Petrographic Modal Analysis

An elementary statistical appraisal

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Preface

The sound development of petrology, whether naturalistic or experimental, demands considerable information about the quantitative modal composition of rocks. In certain areas of petrographic inquiry, further development—indeed, even the resolution of long-standing controversies—seems almost impossible without an abundance of this kind of information. The amount and caliber of it available to us will also powerfully influence the rate of development in many other areas.

Despite increased interest and activity in the field of modal analysis the subject is ignored in nearly all textbooks on petrology and barely mentioned in most on petrography. The student is obliged to acquire the necessary background from scattered journal articles of uneven quality, mostly rather specialized, and often flatly inconsistent or contradictory. There is no single work to which the advanced undergraduate or beginning graduate—or anyone else, for that matter—may turn for an account sufficiently complete to enable him to decide whether and how he ought to set about using the technique in his own researches.

What the student needs, and what I have attempted to give him, are: (*a*) a clear description of the geometrical basis of the method; (*b*) a review and summary of techniques and instrumentation; (*c*) a careful discussion of reproducibility; (*d*) a definition and numerical characterization of analytical error; (*e*) a sense of the importance of analytical error in the design and planning of sampling experiments.

The subject has now advanced to the stage at which a condensed general treatment of the first three of these is possible; such a treatment is attempted in chapters 1–6, inclusive. One can always define analytical error generally, but it is only in some particular situation that one may put numerical flesh on the bones of the definition. The same difficulty arises in the discussion of the effect of analytical error on experimental design. The experimentalist senses and the mathematician symbolizes the relationship in an a priori way, but specific numerical recommendations can only be developed in a practical situation.

Beginning with chapter 7, and continuing through chapter 10, therefore, the discussion is necessarily confined to a particular rock type, the only one, so far as I know, on which a study of this sort has been conducted. Most of the work presented in these chapters has not appeared before. It is included in the hope that it will serve as a model—albeit one which will certainly require extensive revision—for those who may be interested in developing for other rocks the type of information now rapidly accumulating for the two-feldspar granites. Ultimately something of the sort will probably have to be done for every major rock type. I hope my errors of design and judgment are sufficiently penetrating that others may profit by them.

Chapter 2 is included primarily as an antidote to the habitual and largely uncritical skepticism about the potentialities of modal analysis in the study of sediments and the finer-grained metamorphic rocks. Readers who do not share this skepticism or have no immediate concern with laminated rocks will find that the argument of chapter 2 is not essential to an appreciation of the work of succeeding chapters. Chapter 11, on the other hand, is designed as a warning to those who may be tempted to apply modal analysis to rocks to which it should not be applied.

I should caution the reader that this book is not intended as a literature review. No paper is mentioned merely for the sake of completeness, and except in chapter 3 no particular attention is paid to priority. Indeed, the development of the subject has been so unsystematic that this is scarcely possible. If a recent paper states a problem more clearly or solves it more satisfactorily than an older one, the recent one is given preference in the discussion, and in several cases the older one is not even mentioned. Readers interested in piecing together a history of the subject can get off to a good start with the excellent bibliography given by Larsen and Miller.

It will be obvious even to the casual reader that this book is something of a hybrid. Resting heavily on elementary statistical argument, it is not a book about statistics. Anyone who has participated in the development of modal analysis realizes that a sharp subdivision of the subject into statistical and non-statistical categories is no longer either possible or desirable. This raises the puzzling question of how results reached by statistical methods should be presented to an audience most of whom, even now, have had no training in the subject.

Fortunately, the argument is for the most part both simple and straightforward. I have tried to write so that geologists completely unfamiiiar with statistics (and even those who, whether from choice or necessity, plan to continue in this blissful condition) may nevertheless follow its general outline and make use of its major results. It will take a bit of doing, but I believe it can be done. Most of the commoner terms and phrases that have received special definition in elementary statistics retain enough of their general import so that, with some allowance for lack of rigor, they still convey much the same thing to the non-statistical reader as to the reader with some training in the subject. Technical jargon has been avoided—evaded would perhaps be a better term—whenever this could be managed without undue expansion of the text, but has been used freely, and without extended explanation, whenever necessary. Similarly, numerical results are used liberally, but calculation procedures are described only when they are extremely simple or of a type not likely to be discussed in an elementary statistics textbook.

