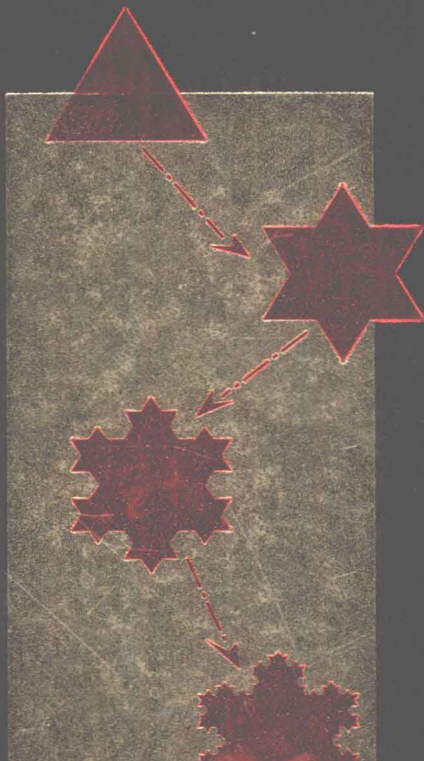
MODERN GEOMETRIES

FOURTH EDITION



JAMES R. SMART

Modern Geometries

Fourth Edition

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Preface

Since the publication of the third edition, the practice of including one or more courses in modern geometry at the junior–senior level in universities has continued. The use of both groups of transformations and sets of axioms to classify geometries remains important. The field of computer graphics has emerged as a major application of geometric ideas. The continuing emphasis on problem solving throughout the mathematics curriculum has shown geometry to be a fascinating source of problems, and has pointed up the relationship between skill in geometry and skill in problem-solving techniques. Increasingly, geometry is seen as an applied as well as an abstract study.

The general goals and recommendations from *Geometry's Future*, conference proceedings sponsored by COMAP, Inc., have been carefully considered in planning this new edition, and specific suggestions from the report have been incorporated.

This fourth edition also incorporates many improvements suggested by reviewers and by teachers and students who have used the previous editions. The exercise sets have been extended to include varying levels of difficulty, with more investigative-type problems appropriate for individuals or small groups. Other changes include additional explicit examples and figures, the improvement of many figures, clarification of wording, rewriting of some proofs, the inclusion of many new applications such as fractals and geometric probability, more historical notes, and a stronger bibliography. All of the desirable features of earlier editions have been retained, so that these changes result in a text that is both mathematically sound and highly teachable.

As the title indicates, the central theme of the text is the study of many different geometries, rather than a single geometry. The first two chapters give two ways of classifying some geometries: by means of sets of axioms or by the type of transformation defined. Both ways are used extensively in this text, and sometimes both are used for studying the same geometry. The finite geometries of Chapter 1; the convexity, Euclidean geometry, and constructions of Chapters 3, 4 and 5; the projective geometry of Chapter 7; and the non-Euclidean geometry of Chapter 9 are based on various sets of axioms. In addition to the examples of geometries presented from a transformational point of view in Chapter 2, there are separate chapters (6, 7, 8) on inversion, projective geometry, and topological transformations with the nature of the geometry depending on the transformation allowed.

Throughout, the text emphasizes practical and up-to-date applications of modern geometry. Students should be aware that many of these topics are discussed in current professional journals and that contemporary research mathematicians are seriously involved in the extension of geometric ideas. The major new applied area of computer graphics, with an increased emphasis on matrices for transformations, is covered in greater detail in the fourth edition.

This text is written for students who range widely in their mathematical abilities. While much of the material is appropriate for those with average or weak backgrounds in geometry, students with strong backgrounds will find many new extensions of ideas that will sharpen their problem-solving skills and encourage them to continue their investigations. The text is planned for both majors and minors in mathematics. It is appropriate for students interested in mathematics from the liberal arts standpoint and for those planning to be teachers of mathematics.

