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Some Remarks on This Conference and Its Proceedings

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This volume collects the papers and problems from the Third International Conference on Combinatorial Mathematics held at the Barbizon Plaza Hotel in New York City under the auspices of the New York Academy of Sciences from June 10 through June 14, 1985. Like its two predecessors, held in 1970 and 1978, respectively, this meeting was held in recognition of the current dynamic evolution of combinatorics, as this fundamental subject continues to mature and develop into a major component of the mathematical mainstream. The geographical scope of the conference was truly international, with participants coming from Australia, Austria, Belgium, Canada, China, Czechoslovakia, Denmark, England, France, Holland, Hungary, Israel, Japan, Puerto Rico, Poland, South Africa, Sweden, West Germany, and other countries, as well as from every region of the United States. The combinatorial scope of the conference was equally broad, as can be seen from the 54 papers in this volume. These papers explore aspects of such topics as structural graph theory, extremal set theory, Ramsey theory, combinatorial group theory, random graphs, matroids, finite geometries, game theory, block designs, coding theory, polyhedral combinatorics, irregularities of distribution, and combinatorial number theory, to name a few, as well as a healthy dose of the increasingly important algorithmic aspects of these various subjects. The "Collection of Open Problems" presented during an organized problem session contains a taste of the many unsolved problems discussed at the conference; some of the many other open issues discussed at the meeting are contained within the papers of this volume.

We gratefully acknowledge the assistance we received from the staff of the New York Academy of Sciences, and of the cooperation of The City College of New York, of York College, and of AT&T Bell Laboratories. The success of the Conference was greatly enhanced by the financial support given by the Air Force Office of Scientific Research, the Office of Naval Research, The City College, and York College.

We hope the reader enjoys and benefits from the mathematical contributions in this volume as much as the conference participants did. In the end, we again thank all of those participants, the authors of the papers, the Editorial Committee, who refereed the papers, and everyone who helped make this *Annals* available to those of us who are fascinated by the beauty of combinatorial mathematics.

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Editors and Conference Organizers

GARY S. BLOOM, RONALD L. GRAHAM, AND JOSEPH MALKEVITCH

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Disjoint Simplices and Geometric Hypergraphs

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INTRODUCTION

Let A be a set of $2n$ points in general position in the Euclidean plane R^2, and suppose n of the points are colored red and the remaining n are colored blue. A celebrated Putnam problem (see [6]) asserts that there are n pairwise disjoint straight line segments matching the red points to the blue points. To show this, consider the set of all $n!$ possible matchings and choose one, M, that minimizes the sum of lengths $l(M)$ of its line segments. It is easy to show that these line segments cannot intersect. Indeed, if the two segments v_1, b_1 and v_2, b_2 intersect, where v_1, v_2 are two red points and b_1, b_2 are two blue points, the matching M' obtained from M by replacing v_1b_1 and v_2b_2 by v_1b_2 and v_2b_1 satisfies $l(M') < l(M)$, contradicting the choice of M. Our first result in this paper is a generalization of this result to higher dimensions.

THEOREM 1: Let A be a set of $d \cdot n$ points in general position in R^d, and let $A = A_1 \cup A_2 \cup \cdots \cup A_d$ be a partition of A into d pairwise disjoint sets, each consisting of n points. Then there are n pairwise disjoint $(d-1)$-dimensional simplices, each containing precisely one vertex from each A_i, $1 \le i \le d$.

We prove this theorem in the next section. The proof is short but uses a nonelementary tool: the well-known Borsuk-Ulam theorem.

Combining Theorem 1 with an old result of Erdös from extremal graph theory we obtain a corollary dealing with geometric hypergraphs. A *geometric d-hypergraph* is a pair $G = (V, E)$, where V is a set of points called vertices, in general position in R^d, and E is a set of (closed) $(d-1)$-dimensional simplices called edges, whose vertices are points of V. If $d = 2$, G is called a *geometric graph*. It is well known (see [3], [5]) that every geometric graph with n vertices and $n + 1$ edges contains two disjoint edges, two nonintersecting edges, and this result is the best possible. The number of edges that guarantees l pairwise disjoint edges is not known for $l > 2$, although Perles [7] determined the exact number for the case that the set of vertices

V is the set of vertices of a convex polygon. The situation seems much more difficult for geometric d-hypergraphs, when $d > 2$. Even the number of edges that guarantees two disjoint simplices is not known in this case. Clearly this number is greater than $\binom{n-1}{d-1}$ (simply take all edges containing a given point) and is at most $\binom{n}{d}$. In the final section we prove the following theorem, that implies that for every fixed d, $l \geq 2$, every geometric d-hypergraph on n vertices that contains no l pairwise nonintersecting edges has $o(n^d)$ edges.