Instead of attempting the usual statistical explanation or laboring the book with footnotes, I have inserted an appendix containing a somewhat annotated statistical bibliography. Readers unfamiliar with the subject will find here references in which the various statistical terms and procedures employed are described in a fashion I could not hope to equal.

The book grows out of a series of lectures delivered to a graduate seminar in petrology at the California Institute of Technology during the winter of 1955. I am grateful to the staff and members of the Division of Geological Sciences for gracious hospitality and stimulating criticism. I am also indebted to several colleagues at the Geophysical Laboratory and to Earl Ingerson and J. D. H. Donnay for careful criticism of parts of the manuscript. The discussion of oriented rocks contained in the original lecture notes was so unsatisfactory that I planned to omit the subject from the published version; what now appears as chapter 2 was written largely because of the insistent encouragement of W. S. MacKenzie. Some of the experimental data used in chapter 8 were described briefly in *Year Book* No. 53 of the Carnegie Institution of Washington, and much of the material in chapters 1, 4, and 5 is reprinted here by permission of the editors of the *American Mineralogist* and the *Journal of Geology*. The substance of chapter 11 appeared originally in the *Mineralogical Magazine* and is reprinted here by permission of the councilors of the Mineralogical Society of London. Finally, it is pleasant to record my gratitude to the staff of the Statistical Engineering Laboratory of the National Bureau of Standards, and particularly to J. M. Cameron, for advice, assistance, and encouragement extending over several years.

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Introduction

This book is intended for petrographers but may also be read by practical statisticians who have no knowledge of petrography. Modal analysis is still so sparingly used in geology that many readers of both types may appreciate a brief statement of the character and purpose of the procedure.

A rock is a mineral aggregate. To the petrologist, the kinds and amounts of mineral species it contains are matters of first importance. With regard to determination of the *kinds* of minerals present, petrography is a highly developed descriptive science, and we shall not be further concerned here with the general problem of qualitative identification.

The composition of rock expressed in terms of the relative *amounts* of minerals actually present is called a *mode*. We refer to a procedure which yields such a statement, and usually to the statement itself, as a *modal analysis*. Modes may be obtained by recalculation from bulk chemical analysis, by the counting of crushed fragments, or by the measurement of relative areas underlain by each of the mineral species in a polished slab or thin section of the rock.

The compositions of the constituent minerals are rarely well enough known so that much reliance can be placed on modes recalculated from bulk analyses. Although the procedure of counting sized, crushed fragments seems quite straightforward, the results are of questionable value because of sampling difficulties which have not yet been carefully evaluated. At present very few modes are determined by fragment counting.

Modes were determined by areal measurements on polished slabs before the development of the thin section—or, at any rate, before the thin section became a common adjunct of petrography—and this is still the preferred procedure in rather special circumstances. Discrimination between some of the rock-forming minerals is difficult or impossible under reflected light, however, and the number of reliable modes obtained by measurements made on polished slabs is almost vanishingly small.

Thus, although any procedure which estimates the actual mineral composition of a rock is, strictly speaking, a modal analysis, nearly all modes are estimated by areal measurement performed on thin sections under the microscope. The instruments used for this purpose are now fairly numerous and quite varied in design and construction. Their proper application always has the same goal, viz., a reliable estimate of the relative proportions of the measurement area underlain by minerals of different species, and they all secure this information in one of two ways. Either they cumulate intercept lengths along a set of parallel equidistant lines, or they tally the frequencies with which the members of a symmetrical point grid are underlain by minerals of each species.