Many of the first exercises in each set can serve as the basis for classroom discussion; in that way, the instructor can make certain that fundamental concepts are understood. Later exercises allow extensive practice in providing independent proofs. The open-ended exercises provide extensive opportunities for further study. New in this edition is the designation of some of these investigative–type questions for use in cooperative learning situations. Chapter review exercises provide additional practice of many concepts and skills. Some instructors may also wish to use these as practice tests.

The various chapters of the text are largely independent. This and the arrangement of topics within each chapter allow for great flexibility in their use, according to the needs of the class and the desires of the instructor.

▲ Table Suggesting Some Possible Course Arrangements

I. Use of entire text for two-semester or three-quarter course

Plan A. (Semester or quarter system) Follow sequence of text

Plan B. (Semester system) One semester for geometry of transformations (Chapters 2, 6, 7, 8) and another semester for axiomatic systems (Chapters 1, 3, 4, 5, 9)

Plan C. (Quarter system) One quarter for geometry of transformations (Chapters 2, 7, 8), another quarter for axiomatic systems (Chapters 1, 6, 9), another quarter for extensions of Euclidean geometry (Chapters 3, 4, 5)

II. Use of portions of text for courses covering less than an academic year

Plan D. (Two-quarter) Any two quarters from plan C

Plan E. (Two-quarter) All of text except Chapters 5, 6, 9

Plan F. (Two-quarter) All of text except Chapters 3, 5, 8

Plan G. (One-semester) Either semester from plan B

Plan H. (One-semester) Chapters 1–4

Plan I. (One-semester) Chapters 4, 5, 7 and one other selected chapter.

Plan J. (One-quarter) Any one quarter from plan C

Plan K. Survey as many chapters as time permits by choosing first sections and omitting later sections of each chapter.

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I would also like to thank Robert Wisner, Jeremy Hayhurst, Elizabeth Barelli, Joan Marsh, Carline Haga, and all the other people at Brooks/Cole Publishing Company who helped in the preparation of this edition. Finally, let me thank Daryl Angney and the others at TECH*arts* for the great job of making work on the book such a pleasant experience.

James R. Smart

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Sets of Axioms and Finite Geometries

1.1 Introduction

Today, the great increase in practical applications of geometry is leading to interest in many new topics. The idea that geometry consists only of the typical high school geometry course is completely inadequate. Your concept of the field of geometry should change dramatically as you study this text.

The word *geometry* literally means "earth (*geo*) measure (*metry*)." Although this literal meaning is far too narrow to include the various modern geometries explored in this text, the idea of earth measure was important in the ancient, pre-Greek development of geometry. These practical Egyptian and Babylonian applications of geometry involved measurement, to a great extent, and they were not complicated by formal proofs.

Hundreds of baked clay tablets from ancient Babylonia include mathematical ideas. The analysis of a tablet designated Plimpton 322 (catalog number 322 in the Plimpton collection at Columbia University) has shown the Babylonians, as early as about 1900–1600 B.C., had extended their knowledge of right triangles to a compilation of primitive Pythagorean triples. For example, the numbers 3, 4, 5 are primitive triples because they can be used as the lengths of the sides of a right triangle, and have no common integral factors other than one. The Babylonians used rules for finding areas and volumes of some figures, and are also given credit for dividing the circle into 360 parts.

Two ancient Egyptian papyruses dating from about the same period, the Moscow papyrus and the Rhind papyrus, contain a total of 26 geometric problems, most of which involve various mensuration formulas used for computing land area and the volume of granaries. Some problems concerned finding the slope of the face of a pyramid.

During the Greek period, the science of earth measure became more refined. About 230 B.C., Eratosthenes made a remarkably precise measurement of the size of the earth. According to the familiar story, Eratosthenes knew that at the summer solstice the sun shone directly into a well at Syene at noon. He found that at the same time, in

Alexandria, approximately 787 km due north of Syene, the rays of the sun were inclined about 7.2° from the vertical (Figure 1.1). With these measurements, Eratosthenes was able to find the diameter of the earth.