THEOREM 2: Every geometric d-hypergraph with n vertices and at least $n^{d-(1/l^{d-1})}$ edges contains l pairwise nonintersecting edges.

It is worth noting that the following, much stronger conjecture seems plausible.

CONJECTURE 1: For every l, $d \geq 2$ there exists a constant $c = c(l, d)$ such that every geometric d-hypergraph with n vertices and at least $c \cdot n^{d-1}$ edges contains l pairwise nonintersecting edges.

We do not know how to prove this conjecture, even for $d = 2$, $l = 3$.

PROOF OF THEOREM 1

We need the following lemma, sometimes called the "Ham-Sandwich theorem," which is a well-known consequence of the Borsuk-Ulam theorem (see [1], [2]).

LEMMA 1: Let $\mu_1, \mu_2, \ldots, \mu_d$ be d continuous probability measures in R^d. Then there exists a hyperplane H in R^d that bisects each of the d measures, that is, $\mu_i(H^+) = \mu_i(H^-)(= \frac{1}{2})$ for all $1 \leq i \leq d$, where H^+ and H^- denote, respectively, the open positive side and the open negative side of H.

Theorem 1 will be derived from the following lemma.

LEMMA 2: Let $A, A_1, A_2, \ldots, A_d$ be as in Theorem 1. Then there exists a hyperplane H in R^d such that

$$|H^+ \cap A_i| = [n/2] \quad \text{and} \quad |H^- \cap A_i| = [n/2] \quad \text{for all } 1 \leq i \leq d. \quad (1)$$

(Notice that if n is odd (1) implies that H contains precisely one point from each A_i.)

Proof: Replace each point $p \in A$ by a ball of radius ε centered in p, where ε is small enough to guarantee that no hyperplane intersects more than d balls. Associate each ball with a uniformly distributed measure of $1/n$. For $1 \leq i \leq d$ and a (lebesgue)-measurable subset T of R^d, define $\mu_i(T)$ as the total measure of balls centered at point of A_i captured by T. Clearly $\mu_1, \mu_2, \ldots, \mu_d$ are a continuous probability measure. By Lemma 1 there exists a hyperplane H in R^d such that $\mu_i(H^+) = \mu_i(H^-) = \frac{1}{2}$ for all $1 \leq i \leq d$. If n is odd, this implies that H intersects at least one ball centered at a point of A_i. However, H cannot intersect more than d balls altogether, and thus it intersects precisly one ball centered at a point of A_i, and it must bisect these d balls. Hence, for odd n, H satisfies (1). If n is even, H intersects at most d balls, and by slightly rotating H we can divide the centers of these balls between

H^+ and H^- as we wish, without changing the position of each other point of A with respect to H. One can easily check that this guarantees the existence of an H satisfying (1). □

We can now prove Theorem 1 by induction on n. For $n = 1$ the result is trivial. Assuming the result for all n', $n' < n$, let $A, A_1, A_2, \ldots, A_d$ be as in Theorem 1 and let H be a hyperplane, guaranteed by Lemma 2, satisfying (1). Put $B_i = H^+ \cap A_i$ and $C_i = H^- \cap A_i$ for $1 \le i \le d$, $B = B_1 \cup \cdots \cup B_d$ and $C = C_1 \cup \cdots \cup C_d$. By applying the induction hypothesis to $B, B_1, \ldots, B_d$ and $C, C_1, \ldots, C_d$, we obtain two sets S_1 and S_2 of $[n/2]$ pairwise disjoint simplices each, where each simplex of S_1 contains precisly one vertex from each B_i and each simplex of S_2 contains precisely one vertex from each C_i. Clearly, all the simplices in S_1 lie in H^+ and all those in S_2 lie in H^-.

We thus obtained $2 \cdot [n/2]$ pairwise nonintersecting simplices. These, together with the simplex spanned by $A_i \cap H$ if n is odd, complete the induction and the proof of Theorem 1. □

PROOF OF THEOREM 2

We need the following result of Erdös.

Lemma 3 [4]: Every d-uniform hypergraph with n vertices and at least $n^{d-(1/l^{d-1})}$ edges contains a complete d-partite subhypergraph on d classes of l vertices each.