The equivalence of areal proportions to volumetric proportions was suspected and announced by Delesse in 1848. (It may have been known before this in other sciences, but there is no earlier mention of it in the literature of geology.) Though Delesse used the relation to good advantage, he did not actually prove it. Nor did any other geologist. As a consequence it was always regarded with considerable skepticism. Those who placed any confidence in it might justify their credulity by pointing to an experiment, commonly too small to demonstrate anything at all, in which the bulk chemical composition calculated from the mean of a few modes of dubious quality agreed fairly well with an actual chemical analysis of unknown quality. Sometimes the procedure was reversed, and measured modes were compared with modes calculated from chemical analyses. Occasional tests of this kind could convince only those who had a powerful will to believe. And geology is by tradition an agnostic science.

The development of the subject was correspondingly slow. Indeed, what little there was of it was primarily concerned with instrumentation. In the century following Delesse's announcement of the method it is difficult to cite a single geological research in which critical issues were either illuminated or decided by means of modal analyses. Even the currently increasing popularity of quantitative minerological rock classifications, in which the very basis of classification is modal composition, has so far proved insufficient to stimulate activity in this field.

The principal problems of modal analysis are: (*a*) the equivalence of areal and volumetric proportions, (*b*) the reproducibility of estimates of areal proportions, and (*c*) the sampling efficiency of thin sections. These are all problems which are readily susceptible of statistical examination and difficult though perhaps not impossible to study satisfactorily in any other way. During the first century of its career modal analysis enjoyed a kind of extra-statistical existence during which it promised much and accomplished practically nothing. Since 1945 it has been subjected to a persistent though rather elementary statistical reorientation and has already begun to stand on its own feet as an independent discipline; if the trend continues we may reasonably expect that it will soon assume its rightful place as the simplest, quickest, and cheapest analytical procedure available to the petrologist. The immediate future holds the promise of a development of quantitative petrography as brilliant and as productive as the great flowering of qualitative petrography at the close of the last century.

1

The geometrical basis of modal analysis

The lack of a satisfactory and easily comprehended analytical demonstration of the validity of thin-section analysis has probably been the most important deterrent to the development of the subject. Both Delesse and Rosiwal, by whom the technique was first proposed, were aware of the weakness of their analytical arguments. Despite occasional attempts since their day, no satisfactory solution of the general problem is available in the geological literature.

Delesse's original announcement of the method attracted little attention. His procedure was hopelessly time-consuming, yet a half century elapsed before anyone attempted to improve upon it. Rosiwal's improvement reduced the time per analysis to something on the order of many hours for a medium- or fine-grained rock. And the next—really the first—substantial improvement, the Shand recording micrometer, was not announced until 1916, sixty-eight years after Delesse and eighteen after Rosiwal.

The period between the announcement of the Delesse method and the appearance of the Shand micrometer is precisely the golden age of descriptive petrography. The petrographers of that day could have made excellent use of reliable quantitative modes, and many of them were keenly aware of the need for such information. They could have had it—and the petrography of our day would have profited immeasurably thereby—with instrumentation far simpler than was then developing in the sister science of optical crystallography. But the appropriate instrumentation was not forthcoming.

Considering all the circumstances, it is reasonable to suppose that the root of the failure lay in the fact that no one, not even Delesse and certainly not Rosiwal, was really convinced of the validity of the geometrical theory. In the language of today, the petrographers who should have taken prompt and thorough advantage of the Delesse method seem always to have been bothered by the fear that their results would be inconsistent, that differences between analysis would contain large, unknown, and essentially unknowable, contributions which had nothing to do with the real differences between rocks.

This may be reading too much into a long record of indifference and inertia. It cannot be denied, however, that in the last quarter of the nineteenth century the time was ripe for full exploitation of quantitative modal analysis by descriptive petrography, that the method was available, that the necessary instrumentation was within easy reach, and that nothing happened. It is also true that the textbook and lecture material to which the average geology student is exposed today still contains far more of admonition and qualification than of endorsement and encouragement. Since most geologists have actually done very little modal analysis, this attitude can hardly stem from extensive practical experience. Rather, it is a kind of professional memory, an inheritance of the fear that except on very special rocks—and perhaps even on such rocks—the thing really doesn't work. Our first business is to dispel this fear. The question of whether modal analysis is *theoretically* sound has nothing to do with petrography. It is entirely a matter of geometrical probability. In this and the succeeding chapter the discussion will accordingly be much more of geometry than geology.

