FORECASTING ON YOUR MICROCOMPUTER

BY DANIEL B. NICKELL

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Introduction

A famous mentalist used to open his program by saying: "We are all interested in the future because that is where we are going to spend the rest of our lives." The function of scientific forecasting is to provide a rational and acceptable basis for what we believe the future holds for us.

The purpose of this book is to help a wide range of readers use a microcomputer in the process of formulating forecasts over a fairly broad range of human interests. The definition of the average reader of this book is rather difficult to state meaningfully. I purposefully did not write it specifically for stock brokers, math teachers, child prodigies, business planners, or any other group likely to have a vested interest in forecasting. Rather, I wrote it as a resource book for each of the foregoing, as well as for others that might have an interest in some aspect of the subject. There is a section of particular interest to one tracking the stock market or similar data sets. For the amateur astronomer, there is a section that provides some useful routines for computing planetary positions and related information. The gambler in us will find some satisfaction in the section on random events in which an effort is made to put some order into chaos.

In the book are some 94 programs (all in BASIC), which illustrate or support one aspect of forecasting or another. Although some of them are obviously demonstration routines, many of them are intended mainly as utilities to be embedded in programs of your own making. Finally, there are several long programs that present some rather useful analysis of variance (ANOVA), multiple correlation, and tabular computation routines that stand on their own merits.

While this book doesn't presume to address the interests of the youngest or the most sophisticated of computer programmers, it is hoped to reach and be of use to the general population of computer users.

A LITTLE ABOUT THE BOOK

The material in this book is grouped into four main divisions: fundamentals, techniques, applications, and dispositions. Each division consists of two or more chapters. Each chapter contains a main body of information, including related programs and examples; several exercises or questions on the chapter's material; and suggested reading.

Fundamentals

In this section I attempt to cover the numerical and mathematical foundations for modern forecasting and to describe the nature, applications, and limitations of modern forecasting. This division attempts to convey the notion that forecasts vary considerably in their reliability, depending on a number of factors including the quality of the data base. Reliable forecasting requires rationally chosen, accurate data that are appropriately quantified. The types of data, scales, and measures that are used to quantify data are examined. Also presented are the dichotomous concepts of absolute and relative scales, parametric and nonparametric data, and continuous and discrete distributions. It is in this section that I provide you with the tools with which to do a descriptive analysis of a set of data and to compute the probabilities of a variety of conditions. The division also includes a primer on data base management, with an emphasis on data bases suitable for microcomputers. I conclude with a few words on the prudent choice of techniques with which to do accurate forecasting.

Techniques

This book presents four basic families of forecasting techniques: correlation and regression analysis, time-series analysis, modeling and simulations, and numerical techniques. Correlation and regression analysis attempts to fit a smoothed curve through available data, assuming the future data will continue to follow the smoothed curve. Time series analysis assumes that the data found are functions of the passage of time. Curve-fitting routines, in time-functional data, often lead to rather bizarre unnecessary shapes and equations. Time-series equations, on the other hand, concentrate on the state of the data at given instants. Modeling a problem is frequently the easiest way to forecast the state of a complex system after a number of operations or periods of time. Closely related to modeling is creating simulations. The distinctions between the terms is so vague that I am likely to use them interchangeably. Conceptually, a model is thought of as a representation of a real or proposed thing or system (such as an oil factory), whereas a simulation may be used to reflect a less specifically organized set of objects (such as a simulation of waterflow through a swamp land) in which arbitrary starting values are assigned. Numerical analysis is sort of a coverall for all the other miscellaneous techniques that can be used in attempts to fit some sort of conceptual model against the real world. For example, we are given the problem of a lost person in the Arctic. We know where he started and that he appears to be moving clockwise in a circular fashion (it is snowing and drifts often block out the trail) except that the radius of revolutions is increasing. None of the first three techniques will lead quickly to a satisfactory solution. However, by simply fitting one of the spirals of Archimedes noted in Chapter 8, the appropriate search pattern can be developed and the current, most likely location forecast.