Example ➤ Use the measurements of Eratosthenes to find the approximate difference in angle of elevation of the sun in two places 476 km apart in a north-south line, both south of the equator.

$$\frac{787}{7.2} = \frac{476}{x}$$

$$x \approx 4.4^\circ$$

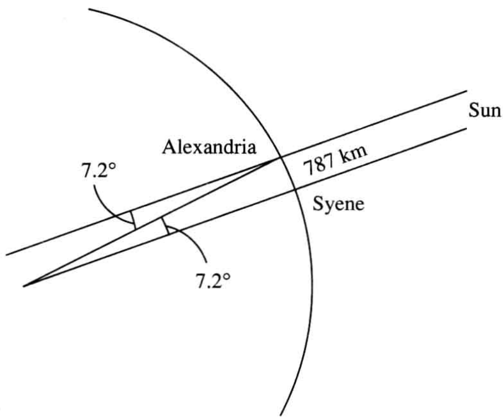

Figure 1.1

Interestingly enough, the earth measurement aspect of geometry has been of recent interest because satellites, instruments placed on the moon, and the U.S. Coast and Geodetic Survey (in producing nautical and aeronautical charts) have been able to provide very precise measurements of the earth.

Today, the science of geodesy includes making precise measurements of the earth's size and shape, as well as determining precise locations of points on the earth's surface. These measurements are used by cartographers in the production of maps. The earth is considered to be a geoidal surface whose composition is not uniform and which has lumps, or bulges, rather than being an exact sphere or even an exact oblate spheroid (with slightly different major and minor axes). Figure 1.2 illustrates two common map projections.

The first is the Mercator projection and the second is the Conic projection. A third type of map is discussed in Chapter 6. The concept of projection appears in Chapter 7.