Now suppose that G is a geometric d-hypergraph with n vertices and at least $n^{d-(1/l^{d-1})}$ edges. By Lemma 3 there is a set A of $l \cdot d$ vertices of G, $A = A_1 \cup \cdots \cup A_d$, where $|A_i| = l$ for each i, and all the l^d $(d-1)$-simplices consisting of one vertex from each A_i are edges of G. The assertion of Theorem 2 now follows from Theorem 1. □

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A Turanlike Neighborhood Condition and Cliques in Graphs

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INTRODUCTION

There have been many conditions placed on graphs to ensure the existence of certain kinds of subgraphs, in particular, conditions on the degrees of vertices have been useful. The following result of Ore is an example of the use of such a degree condition.

THEOREM A [5]: If G is a graph of order $n \geq 3$ such that the sum of degrees of any pair of nonadjacent vertices is at least n, then G is Hamiltonian.

Gould and Jacobson introduced a neighborhood condition that was patterned after the Ore type of degree condition, and that also implies the existence of certain subgraphs. An example of a result using this condition is the following, which parallels the previously cited result of Ore.

THEOREM B [3]: If G is a graph of order $n \geq 3$ such that the union of the neighborhoods of each pair of nonadjacent vertices is of cardinality at least $(2n + 1)/3$, then G is Hamiltonian.

Our purpose is to investigate the neighborhood condition of the preceding type needed to ensure a clique of a fixed order. If $n = km$, then the Turan graph [6], which is the complete k-partite graph, $K_{m, m, \ldots, m}$, does not contain a complete K_{k+1} as a subgraph. However, for $m \geq t \geq 1$, the union of the neighborhoods of any set of t independent vertices has precisely $(k - 1)m = (k - 1)n/k$ vertices. Therefore, the following theorem, which is the main result to be proved, is the best possible of this type.

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THEOREM 1: Let k and t be fixed integers greater than or equal to 2. If any set of t independent vertices of a graph of order $n > n_0(k, t)$ has more than $(k-1)n/k$ vertices in the union of the neighborhoods of the vertices, then G has a clique of order at least $k+1$.

NOTATION

All graphs will be finite and without loops or multiple edges. Notation will generally follow that of [4] unless otherwise stated. Some special notation and terminology will be introduced, and standard notation that is used extensively will be briefly described. For example, the complete multipartite graph with k parts each with m vertices will be expressed as $K(k; m)$, and the special case when $m = 1$, which is the complete graph on k vertices, will be expressed simply as K_k.

Let v be a vertex of a graph H. The neighborhood of v (the vertices that are adjacent in H to v) will be denoted by $N_H(v)$, or simply $N(v)$ when the identity of H is clear. If t is a positive integer, then $H_v(t)$ will denote the graph obtained from H by replacing v with t independent vertices, each with the same neighborhood as v. We will say that $H_v(t)$ is obtained from H by expanding the vertex v into t vertices. The graph obtained when each vertex of H is expanded into t vertices will be denoted by $H(t)$. Therefore, if H has order h (the number of vertices in H), then $H_v(t)$ and $H(t)$ have orders $h + t - 1$ and ht, respectively. Also, with this notation, $K_k(t) = K(k; t)$.

The maximum number of edges a graph G of order n can have without having a copy of a graph H is the extremal number ex(n, H). Additional edges in G will ensure at least one copy, but possibly many copies of H. By $n_G(H)$ we will denote the number of copies of H in G, where H is considered as a labeled graph. If the order of H is p, then $n_G(H) \leq cn^p$ for some $c = c(H)$, because there are at most that many subsets of p vertices of G. If, on the other hand, $n_G(H) \geq c'n^p$ for some $c' = c'(H)$, we will say that H *saturates* G.

We next carefully define the neighborhood condition that appears in the statement of Theorem 1, and is the basis of this investigation.

DEFINITION: For fixed positive integers k and t, a graph G of order n satisfies the *neighborhood condition* $N(k, t)$ if for each set $\{x_1, x_2, \ldots, x_t\}$ of t independent vertices,

$$|\{\cup N_G(x_i): 1 \leq i \leq t\}| > (k-1)n/k.$$

PROOFS

We begin with a restatement of the result to be proved in this section.

THEOREM 1: Let $k, t \geq 2$ be integers. If a graph G of order $n > n_0(k, t)$ satisfies the neighborhood condition $N(k, t)$, then G contains a K_{k+1}.

It should be noted that a graph G of order n that satisfies the neighborhood condition $N(k, t)$ does not necessarily have more than ex(n, K_{k+1}) edges. Thus, Theorem 1 is not a consequence of the extremal result of Turan [6].

The following lemma reduces the proof of Theorem 1 to proving the existence of an expansion of K_k, namely a $K(k; t)$, instead of a K_{k+1}.