Applications

Examples from seven basic areas of human interest in which forecasting techniques can be applied are presented. In addition, there is a brief chapter on miscellaneous techniques that were not otherwise classifiable. It was with a great deal of humility that I undertook these chapters and with an even greater sense of humility that I presumed to finish them. While they each come from some area of my personal and professional experiences, the requisite research to prepare the text led me back into broad domains of human knowledge the vastness of which I had forgotten in my haste to keep pace with the present. Behind each chapter stands several thousands of years of specialized scientific inquiry, which has been cataloged, analyzed, digested and disseminated by hundreds of thousands of scholars—better than eighty percent of them alive since the year 1800AD. In translating their concepts into microcomputer BASIC I hope we have been faithful to the true and underlying concepts.

Dispositions

Many forecasts fail to be implemented due to a misconception, often on the part of the forecaster, as to what to do with the "bottom line" data. In this division the different ways scientific data can be displayed effectively to non-scientific people are illustrated. I also offer some techniques, using the microcomputer, or using results from forecasting efforts to reach meaningful decisions.

The Programs

Listings will always include the complete program written in BASIC. There are short routines that you can simply type in to see what it is I am talking about. There are long programs that do a number of functions. The intent of these longer program listings is to provide complete programs, which you can use intact or from which you can borrow subroutines for other applications. In some instances, I use a technique used by a major retail outlet: I offer a good routine, a better routine, and a best routine. With the longer, more complex programs I also include flowcharts of the program and, if needed, additional documentation showing line references and variable names.

Most of the chapters also include exercises. These are intended to be used either to check your programming skills and understanding of the material just presented or to check student progress in a computer applications course in which this book is used as a text or reference source.

The language used in the book is the Radio Shack TRS-80 Level II version of BASIC. With two exceptions, I have used a programming technique and a dialect of BASIC that is as standard as possible so that most of the programs will run as written on the other leading systems. The first exception is the use of the DEF FN command to define a subroutine used frequently. If your computer does not support this command, you can replace it at each occurrence, for example, X = FNM, with a GOSUB command in which the subroutine performs the same function.

The second exception is that in programs requiring intensive data base support, I have used the Exatron Stringy Floppy (ESF) command vocabulary. With minor modifications, those of you who have disk drive systems can make the necessary adjustments to get the programs to run under most versions of DOS. These are the most important changes:

Exatron	Disk (TRSDOS)
@CLOSE	CLOSE
@CLOSE b	CLOSE b
@PRINT A$	PRINT#b, A$
@INPUT A$	INPUT#b, A$
@OPEN	OPEN"O",b,file spec. for output files, or OPEN"I",b,file spec. for input files.

where b is the designated buffer.

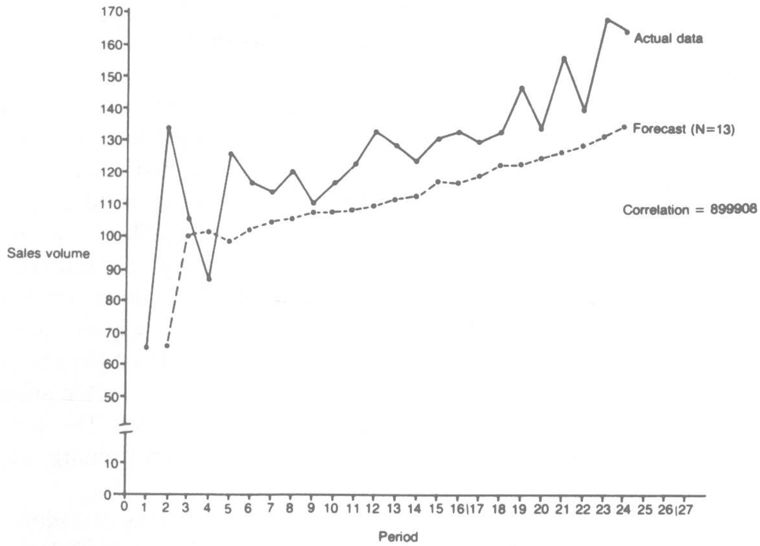
Refer to Appendix A for further information on the Exatron system.

In brief, this book attempts to define what the forecast process is and what the underlying mathematical principles are, to illustrate ways forecasting has been accomplished in the past, and to suggest what to do with a forecast once it's made. By definition, analogy, example, and actual data resources, I attempt to equip the conscientious reader with a basic forecasting toolkit.

NOTE: Most of the programs in this book are written to be used with a printer. If you do not want to print the output of any of these programs, delete all LPRINT statements from them. If there is no printer connected and you do not delete these statements, the programs will not function and no error message will appear.