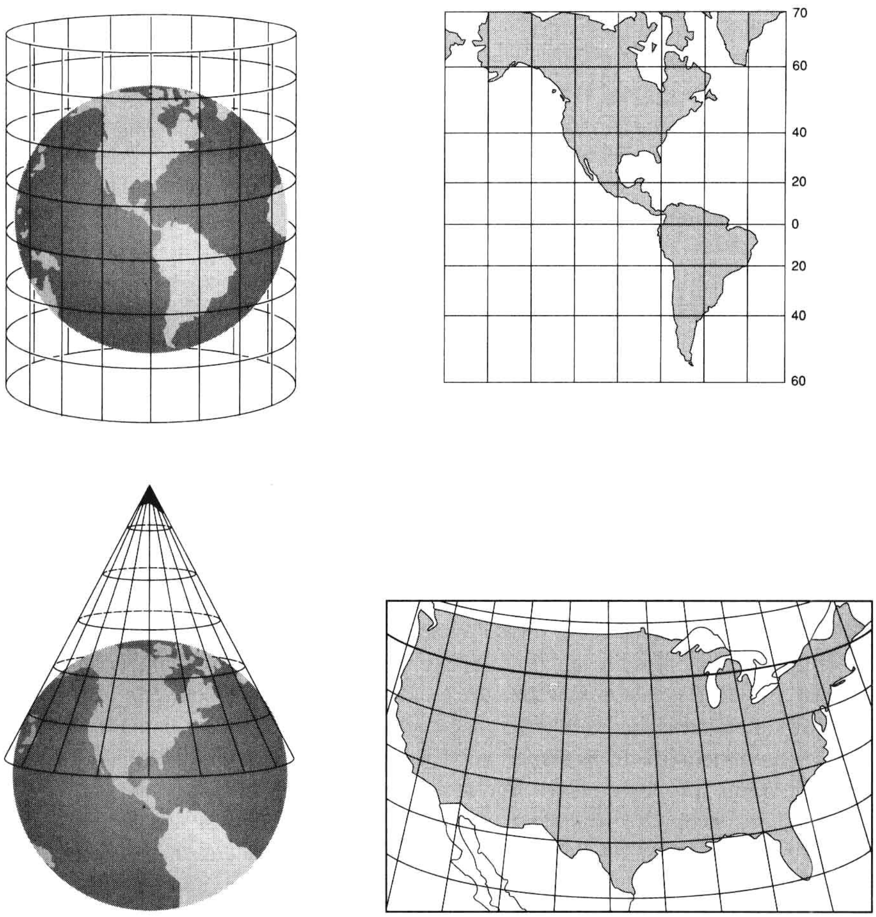

Figure 1.2

To many people today, however, the word *geometry* no longer suggests earth measure: mathematicians might describe the geometries included in this text as the study of the properties and relationships of sets of points. To the typical adult, though, the word *geometry* probably suggests the high school course they took in plane geometry or possibly the use of coordinates for points encountered in the study of algebra or calculus. The ancient Greeks of the period 500 B.C. to A.D. 100 receive much of the credit for developing the demonstrative geometry studied at the high school level. They recognized the beauty of geometry as a discipline with a structure and understood that the proof of a theorem could be even more exciting than the discovery of a practical application.

Greek geometry, called *Euclidean geometry* because of the monumental work of Euclid (300 B.C.), includes *undefined terms, defined terms, axioms or postulates*, and *theorems*. Other geometries studied in this text have the same sort of structure, so sets of axioms are one convenient means of classifying a geometry. The geometries in Chapters 1, 3, 4, 5, and 9 are all approached from the axiomatic viewpoint. In Euclidean geometry, undefined terms, which are arbitrary and could easily be replaced by other terms, normally include terms such as *points*, *lines*, and *planes*; it would also be possible to develop Euclidean geometry using such concepts as *distance* and *angle* and keeping them undefined. Definitions of new words involve using the undefined terms.

Today, the words *axiom* and *postulate* are used interchangeably. In the development of geometry, however, the word *postulate* was used for an assumption confined to one particular subject (such as geometry), while *axiom* denoted a "universal truth," a more general assumption that applied to all of mathematics. The axioms and postulates of Euclid are stated in Section 1.7. The *truth* of axioms or postulates is not at issue. These statements are beginning assumptions from which logical consequences follow. They are analogous to the rules for a game. Since the mathematical system to be developed depends on the axioms, changing the axioms can greatly change the system, just as changing the rules for a game would change the game.

Theorems are statements to be proved by using the axioms, definitions, and previous theorems as reasons for the logical steps in the proof. The theorems of geometry are valid conclusions based on the axioms. A simple theorem typically is stated in the form of an if-then statement such as, "If the sum of the measures of the opposite angles of a quadrilateral is 180 (in degrees), then the quadrilateral can be inscribed in a circle." In logic, this statement is a conditional. Selected ideas from logic are listed in Appendix 4, and selected theorems from plane Euclidean geometry are listed in Appendix 5. The first 28 theorems (propositions) of Euclid are listed in Appendix 6.

The Greek geometry was *synthetic*, which means it did not use coordinates for numbers as in analytic geometry. Really significant advances over the synthetic geometry of the Greeks were made only with the invention of analytic geometry (about 1637) and its subsequent use as a tool in modern analysis. While analytic geometry is not the dominant theme of this text, coordinates of points are used as an alternative to the synthetic approach when convenient.

The title, *Modern Geometries*, with emphasis on the plural, should prepare you for finding out much more about the existence of not just one, but an infinitude, of geometries. As the word *modern* in the title implies, the major emphasis is on newer geometries—those that have been developed since 1800. The emergence of modern algebra, with its theory of groups, and the introduction of axiomatics into algebra paved the way for Felix Klein's classification of geometries in 1872. The basic concept of transformations needed to understand this classification is discussed in Chapter 2. Many geometries can be explained using the basic idea of transformations. In each case, mathematicians are interested in properties that remain the same when sets of points are changed in some way. The geometries in Chapters 2, 6, 7, and 8 are all classified by means of transformations.

The latter part of the nineteenth century witnessed a revival of interest in the classical geometry of the circle and the triangle, with the result that the Greek geometry was extended by many significant additions (Chapters 4 and 5). Projective geometry (Chapter 7) was invented about 1822; material on non-Euclidean geometry (Chapter 9) was in print by about 1830. Inversive geometry (Chapter 6) was developed about the same time.