LEMMA 1: Let $k, t \geq 2$ be integers. If a graph G of order n satisfies the neighborhood condition $N(k, t)$ and contains a $K(k; t)$, then G contains a K_{k+1}.

Proof: Let $A_1, A_2, \ldots, A_k$ be the vertices in the k parts of the complete multipartite graph $K(k; t)$, and let A be the remaining $n - kt$ vertices of G. We will assume that G does not contain a K_{k+1}, and show that this leads to a contradiction.

The vertices in each A_i are independent, and no vertex of A is adjacent to at least one vertex in each A_i $(1 \leq i \leq t)$, since there is no K_{k+1} in G. There is no loss of generality in assuming that there are $|A|/k = (n - kt)/k = (n/k) - t$ vertices of A with no adjacencies in A_1. Therefore, the t independent vertices of A_1 have a combined neighborhood of at most $n - (n/k)$ vertices, which implies that G does not satisfy the neighborhood condition $N(k, t)$. This contradiction completes the proof of Lemma 1. □

Our next objective is to show that a graph G that satisfies the neighborhood condition $N(k, t)$ contains a $K(k; t)$. We will show something stronger, namely that $K(i; t)$ saturates G for $(1 \leq i \leq k)$. The following lemma will be used in an inductive proof of the preceding statement. Lemma 2, and its proof, are patterned after a result of Erdös and Simonovits in [2].

LEMMA 2: Let t be a fixed positive integer and H a fixed graph of order p. If G is any graph of order n with

$$n_G(H) = m,$$

then there is a constant $c = c(p, t)$ such that

$$n_G(H_v(t)) \geq [(cm^t)/(n^{(p-1)(t-1)})]$$

for any vertex v of H.

Proof: Let $H' = H - v$, and $\{H'_r: r \in R\}$ be the copies of H' contained in G. For each copy H'_r, let L_r be the vertices of $G - H'_r$ that are adjacent in G to the neighborhood $N_H(v)$ of v in H'_r. If $l_r = |L_r|$, then $\sum_{r \in R} l_r = m$. Each subset of L_r with t vertices will give a copy of $H_v(t)$ in G. Therefore,

$$n_G(H_v(t)) = \sum_{r \in R} \binom{l_r}{t} \geq |R| \binom{m/|R|}{t}$$

$$\geq [c'(p, t)(m^t/|R|^{t-1})].$$

Since H' has order $p - 1$, $|R| \leq c'' n^{p-1}$ and

$$n_G(H_v(t)) \geq [c(p, t)m^t/(n^{(p-1)(t-1)})].$$

This completes the proof of Lemma 2. □

The special case of Lemma 2 when H saturates G gives the following two corollaries, which are expressed in the form that we will apply them in the proof of Proposition 1.

COROLLARY 1: If $m = c'n^p$, then $n_G(H_p(t)) \geq [cn^{p+t-1}]$.

COROLLARY 2: If $m = c'n^p$, then $n_G(H(t)) \geq [cn^{pt}]$.

The proof of Theorem 1 will be complete with the proof of the following result, which states that $K(k; t)$ saturates any graph that satisfies the neighborhood condition $N(k, t)$.

PROPOSITION 1: Let $t \geq 2$, $k \geq 1$ be integers and let G be a graph of order n, which satisfies $N(k, t)$. Then, there exist positive constants $c = c_{k,t}$ and $c' = c'_{k,t}$ such that

$$n_G(K_k) \geq [cn^k] \tag{1}$$

and

$$n_G(K(k; t)) \geq [c'n^{kt}]. \tag{2}$$

Proof: The proof is by induction on k with t fixed throughout the proof. For $k = 1$, both (1) and (2) are trivially true. We assume that (1) and (2) are true for $k = r \geq 1$ and verify them for $k = r + 1$. Thus, we assume G satisfies the neighborhood condition $N(r + 1, t)$. We can also assume that n is large, because appropriate choice of constants c and c' make the result trivial for small values of n.

Since property $N(r + 1, t)$ implies $N(r, t)$, we have that both (1) and (2) are true for $k = r$, so G contains at least $[c'n^{rt}]$ copies of $K(r; t)$. There are two types of copies of $K(r; t)$: there are those with no edges in each of their parts and those with at least one edge in some part.

First consider the case of a copy of $K(r; t)$ with parts $A_1, A_2, \ldots, A_r$, each of which is independent. Let A be the remaining vertices of G. For each i ($1 \leq i \leq r$), let B_i be the vertices of A that have no adjacencies in A_i. Let B be the remaining vertices of A. The neighborhood condition $N(r + 1, t)$ implies $|B_i| < |A|/(r + 1)$, and hence

$$|B| \geq |A|/(r + 1) \geq c''n$$

for some positive constant c''. Note that each vertex in B will give at least one copy of a K_{r+1} in G using precisely one vertex from each A_i.