Contents

Chapter 1



Forecasting Philosophies

Forecasting is an art that exploits science in an effort to identify future events or conditions. Later in this book we will stress the necessity for obtaining as much precise data as possible and emphasize the role of mathematics in forecasting. Nonetheless, forecasting remains an art. For virtually every project, the data available for the solution of the problem exceeds both the memory capacity of our computers and the time allotted to reach a forecast; we must artfully select a sample from the data and forecast from that.

SELECTING DATA SETS

While there are techniques, which will be described herein, that help in the process of selecting the more appropriate data sets, there is no practical way of knowing that the best data and only the best data have been chosen for processing. The quality of the forecast is directly related to the quality of the input data we elect to use. It is very much like the computing maxim: "garbage in—garbage out." If we select relevant data sets and use appropriate processing techniques, our forecasts will be as responsible as anyone could expect. In preparing a weather forecast, for example, we would probably want to consult historic weather records for the area of coverage, current temperatures, humidity, and wind direction. According to some weather forecasting techniques, we may even want to determine the current sunspot condition. On the other hand, we have no reason to believe any of these data bases would have any utility in the forecasting of future gold prices. The dividing line between the merely good and the truly great forecasters is drawn according to the skill of the forecaster in selecting data bases that will lead consistently to accurate predictions.

Why can't we just crank in all the data bases and pick out the best predictors? In the first place, the volume of data available in the world at any given moment, even in the available, processable form, is tremendous. For one person to attempt to enter even a fraction of some of these data bases into a microcomputer would be an undertaking of several lifetimes. Secondly, assuming we had some sort of device which could handle all of these data for us, the processing needed to determine the most appropriate sets to use quickly becomes unmanageable. Given N sets of data, the number of unique combinations or *permutations* of these sets taken R sets at a time is given by the equation

$$P = \frac{N!}{R!\,(N\text{-}R)!}$$

N! is an expression which means "N factorial" and is computed by $N! \times 1 \times 2 \times 3 \times 4 \ldots . \times N$. For example, if we have 10 people and 6 chairs, we compute the number of unique permutations of seating 10 people, 6 at a time, as

$$P = \frac{}{6!(10-6)!} = \frac{3628800}{720 \times 24} = 210.$$

There are 210 ways to seat 10 people 6 at a time without repeating any possible combination. If we have 10 data bases, there are 252 unique ways we can test the utility of them taking them 4 at a time. But, to check out the utility of the data bases properly, we first need to check them one at a time. This requires 10 evaluations, one at a time, then 45 evaluations doing 2 at a time, 120 evaluations with 3 at a time, and so forth. To check all possible permutations, from 1 to N sets at a time, requires $2^N - 1$ evaluations. In the case of 10 data sets, there must be at least 1023 evaluations. Adding just one data set, making the total, N, 11, doubles the number of evaluations required: $2^{11} - 1 = 2047$. The 1975 *Statistical Abstract of the United States* offers well over 1400 tables of data. To evaluate just these tables, even with very fast microcomputers using optimum programs, would require thousands of years of processing. The working forecaster, of course, rejects such an approach and, by some intuitive process, selects perhaps two or three tables from which to compute a prediction. It is for this reason that forecasting is as much an art as a science.

THE QUALIFICATIONS OF A FORECASTER

The difficulties and complexities just alluded to notwithstanding, it appears that forecasting is a skill or craft a number of so-called laymen can master. A few years ago a study was conducted to determine, among other things, the impact of training and experience on the accuracy of forecasts. The initial hypothesis was that as the experience or training level of a given forecaster increased, there would be a corresponding increase in the accuracy of his or her forecasts. It was anticipated that the graph of the accuracy/experience table would appear very much like that in Fig. 1-1:

Interestingly, however, the data obtained in the study failed to support the hypothesis. Instead, the data suggested that almost anyone with average intelligence and reasonable judgement can become an effective forecaster. The actual relationship between accuracy and training are illustrated in Fig. 1-2:

The implications are clear. You don't necessarily need to have a Ph.D in economics to do financial forecasting, nor do you need advanced degrees in political science to make useful contributions in voter preference studies.

One explanation of the unusual outcome of the study is that the future is really the composite outcome of such a multitude of factors that, beyond a certain level, the precise computations of the highly trained and expensively equipped add little over common sense and a more generalized approach to a problem. With reasonable technical preparation and modest computing gear (such as microcomputers), you can become a very credible forecaster.