During the twentieth century, studies in the axiomatic foundations of geometry and the finite geometries (Chapter 1), the geometry of convexity (Chapter 3), and geometric topology (Chapter 8) have all been added to the great body of geometry that is relatively independent of analysis. A recent development, largely since 1975, has been the application of many ideas from geometry in the new applied area of computer graphics. Other recent developments include insertion of material on fractals, tessellations, and geometric probability in many geometry books.

Even this brief sketch of some of the major steps in the history of geometry should convince you that a discussion of modern geometries must deal with many different kinds of geometry. This dominant theme should be remembered each time a new geometry is encountered. It is an acknowledgment of the diversity of mathematical systems titled "geometry" that distinguishes a book on modern geometries from a traditional college geometry text of a quarter century ago, which concentrated only on a restudy and direct extension of the Euclidean geometry of the high school. Still other geometries might have been included. For example, in the early nineteenth century, Gauss, Riemann, and others applied calculus and analysis to geometry, spawning a specialty known as differential geometry. This approach was very fruitful, and led to results that have been crucial to the development of modern physics and cosmology.

Computer graphics is the use of a computer to produce pictorial output. Aspects of this production include defining the data points, manipulating the figure, and presenting it. The use of animation and the depiction of fractals are extensions of these ideas. Computer graphics solves the problem of creating a particular picture or pictures on the computer screen. Computer graphics may be studied using a small personal computer with graphics capability, but there are also complex and expensive computers especially designed for computer graphics use. Applications of computer graphics can enrich regular geometry texts and courses so that students see this important application at an earlier age.

The concepts of geometry such as points, lines, and slope are modified when used in computer graphics. For points, the use of integral coordinates on the computer screen means pictures composed of *pixels*, the smallest location on the screen that can be addressed. The computer graphics screen is a grid of pixels, though they often are called *dots* or *points* in computer graphics. A graphics picture is created by selecting and energizing some pattern of pixels.

The orientation and scale of axes are concepts that may be used differently in computer graphics than in high school geometry. For example, the positive y-axis may be directed downward on some computer screens, so that the origin is in the upper left hand corner. Because the horizontal–to–vertical ratio of pixels may be about 2 to 1 instead of 1 to 1, the effect on graphing is to change the looks of figures. For example, a line with slope of 1 and with positive y-axis pointed downward may look like Figure 1.3a. Circles may look like ellipses, and figures that are reflected about an axis may not look congruent. Although it is possible to make changes in the programs to compensate for this "distortion" on the graphics screen, geometry students should expect it and should learn to accept various representations as correct. One advantage of this application is to convince students of geometry that trying to prove something from a picture alone is dangerous.

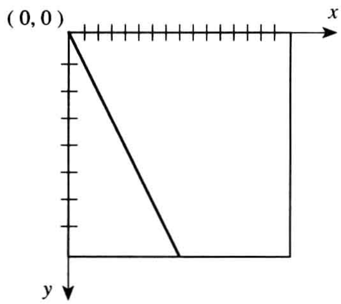

Figure 1.3a

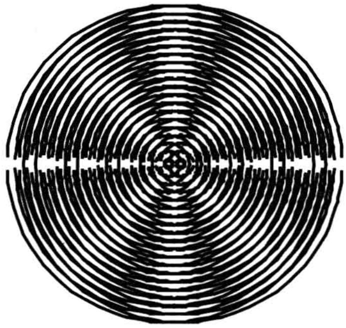

Figure 1.3b

A specific application of computer graphics that takes advantage of the "ragged" appearance of lines and some curves in earlier inexpensive computers is the Moiré pattern, an example of which is shown in Figure 1.3b. These graphic designs simulate art produced by slight variations in position of a basic figure.

▲ EXERCISES 1.1 ▲

(Answers to selected exercises are given at the back of the text.)

1. Verify that the Pythagorean theorem $a^2 + b^2 = c^2$, holds for the primitive triple 3, 4, 5.
2. Find the perimeter of a right triangle with shorter sides measuring 5 and 12 units.
3. Find the perimeter of a right triangle with shorter sides measuring 9 and 12 units.
4. Why could a right triangle with integers for lengths of sides not have shorter sides measuring 5 and 10 units?
5. Use the measurements of Eratosthenes to find the approximate diameter of the earth.