If at least one half of the copies of $K(r; t)$ in G are of the first type, then there will be at least $[(c''n)(c'n^{rt})/2]$ copies of a K_{r+1}, counting multiplicities. However, any such K_{r+1} can come from at most n^{rt-r} different copies of a $K(r; t)$. Thus G would contain at least $[(c''c'n^{r+1})/2]$ copies of a K_{r+1} in this case.

We can now assume that at least one half of the copies of $K(r; t)$ in G are of the second type and have at least one edge in one of their parts. Associated with each of the $(c'n^{rt})/2$ copies of $K(r; t)$ of this type there is a copy of K_{r+1} in G. Also, any such K_{r+1} will arise from at most n^{rt-r-1} different copies of a $K(r; t)$. Hence there are at least

$$[(c'n^{rt})/(2n^{rt-r-1})] = [(c'n^{r+1})/2]$$

copies of a K_{r+1} in G. This verifies (1) for $k = r + 1$.

Since $K(r + 1, t) = K_{r+1}(t)$, Corollary 2 and (1) verify that (2) is true when $k = r + 1$. This completes the proof of Proposition 1. □

The proof of Theorem 1 is an immediate consequence of Proportion 1 and Lemma 1.

PROBLEMS

There are numerous unsolved problems related to neighborhood conditions like the one just considered. In [3] and [4] neighborhood conditions for nonadjacent pairs of vertices are used to ensure the existence of certain types of subgraphs. Theorem B is an example of one of these results. It would be nice to replace each of these conditions by a neighborhood condition involving t independent vertices where $t \geq 3$. Also, one can be concerned not with just the existence of a certain subgraph, but with how many subgraphs of this type there are. Proposition 1 is an example of a result of this type.

Bondy and Chvátal considered a "degree" closure that generalized results of the type given in Theorem A. Does there exist a "neighborhood" closure analogous to the "degree" closure that would generalize the results using neighborhood conditions?

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Families in Which Disjoint Sets Have Large Union

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INTRODUCTION AND STATEMENT OF THE RESULTS

Let $X = \{1, 2, \ldots, n\}$ and F be a family of subsets of X, that is $F \subset 2^X$. For $1 \le i \le j \le n$ set $[i, j] = \{i, \ldots, j\}$. For integers k, m with $k \ge 2$, $0 \le m \le n$, we say that F has property $P(k, m)$ if any k pairwise disjoint members of F have union of size greater than m. Thus $P(k, n)$ means simply that F contains no k pairwise disjoint sets.

Let us write m in the form $m = kt - r$, where $1 \le r \le k$. Define

$$F(n, k, m) = \{\mathbf{F} \subseteq X : |\mathbf{F}| + |\mathbf{F} \cap [1, r-1]| \ge t\}.$$

It is easy to check that $F(n, m, k)$ has property $P(k, n)$. In fact, if $\mathbf{F}_1, \ldots, \mathbf{F}_k$ are pairwise disjoint members of F, then

$$|\mathbf{F}_1 \cup \cdots \cup \mathbf{F}_k| = |\mathbf{F}_1| + \cdots + |\mathbf{F}_k| \ge kt - \sum_{1 \le i \le k} |\mathbf{F}_i \cap [1, r-1]| \ge kt - (r-1)$$

holds.

Note that for $m = kt - 1$ one has simply $F(n, k, m) = \{\mathbf{F} \subseteq X : |\mathbf{F}| \ge t\}$.

THEOREM 1: Suppose $F \subset 2^X$, F has $P(k, m)$. Then $|F| \le |F(n, k, m)|$ holds in each of the following cases.

(a) $m = kt - 1$.
(b) $k = 2, m = 2t - 2$,
(c) k, r arbitrary, $n > 2m^3$. Moreover, $|F| = |F(n, k, m)|$ is possible only if F is isomorphic to $F(n, k, m)$.

Let us mention that the condition $n > n_0(m)$ cannot be completely removed in (c). In fact, Kleitman [8] proved that for $n = m = kt - k$ the maximum size of a family having $P(n, k, n)$ is attained by $F = \{\mathbf{F} \subseteq X : |\mathbf{F} \cap \{1, 2, \ldots, n-1\}| \ge t - 1\}$.

Let us also note that if (c) holds for some triple (n, k, m), then it also holds for all (n', k, m) with $n' > n$—this will be clear from the inductive proof of (a) and (b).

The following old conjecture of Erdös is related to our problem.