The point here is to encourage you to tackle forecasting seriously and not to be intimidated by a shortfall in formal training or professional credentials. The forecaster's best credential is a demonstrated forecasting accuracy.

PRECISION IN FORECASTING

It is best to avoid the lure of the double-precision mathematics capabilities of most of the current microcomputers. There is no point in computing to 16 decimal places data that are meaningful to only three or four. Curve-fitting routines, for example, lead to equations that generate smooth

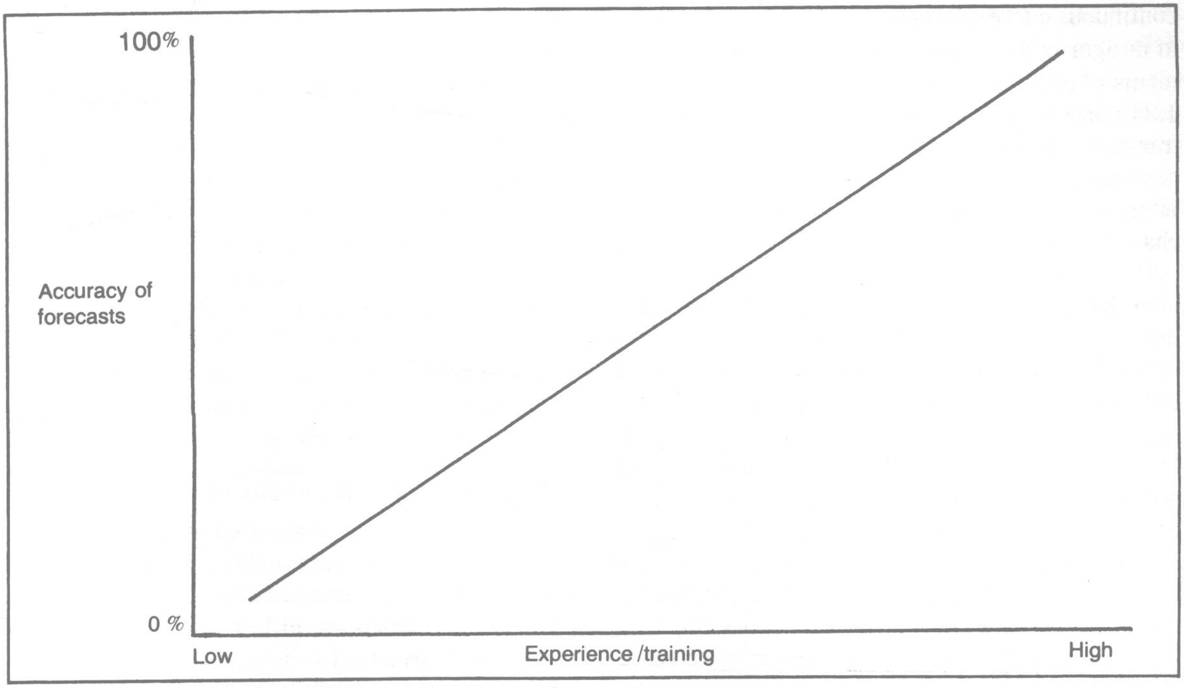


Fig. 1-1. Hypothetical relationship between training and accuracy of forecasts.