1. Point Sums as Estimators of Relative Areas

In Fig. 1 a small irregular area (B) is enclosed in a large irregular area ($B + W$). The probability that a point located simply at random in ($B + W$)[1] will also lie in B is, by definition,

$$p = \left(\frac{A_B}{A_{(B+W)}}\right)$$

the ratio of the two areas.

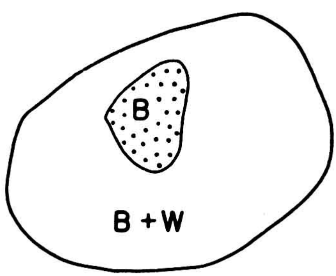

Fig. 1. Small area (B) enclosed in large area ($B + W$), the ratio of the areas to be estimated by the sums of points chosen simply at random in the region ($B + W$).

[1] I.e., in such fashion that each point in the area ($B + W$) has the same probability of being selected as any other point.

The expected value of the number of points, $S(X)$, which fall in B in a particular sample containing n points, is

$$E(S(X_b)) = np = n\left(\frac{A_B}{A_{(B+W)}}\right) \tag{1.1}$$

As a proportion, μ, of the total count, this is

$$\mu = \frac{1}{n}E(S(X_b)) = p = \frac{A_B}{A_{(B+W)}} \tag{1.2}$$

Since its expected value is the ratio of the smaller to the larger area, the proportion of the total count that falls in the smaller area is an unbiased estimate of that ratio. (In the language of thin-section analysis, $A_{(B+W)}$ is the total measurement area available in any thin section, and p is the proportion of that area occupied by mineral B, whether as a single large grain or many small ones.)

2. Parallel Lines as Estimators of Relative Areas

The areas under the curves in Fig. 2 are obviously

$$A_1 = \int_a^d y_1\,dx, \qquad A_2 = \int_b^c y_2\,dx$$

Let us suppose that ordinates are to be erected at points along OX chosen simply at random in the region $a < x < d$. The element of frequency is thus dx, and the total frequency is

$$F = \frac{1}{d-a}\int_a^d dx = 1 \tag{1.3}$$

The expected value of y_1 is then

$$E(y_1) = \frac{1}{d-a}\int_a^d y_1\,dx = \frac{A_1}{d-a} \tag{1.4}$$

and for y_2

$$E(y_2) = \frac{1}{d-a}\int_b^c y_2\,dx = \frac{A_2}{d-a} \tag{1.5}$$

(In the regions $x < b$ and $x > c$, $g(x)$ is undefined and y_2 is zero.)

The ratio of the average ordinates is thus a consistent estimate of the ratio of the areas, for it is an estimate of the parent value μ_2/μ_1, and

$$\frac{\mu_2}{\mu_1} = \frac{E(y_2)}{E(y_1)} = \frac{A_2}{(d-a)}\cdot\frac{(d-a)}{A_1} = \frac{A_2}{A_1} \tag{1.6}$$

Now the expected value of the sum of the ordinate lengths under either curve is

$$E[\sum(Y_i)] = N\mu_i = NE(y_i) \tag{1.7}$$

where N is the total number of traverses made in a particular random

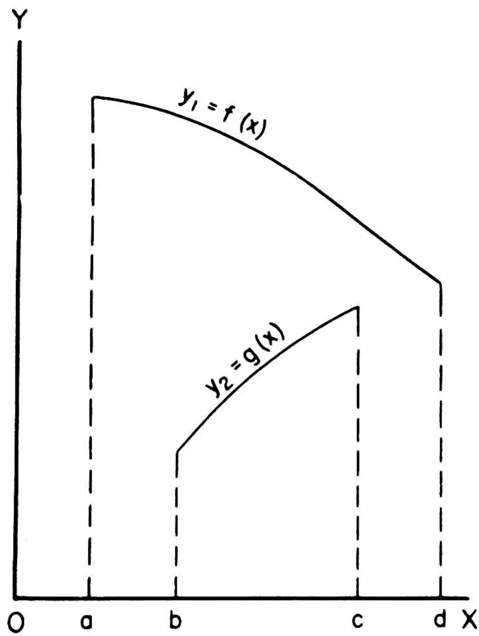

Fig. 2. Open areas under two curves, the ratio of the areas to be estimated by sums of ordinates chosen simply at random in the region $a < x < d$.