6. Use the measurements of Eratosthenes to find the approximate difference in angle of elevation of the sun at two places 1000 km apart in a north-south direction, both north of the equator.

For Exercises 7 and 8, assume that the method of Eratosthenes could be used for other planets.

7. If the distance were 300 miles and the angle difference 6°, what would be the circumference of the planet?

8. If the distance were 400 miles and the angle difference 8°, what would be the circumference of the planet?

For Exercises 9–14, answer true or false; then explain what is wrong with each false statement.

9. High school geometry owes more to the ancient Egyptians than to the ancient Greeks.

10. Euclid used the word *postulate* for an assumption confined to one particular subject.

11. Definitions must use only words that have previously been defined, not undefined terms.

12. Analytic geometry was invented before the development of finite geometries.

13. The latter part of the nineteenth century witnessed a revival of interest in the classical geometry of the circle and the triangle.

14. Traditional college geometry of a quarter of a century ago included the study of more different geometries than are included today.

Throughout the text, exercises preceeded by the symbol ▷ are investigative-type projects that may require checking references, doing research, making generalizations, and other supplementary activities to extend the coverage of the section. Some investigative-type exercises are designated by ▶▷ to show they are particularly appropriate for use with cooperative or small-group learning situations.

▷ **15.** How could instruments on the moon be used to make precise measurements of the earth? Try to discover an answer before consulting a reference.

▷ **16.** Read in a history of mathematics text about the geometry of ancient Babylonia and Egypt; then work several geometric problems found in the papyruses.

▶▷ **17.** Read more about the subject of geodesy and discuss it as an application of geometry.

1.2 Introduction to Finite Geometries

The Euclidean plane has an infinitude of points and lines in it, and a rich collection of theorems that continues to increase over the years. By contrast, "miniature" geometries have just a few axioms and theorems and a finite number of elements. These geometries are *finite geometries*, and the purpose of including them here is that they provide excellent opportunities for the study of geometries with a simple structure.

All the geometries studied in this text have a finite number of axioms and a finite number of undefined terms. Thus, those features do not make a geometry finite. Instead, a finite geometry has a finite number of elements—that is, points or lines or "things to work with." For the geometries studied in this chapter, these elements can be considered points and lines.

Historically, the first finite geometry was a three-dimensional geometry, each plane of which contained seven points and seven lines. The modernity of finite geometries is emphasized by the fact that Gino Fano explored this first finite geometry in 1892, although some ideas can be traced back to von Staudt (1856). It was not until 1906 that finite projective geometries were studied by Veblen and Bussey. Since that time, a great many finite geometries have been or are being studied. Many sets of points and lines that were already familiar figures in Euclidean geometry were investigated from this new point of view. Several of the finite geometries are an integral part of projective geometry, and a knowledge of the finite geometries of Chapter 1 will help in the study of some of the basic set of points and lines used in Chapter 7. However, at the present time it is quite possible for a mathematics major to graduate without ever encountering finite geometries, although it is also true that finite geometries are being used increasingly as enrichment topics and extension units at the high school level. Finite geometries also find a practical application in statistics.

All the finite geometries in this chapter have *point* and *line* as undefined terms. The connotation of *line* is not the same in finite geometry as in ordinary Euclidean geometry, however, since a line in finite geometry is assumed to have more than one, but only a finite number, of points.

The first simple finite geometry to be investigated, called a *three-point geometry* here for identification, has only four axioms:

Axioms for Three-Point Geometry

1. There exist exactly three distinct points in the geometry.
2. Two distinct points are on exactly one line.
3. Not all the points of the geometry are on the same line.
4. Two distinct lines are on at least one point.

Assume that the words *point*, *line*, and *on* are undefined terms. In Axiom 4, the two lines with a point in common are called *intersecting lines*.

Try to discover the answers to these questions before reading further:

- What kinds of figures or models could be drawn to represent the geometry?