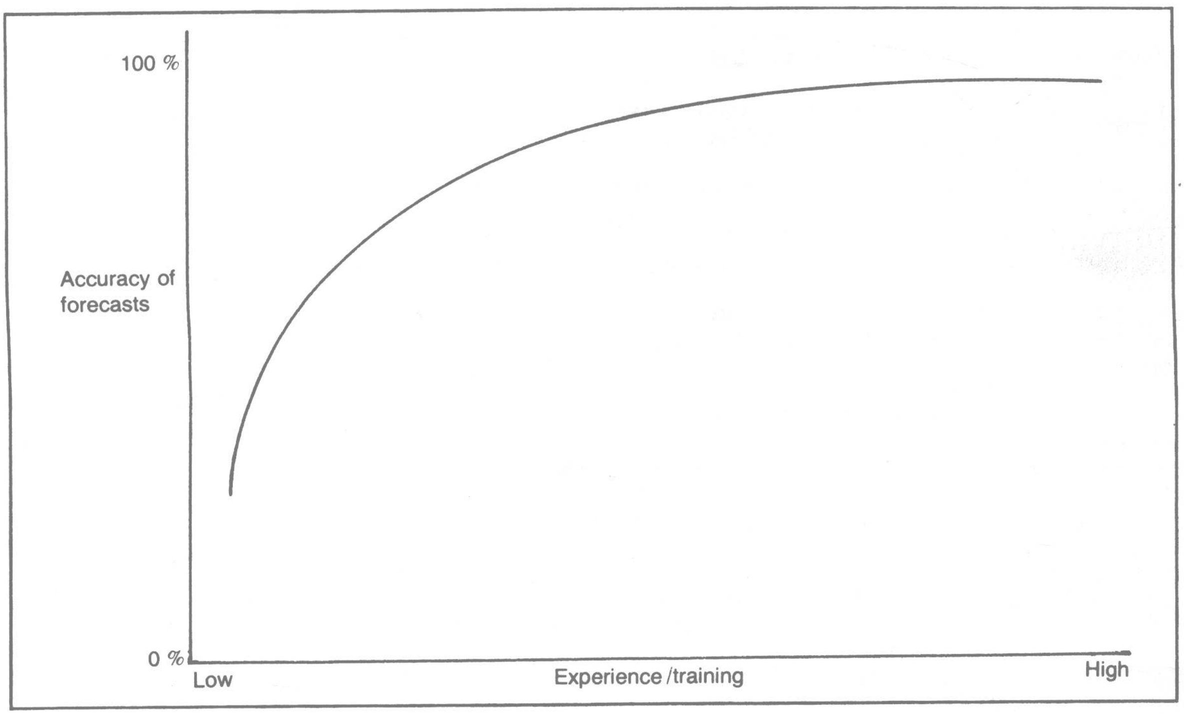


Fig. 1-2. Actual relationship between training and accuracy of forecasts.

continuous curves. People, on the other hand, come in integer units. In the real world we must deal in terms of one person or two, but never in terms of 1.34 persons. It is seldom meaningful to compute fractional human beings, certainly not beyond two or three decimal points in most applications. If the effort is to determine the amount of grain to purchase to feed a million people, it is useful to compute grain consumption only to the degree of precision that will affect the decision to buy one bag of grain more or less. To compute grain requirements to fractions of one grain is meaningless. Be careful to be precise enough so that in adding or multiplying (or subtracting or dividing) the final result is still accurate. Be especially careful when using exponential tools wherein values are raised by powers of numbers. Be precise enough to be accurate, but not tedious. A candidate for political office, for example, looks to the forecaster for an estimate of the number of votes pro and con. In the final analysis, this can and should be an integer value. 501 votes for a candidate are significantly greater than the 499 against: 500.0001, on the other hand, is effectively equal to 499.9999. Additionally, the double-precision feature dramatically increases computational time. As a demonstration, run Listing 1-1 on your system.

The lesson here is that, except for short programs or programs requiring extremely precise computations, double-precision math can complicate and frustrate your data processing more than it can solve your problems. For example, Chapter 12 contains a simplified program to compute planetary positions (Listing 12-6). The technique used provides an acceptable degree of accuracy for the astronomical hobbyist who can accept a half a degree of arc error here and there.

THE DECISION-MAKING PROCESS

Forecasting involves two philosophies: an internal philosophy that enables the forecaster to remain calm and serene when it appears that the available data bases are either incomplete or inappropriately formatted—thus creating the need for original research before the project can continue—or when it appears the decision-maker is not

Listing 1-1. Double Precision Demonstrator

```
1 '****************************************
2 'DOUBLE-PRECISION DEMONSTRATOR          *
3 'LISTING 1-1                            *
4 '****************************************
5 '
10 CLS:GOTO100
20 PRINT"WHEN READY, TOUCH ANY KEY AND START TIM
    ER."
30 Q$=""+INKEY$:IFQ$=""THEN30
40 PRINT"SYSTEM WILL COUNT TO 1000"
50 FOR I=2 TO 10
60   X=3/LOG(I)
70   Y=SIN(X):PRINT@448,I;Y,
80 NEXT I
90 PRINT:PRINT"STOP TIMER, NOW!":RETURN
100 GOSUB20
110 INPUT"ENTER NUMBER OF SECONDS";S1
120 DEFDBL X,Y
130 GOSUB20
140 INPUT"ENTER NUMBER OF SECONDS";S2
150 PRINT:PRINT"BY SIMPLY CALLING FOR DOUBLE PRE
    CISION IN LINE 110, YOUR
COMPUTER'S PROCESSING TIME IS INCREASED";100*((S
    2/S1)-1);" %"
160 END
```

inclined to be rational this month; and an external philosophy that is skeptical and inquiring at its foundation, but includes a reasonable faith in properly collected and processed data as potentially rational indicators of future events or conditions.