sampling of the region $a < x < b$. Again the *ratio of the sums of ordinates is a consistent estimate of the ratio of the areas, for*

$$\frac{E[\sum(Y_2)]}{E[\sum(Y_1)]} = \frac{N\mu_2}{N\mu_1} = \frac{A_2}{A_1} \qquad (1.8)^2$$

If in Fig. 2 the axes of reference are rotated about O, the area between the curves and OX will of course change, and so will the ratio of these areas. Thus, although $\frac{\sum(Y_2)}{\sum(Y_1)}$ is a consistent estimate for any particular orientation, its value will change with any change in the orientation of the axes.

[2] Equations (1.6) and (1.8) do not quite establish consistence. For this it is necessary that the variances of y_1 and y_2 be finite, but this is obviously the case since each varies over a finite range. See section 3 below.

Since the areas in Fig. 2 *do* change with rotation of the axes, any reliable estimate of areal ratios is also bound to change. If the areas in question are insensitive to location or rotation of the reference axes, however, eqs. (1.4–1.8) apply *regardless* of the position or orientation of the reference axes. Each of the closed curves in Fig. 3 may be divided into two parts by lines tangent to it and parallel to any chosen ordinate axis. In Fig. 3,

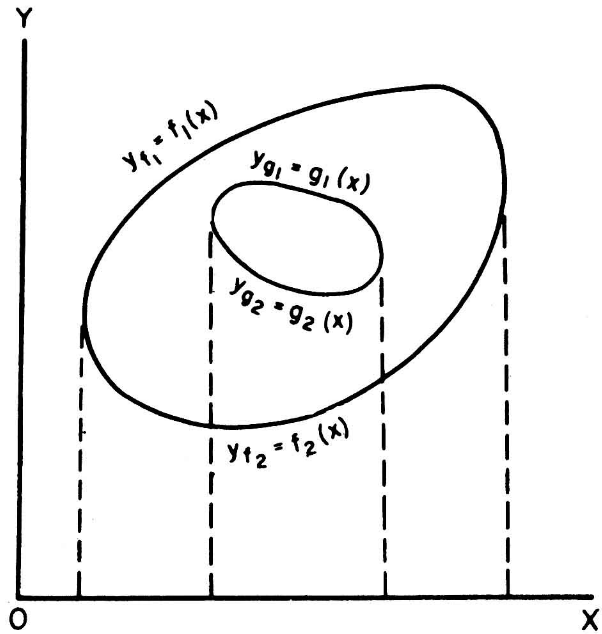

Fig. 3. Areas enclosed by two curves, the ratio of the areas to be estimated by sums of ordinates chosen simply at random in the region between the ordinates tangent to the larger area.

OY is used, and the tangents divide each curve into segments, $f_1(x)$, $f_2(x)$, and $g_1(x)$, $g_2(x)$, under each of which we can find $E(y)$ by means of eq. (1.4) or (1.5). The expected value of a difference is equal to the difference of the expected values of which it is formed, so we have at once that

$$E(y_{f_1} - y_{f_2}) = E(y_{f_1}) - E(y_{f_2}) = \frac{A_1}{d - a}$$

where A_1 is the area enclosed by the outer curve. Similarly

$$E(y_{g_1} - y_{g_2}) = \frac{A_2}{d - a}$$

where A_2 is the area enclosed by the inner curve. The ratio we seek is

$$\frac{E(y_{g_1} - y_{g_2})}{E(y_{f_1} - y_{f_2})} = \frac{A_2}{A_1}$$

and this, being a function only of the enclosed areas, is obviously insensitive to the choice of axis.

