutator* of g and h is

$$[g, h] = ghg^{-1}h^{-1}:$$

for our purposes this should be viewed as the composition of g and a conjugate of g^{-1}.

Let G be a group with subgroups G_i (i belonging to some indexing set). We assume that the union of the G_i generate G and that different G_i have only the identity in common. Then G is the *free product* of the G_i if and only if each g in G has a unique expression as $g_1 \cdots g_n$ where no two consecutive g_i belong to the same G_j. Examples of this will occur later in the text.

§1.4. Topology

We shall assume a knowledge of topology sufficient, for example, to discuss Hausdorff spaces, connected spaces, compact spaces, product spaces and homeomorphisms. In particular, *if f is a 1–1 continuous map of a compact* *space X onto a Hausdorff space Y, then f is a homeomorphism.* As special examples of topologies we mention the discrete topology (in which every subset is open) and the topology derived from a metric ρ on a set X. An isometry f of one metric space (X, ρ) onto another, say (Y, σ), satisfies

$$\sigma(fx, fy) = \rho(x, y)$$

and is necessarily a homeomorphism.

Briefly, we discuss the construction of the quotient topology induced by a given function. Let X be any topological space, let Y be any non-empty set and let $f: X \to Y$ be any function. A subset V of Y is *open* if and only if $f^{-1}(V)$ is an open subset of X: the class of open subsets of Y is indeed a topology $\mathscr{T}_f$ on Y and is called the *quotient topology induced by f*. With this topology, f is automatically continuous. The following two results on the quotient topology are useful.

Proposition 1.4.1. *Let X be a topological space and suppose that f maps X onto Y. Let $\mathscr{T}$ be any topology on Y and let $\mathscr{T}_f$ be the quotient topology on Y induced by f.*

(1) *If $f: X \to (Y, \mathscr{T})$ is continuous, then $\mathscr{T} \subset \mathscr{T}_f$.*
(2) *If $f: X \to (Y, \mathscr{T})$ is continuous and open, then $\mathscr{T} = \mathscr{T}_f$.*

PROOF. Suppose that $f: X \to (Y, \mathscr{T})$ is continuous. If V is in $\mathscr{T}$, then $f^{-1}(V)$ is in open in X and so V is in $\mathscr{T}_f$. If, in addition, $f: X \to (Y, \mathscr{T})$ is an open map then V in $\mathscr{T}_f$ implies that $f^{-1}(V)$ is open in X and so $f(f^{-1}V)$ is in $\mathscr{T}$. As f is surjective, $f(f^{-1}V) = V$ so $\mathscr{T}_f \subset \mathscr{T}$. □

Proposition 1.4.2. *Suppose that f maps X into Y where X and Y are topological spaces, Y having the quotient topology $\mathscr{T}_f$. For each map $g: Y \to Z$ define $g_1: X \to Z$ by $g_1 = gf$. Then g is continuous if and only if g_1 is continuous.*

PROOF. As f is continuous, the continuity of g implies that of g_1. Now suppose that g_1 is continuous. For an open subset V of Z (we assume, of course, that Z is a topological space) we have

$$(g_1)^{-1}(V) = f^{-1}(g^{-1}V)$$

and this is open in X. By the definition of the quotient topology, $g^{-1}(V)$ is open in Y so g is continuous. □

An alternative approach to the quotient topology is by equivalence relations. If X carries an equivalence relation R with equivalence classes $[x]$, then X/R (the space of equivalence classes) inherits the quotient topology induced by the map $x \mapsto [x]$. Equally, any surjective function $f: X \to Y$ induces an equivalence relation R on X by xRy if and only if $f(x) = f(y)$ and Y can be identified with X/R. As an example, let G be a group of homeomorphisms of a topological space X onto itself and let f map each x in X

to its G-orbit $[x]$ in X/G. If X/G is given the induced quotient topology, then $f: X \to X/G$ is continuous. In this case, f is also an open map because if V is open in X then so is

$$f^{-1}(fV) = \bigcup_{g \in G} g(V).$$

Finally, the reader will benefit from an understanding of covering spaces and Riemann surfaces although most of the material in this book can be read independently of these ideas. Some of this is discussed briefly in Chapter 6: for further information, the reader is referred to (for example) [4], [6], [28], [50], [63] and [100].