Perhaps the dichotomy between the analyst and the decision-maker is created by otherwise useful personality differences. By nature the analyst proceeds in an orderly and frequently plodding manner, working through the various steps of a problem very methodically and conscientiously. By the time the final analysis is done and conclusions are reached, it is often too much of a temptation to lead the decision-maker through the same process one step at a time. The decision-maker, on the other hand, frequently cares about neither the process nor the standard errors and deviations surrounding the bottom line conclusions. Will the widget sell? Yes or no? Will sufficient units be sold to offset all production and sales costs so an acceptable net profit can be realized? Yes or no? All too often, the decision-maker expects the forecaster to simply enter the office, and announce: "Sell (or Don't Sell) the Widget!" and promptly leave. If the original instructions included words and guidance that facilitated such decision-making on the part of the forecaster, the expectation above is valid; otherwise, it is not. The decision could be made if, for example, the Widget marketing task was accompanied by these instructions:

A. If the marketing data can be applied to 95% (or better) of the total market place, with a standard error of plus or minus three,

and

B. if the forecast is for a minimum of 1,000 sales in the first year, 1250 the second year, and, in the third and subsequent years, growth at a rate of 50% per year,

then

C. recommend we sell the Widget; otherwise, recommend we do not sell the Widget.

The point is that the rational way to handle scientific data is through some orderly process. While Chapter 18 goes into some detail concerning decision-making techniques, this concept is introduced now in hope that the forecaster in you will be mindful that at the end of all of the analysis, someone has to make a decision on what to do based on the information. Sometimes it can be precisely expressed as above. These kind of criterion can often be developed independent of any market research. A good plan designer, knowing the function each division and machine will have to perform, and in what sequence, can compute the cost of equipment and operators based on information concerning a nominal number of items under production at that particular time. It is the market analyst who must forecast what the most likely product demand volume will be.

EXERCISES

1. Identify conditions or circumstances that affect the reliability of forecasts.
2. Give examples of areas in which precise forecasts can be made. Give an example of an area in which forecasting is probably a waste of time.
3. Identify an area in your personal life for which you could develop a useful forecasting tool.
4. Select a published forecasting effort (economic, business, or weather, for example) and keep a record of the forecasts and actual performance for a month or two. How well do the forecasters do? Can you explain their successes or failures?
5. Develop a convincing case for conducting and publishing political surveys. Develop a convincing case for not doing political surveys.

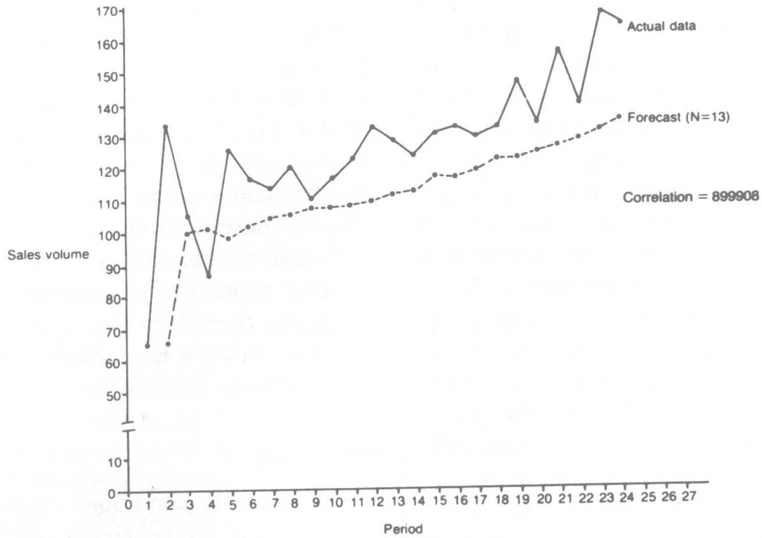
SUGGESTED READING

Ayers, R. U. 1969. *Technological Forecasting and Long Range Planning.* New York: McGraw-Hill.