In the language of thin-section analysis, OY is the traverse path or direction, OX is the traverse normal, and the unchanging areas are, respectively, the total area of measurement and the portion of it occupied by a particular mineral.

3. Bias and Consistence

In accordance with our announced intention of relying on the popular connotations of statistical terms whenever possible, we have so far neither defined nor distinguished between consistence and bias. The reader will have noted that estimates based on parallel lines were characterized as consistent whereas those based on points were called unbiased.

The sample average $\bar{x}$ is said to be a *consistent estimator* of the true, or population, mean μ if

$$Pr\{|\mu - \bar{x}| > \xi\} < \eta \quad \text{as} \quad n \to \infty$$

however small ξ and η.

The sample average $\bar{x}$ is said to be an *unbiased estimator* of μ, on the other hand, if the expected or most probable value of $\bar{x}$ is μ for *any* $n \geqslant 1$.

Lack of bias is obviously the more desirable property. Estimates of areal ratios based on the counting of randomly located points are both consistent and unbiased. Those based on parallel continuous lines are consistent but may be biased. The effect is easily shown by an example. Figure 4 shows a square inscribed in a right isosceles triangle; the area of the triangle is twice that of the square. Proceeding as before, we measure intercepts in each figure along randomly chosen lines parallel to the altitude of the triangle. By using eqs. (1.3–1.6) the student should be able to show that

$$\frac{E(y_2)}{E(y_1)} = 0.5 = \frac{A_s}{A_T} \tag{1.9}$$

where A_s is the area of the square and A_T that of the triangle. From the point of view of modal analysis, this is, of course, the correct answer. The summation extends over all ordinates for each intercept and is completed before the ratio is calculated. Equation (1.9) suggests that the ratio of observed average (or total) intercept distances is a consistent estimator of A_s/A_T but, since it does not describe the situation for any

$n < \infty$, cannot of itself establish consistence. For this we have to rely on the central-limits theorem. The range, and hence the variance, σ^2, of each expected value is finite. From this it follows that with increase in n the distributions of the observed means of the numerator and denominator of eq. (1.9) approach normality with variances σ_1^2/n, σ_2^2/n. And from this, finally, it follows that for n sufficiently large the inequality used as the definition of consistence does in fact hold.

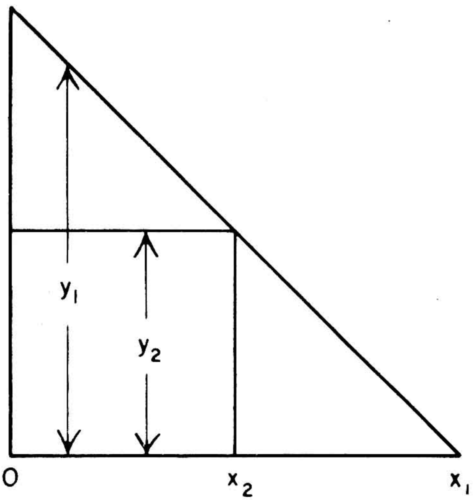

Fig. 4. Square inscribed in isosceles triangle, the expected value of the ratio y_2/y_1 and the ratio (expected value of y_2)/(expected value of y_1) to be estimated from ordinates chosen simply at random in the region $O < x < x_1$.

Suppose, however, that, instead of summing y_2 and y_1 separately before finding the ratio, we calculated the ratio at each ordinate. If $R_1 = y_2/y_1$, then, by construction

$$R_1 = \begin{cases} \dfrac{1}{2-x} & 0 < x \leqslant 1 \\ 0 & 1 < x \leqslant 2 \end{cases}$$

Following the same reasoning as before, we have that

$$E(R_1) = \frac{1}{2}\int_0^1 \frac{dx}{(2-x)} = 0.347 \qquad (1.10)$$

Thus, although A_s/A_T *is* the ratio of the expected values of the ordinates, it is clearly *not* the expected value of the ratio of ordinates.