§1.5. Topological Groups

A *topological group* G is both a group and a topological space, the two structures being related by the requirement that the maps $x \mapsto x^{-1}$ (of G onto G) and $(x, y) \mapsto xy$ (of $G \times G$ onto G) are continuous: obviously, $G \times G$ is given the product topology. Two topological groups are *isomorphic* when there is a bijection of one onto the other which is both a group isomorphism and a homeomorphism: this is the natural identification of topological groups.

For any y in G, the space $G \times \{y\}$ has a natural topology with open sets $A \times \{y\}$ where A is open in G. The map $x \mapsto (x, y)$ is a homeomorphism of G onto $G \times \{y\}$ and the map $(x, y) \mapsto xy$ is a continuous map of $G \times \{y\}$ onto G. It follows that $x \mapsto xy$ is a continuous map of G onto itself with continuous inverse $x \mapsto xy^{-1}$ and so we have the following elementary but useful result.

Proposition 1.5.1. *For each y in G, the map $x \mapsto xy$ is a homeomorphism of G onto itself; the same is true of the map $x \mapsto yx$.*

A topological group G is *discrete* if the topology on G is the discrete topology: thus we have the following Corollary of Proposition 1.5.1.

Corollary 1.5.2. *Let G be a topological group such that for some g in G, the set $\{g\}$ is open. Then each set $\{y\}$ $(y \in G)$ is open and G is discrete.*

Given a topological group G, define the maps

$$\phi(x) = xax^{-1}$$

and

$$\psi(x) = xax^{-1}a^{-1} = [x, a],$$

where a is some element of G. We shall be interested in the iterates ϕ^n and ψ^n of these maps and with this in mind, observe that ϕ has a unique fixed point, namely a. The iterates are related by the equation

$$\phi^n(x) = \psi^n(x)a,$$

because (by induction)

$$\begin{aligned}\phi^{n+1}(x) &= [\psi^n(x)a]a[\psi^n(x)a]^{-1}\\ &= \psi^n(x)a[\psi^n(x)]^{-1}\\ &= \psi^{n+1}(x)a.\end{aligned}$$

In certain circumstances, the iterated commutator $\psi^n(x)$ converges to the identity (equivalently, the iterates $\phi^n(x)$ converge to the unique fixed point a of ϕ) and if the group in question is discrete, then we must have $\phi^n(x) = a$ for some n. For examples of this, see [106], [111: Lemma 3.2.5] and Chapter 5 of this text.

Finally, let G be a topological group and H a normal subgroup of G. Then G/H carries both the usual structures of a quotient group and the quotient topology.

Theorem 1.5.3. *If H is a normal subgroup of a topological group G, then G/H with the usual structures is a topological group.*

For a proof and for further information, see [20], [23], [39], [67], [69] and [94].

§1.6. Analysis

We assume a basic knowledge of analytic functions between subsets of the complex plane and, in particular, the fact that these functions map open sets of open sets. As specific examples, we mention Möbius transformations and hyperbolic functions (both of which form a major theme in this book).

A map f from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$ is *differentiable* at x if

$$f(y) = f(x) + (y - x)A + |y - x|\varepsilon(y),$$

where A is an $n \times n$ matrix and where $\varepsilon(y) \to 0$ as $y \to x$. We say that a differentiable f is *conformal* at x if A is a positive scalar multiple $\mu(x)$ of an orthogonal matrix B. More generally, f is directly or indirectly conformal according as det B is positive or negative. If f is an analytic map between plane domains, then the Cauchy–Riemann equations show that f is directly conformal except at those z where $f^{(1)}(z) = 0$.