Makridakis, S. and S. C. Wheelwright, 1976. *Forecasting Methods & Applications.* New York: John Wiley & Sons.

Reichard, R. S., 1966. *Practical Techniques of Sales Forecasting.* New York: McGraw-Hill.

Chapter 2



Underlying Mathematical Principles

Related to practically any subject selected for forecasting are all manner of data that may range from the purely subjective ("Yeller was a good ole' dog . . .") to the precise and absolute ("Water freezes at 273° Kelvin (K) and boils at 373°K.) I am hard pressed, in this book, to help anyone to do forecasting from the subjective or anecdotal end of the data spectrum. At the same time the data need not all be from absolute scales of measurement. We can frequently (and more often than not, in fact do) make sense and produce effective forecasts from a variety of data types as long as we can count, measure, or by some other means, quantify the data. The key is quantification. If by some rational means or process, you can assign some numerical value to an element under investigation, the data are quantifiable and, more than likely, forecastable to some degree. All that follows, then, deals with numerical or quantified data.

SCALES

Scalar data (those data that describe things in terms of hot to cold, good to bad, little to much, and so forth) can be categorized into four types of measurement scales: *nominal, ordinal, interval,* and *ratio*. These scales are listed in this order for a purpose. There are a certain number of mathematical operations one may do with nominal data, but only relatively few. With ordinal data, however, one may perform all of those operations, plus a number more. This rule continues through interval and ratio scale data as well. It is important to be sure to use mathematical operations appropriate to the data. Failure to do so results in meaningless forecasts.

Nominal Scales

A nominal scale consists of two or more categories or classifications into which the subjects of the study can be divided. One example of the use of a nominal scale is a census of a population described by nationality. At a particular university there are 3256 Americans, 245 British, 178 French, 89 Italians, 78 Gambians, and 2 Swiss. These categories can be totaled; percentages can be computed, and other similar simple numerical relationships can be established. Assignment to a category is not a value judgement, nor are the numbers that are generated meaningful beyond a limited application. It is entirely proper, however, to use such scales in forecasting. Given a series of university census records taken over a period of time, it is certainly feasible to compute the overall growth (or decline) of the university population and the changing ethnic makeup of the student body, and to draw meaningful conclusions from these data.

Ordinal Scales

An ordinal scale is one in which elements of the population under study are placed in some sort of relative order. Different brands of laundry soap, for example, may be placed on an ordinal scale by a panel of judges. The key feature of the scale is that the ratings or measures tend to be adjectival; that is, the items are rated as good or bad or satisfactory or unsatisfactory according to some criterion (smell, shape, apparent utility). The researcher may even attempt to define a certain precision in all this subjectivity by having the item rated numerically, as "on a scale of 1 to 10, how do you rate the President's foreign policy?" The essential element to remember, though, is that the interval between these scale increments is not constant; rather, it is elastic. In reality, the scale for the responses to the foreign policy question are distributed along a scale that is not measured evenly like this:

```
1    2    3    4    5    6    7    8    9
Poor                Average           Good
```

but one much like this:

```
1 2 3      4        5 6 7       8 9
Poor                Average     Good
```

For this reason, the complexity of the scale will serve to confound rather than aid the analyst, and the recommendation is to reduce the number of increments to three or five. It is even more important to avoid any computations that assume even intervals. It would be irresponsible to imply that a rating of four is just one notch below five, or that a rating of two is to four as four is to six. Such relationships will exist only coincidentally. A very common misuse of ordinal scales occurs in schoolrooms across the nation. Students are frequently given spelling tests, mathematics quizzes, and history exams that consist of a certain number of items or questions. The grade or score is then computed as a percentage of the number of items answered correctly. Unless the test is constructed with extraordinary care and skill, it is most unlikely that a student who scores an 89 is actually 2 percent "smarter" on the topic than a student receiving an 87. The fairest and safest thing we can say is that the first student responded to the test items "better" than the second student. It would be wrong to make much of the quantifiable differences between them.

Testing and grading have demonstratable utility and worth. Tests have been constructed such that, when properly administered, a group of subjects can be reliably divided into those who are likely to succeed at some future effort from those who are likely to fail. One who cannot carry a fifty-pound weight ten paces in a test cannot reliably be expected to carry a hundred and fifty-pound weight from a real burning building. Likewise, one who completes examinations in analytic geometry and algebra well above a responsibly established cutoff score can be expected to do well in calculus. To infer anything of consequence from the fine gradations between scores, however, is quite risky.

Interval Scales

Interval scales are those upon which two or more items can be compared in units of constant measure. An example often given is that of temperature as measured on the centigrade scale. Zero is established as the freezing point of water and one hundred degrees, as the boiling point of water. All other aspects of the scale are derived from this interval. To say that something 20°C is twice as cold as something 10°C is incorrect. At the same time, using ratios of differences is acceptable. It is correct to say there is twice as much change in a score that moves from 3 to 7 as in a score that moves from 3 to 5. The essential element of an interval scale is that it has no absolute zero base or starting point. Only with such a point can we say that twenty of something is twice ten of that thing. Twenty people is twice ten people because there is a condition of absolutely zero people. 10°C is not half 20°C because 0°C is not really the bottom of the scale, but only an arbitrary point on it selected for convenience. (Degrees on the Kelvin scale, however, are measures of absolute temperature because the starting or zero point of the scale is at the point of absolute cold, the point at which nothing, theoretically, can be any colder.)

Ratio Scales

The main distinction between interval and ratio scales is that the ratio scale always has a zero point. Units of weight, time, length, area, volume, angular measure, and the cardinal numbers used to count people, eggs, and money are examples of ratio scales.

All of the mathematical operations that can be applied to the foregoing scales can be used with ratio scales, as well as all the remaining statistical tests and measures, especially those requiring ratio differences from an absolute base.

SUMMATION RULES

Numerical tables are very frequently used in statistical studies and projections. Equally as common is the symbol: ΣX. This means to sum up the values of X, given a set or listing of X values. If there is a set of numbers [1, 4, 8, 2, 4, 8, 3, 7], then $\Sigma X = 1 + 4 + 8 + 2 + 4 + 8 + 3 + 7 = 37$. Consider the matrix below:

i (columns)

		1	2	3	4	Total
j (rows)	1	X_{11}	X_{21}	X_{31}	X_{41}	X_{i1}
	2	X_{12}	X_{22}	X_{32}	X_{42}	X_{i2}
	3	X_{13}	X_{23}	X_{33}	X_{43}	X_{i3}
	4	X_{14}	X_{24}	X_{34}	X_{44}	X_{i4}
Total		ΣX_{1j}	ΣX_{2j}	ΣX_{3j}	ΣX_{4j}	$\sum_{j=1}^{4} \sum_{i=1}^{4} X_{ij}$

The notation $\sum_{i=1}^{N} X_i$ means to sum or add up all of the values from 1 to N. The notation, from the matrix above, $\sum_{j=1}^{3} \sum_{i=2}^{4} X_{ij}$ means to add up all of the values in columns 2 to 4 and rows 1 to 3: $X_{21} + X_{31} + X_{41} + X_{22} + X_{32} + X_{42} + X_{23} + X_{33} + X_{43}$. There are three principles in summation that make the task a little easier and less error-prone:

1. Always work from right to left as illustrated above. That is, when two or more summation symbols are given, follow the direction of the symbol on the right first and then move to the left. In the illustration above, the values in columns 2 through 4, row 1, were summed; then all the values in columns 2 through 4, row 2, were summed, and so forth.
2. Complete all work within the parentheses first.

$$\sum_{i=1}^{3} X_i^2 = X_1^2 + X_2^2 + X_3^2$$

$$\left(\sum_{i=1}^{3} X_i \right)^2 = (X_1 + X_2 + X_3)^2$$

3. $\sum_{i=1}^{N} KX_i = K \sum_{i=1}^{N} X_i$ where K is some constant; e.g.,

$$KX_1 + KX_2 + KX_3 = K(X_1 + X_2 + X_3)$$

The following are examples of the summation principles:

$$\sum_{i=1}^{N} (X + Y_i) = \sum_{i=1}^{N} X_i + \sum_{i=1}^{N} Y_i$$

$$\sum_{i=1}^{N} (X_i + 2) = \sum_{i=1}^{N} X_i + 2N$$

$$\sum_{i=1}^{N} (Y - a)^2 = \sum_{i=1}^{N} (Y^2 - 2aY + a^2)$$

$$\sum_{i=1}^{N} Y^2 - 2a\sum_{i=1}^{N} Y + Na^2$$