Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

Subseries: Mathematisches Institut der Universität und
Max-Planck-Institut für Mathematik, Bonn – vol. 9

Adviser: F. Hirzebruch

1244

Werner Müller

Manifolds with Cusps of Rank One

Spectral Theory and L^2-Index Theorem

Springer-Verlag

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

Subseries: Mathematisches Institut der Universität und
Max-Planck-Institut für Mathematik, Bonn – vol. 9
Adviser: F. Hirzebruch

1244

Werner Müller

Manifolds with Cusps of Rank One

Spectral Theory and L^2-Index Theorem

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo

Author

Werner Müller
Akademie der Wissenschaften der DDR
Karl-Weierstraß-Institut für Mathematik
Mohrenstraße 39, DDR – 1086 Berlin, German Democratic Republik

Mathematics Subject Classification (1980): 58G10, 58G11, 58G25

ISBN 3-540-17696-9 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-17696-9 Springer-Verlag New York Berlin Heidelberg

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

INTRODUCTION

Let G be a connected real semisimple Lie group of noncompact type, K a maximal compact subgroup of G and G/K the associated globally symmetric space. Consider a discrete torsion-free subgroup Γ of G with finite covolume and let $\Gamma\backslash G/K$ be the corresponding locally symmetric space. Let V,W be finite-dimensional unitary K-modules and denote by $\tilde{E}$, $\tilde{F}$ the induced homogeneous vector bundles over G/K. $\tilde{E}$ and $\tilde{F}$ can be pushed down to locally homogeneous vector bundles $E = \Gamma\backslash\tilde{E}$ and $F = \Gamma\backslash\tilde{F}$ over $\Gamma\backslash G/K$. Let

$$\tilde{D}: C^{\infty}(G/K,\tilde{E}) \longrightarrow C^{\infty}(G/K,\tilde{F})$$

be an invariant elliptic differential operator. Then $\tilde{D}$ induces an elliptic differential operator

$$D: C^{\infty}(\Gamma\backslash G/K,E) \longrightarrow C^{\infty}(\Gamma\backslash G/K,F) .$$

It is proved in [61] that D has a well-defined L^2-index which, as in the compact case, depends only on $\mathrm{ch}\,V - \mathrm{ch}\,W$. Using Selberg's trace formula, Barbasch and Moscovici [15] derived an explicit formula for the L^2-index of D if the locally symmetric space $\Gamma\backslash G/K$ has strictly negative curvature or, equivalently, if the real rank of G equals one. It seems to be very interesting to have an explicit formula for the L^2-index in the general case.

In this book we shall investigate the case of a locally symmetric space of $\mathbb{Q}$-rank one. Actually, we shall work with a larger class of manifolds. Each of these manifolds is locally symmetric near infinity with ends generalizing the case of a cusp of a $\mathbb{Q}$-rank one locally symmetric space. This is motivated by our approach to the proof of a conjecture of Hirzebruch (c.f. [63,§6]). In [63] we investigated the signature operator on Hilbert modular varieties $X = \Gamma\backslash H^n$. Here $\Gamma = SL(2,\mathcal{O}_F)$ is the Hilbert modular group of a totally real number field F of degree n. The L^2-index of the signature operator was computed with the help of the Selberg trace formula. The contribution of the cusps in the index formula was given by special values of certain L-series associated to the number field F. These special values of the

L-series occure in the formula conjectured by Hirzebruch relating signature defects of cusps of Hilbert modular varieties and special values of L-series. We proved that for a Hilbert modular variety with a single cusp Hirzebruch's conjecture is a consequence of our index formula. This suggests that an explicit formula for the L^2-index of the signature operator on manifolds investigated in this book will have other interesting applications of this type.

A proof of Hirzebruch's conjecture was given by Atiyah, Donnelly and Singer in [6] . This proof is based on the former work of Atiyah Patodi and Singer [7] on spectral asymmetry.

In the present book we shall give another proof of Hirzebruch's conjecture along the lines briefly sketched in [63,§6] . Actually this will turn out to be one application of the index formula we shall establish in this book.

As indicated above, in this book we are dealing with manifolds which are locally symmetric near infinity with ends generalizing the case of $\mathbb{Q}$-rank one cusps. On manifolds of this type we shall investigate a class of first-order elliptic differential operators which we call **generalized chiral Dirac operators**. One of our purposes is to establish a formula for the L^2-index of these operators. This covers the case of twisted Dirac operators on $\mathbb{Q}$-rank one locally symmetric spaces. It is known that this is sufficient to compute the L^2-index of any locally invariant elliptic differential operator on a $\mathbb{Q}$-rank one locally symmetric space (c.f. [15,p.196]). The main contribution of the cusps in this index formula is again given by a special value of a certain L-series associated to the locally symmetric structure of the ends of the manifolds. This generalizes the L-series arising in the Hilbert modular case.

We shall now give a more detailed description of the content of this book. In §1 we have collected some auxiliary results from the theory of linear operators in Hilbert space. We recall here some results of the Krein-Birman theory of the spectral shift function and also some facts concerning supersymmetric scattering theory [82] . In §2 we introduce the cusps we shall consider in this book. Each cusp is a locally symmetric space $Y = \Gamma\backslash G/K$ of infinite volume. Y is diffeomorphic to a cylinder $\mathbb{R}^+ \times \Gamma\backslash Z$ where Z is a certain homogeneous space and $\Gamma\backslash Z$ is compact. Moreover, $\Gamma\backslash Z$ is a fibration over a compact locally symmetric space $\Gamma_M\backslash X_M$ with fibre a compact nilmanifold. For each $b \geq 0$, we denote by Y_b the submanifold of Y which corresponds to $[b,\infty) \times \Gamma\backslash Z$. Each submanifold Y_b, $b \geq 0$, of Y with the induced Riemannian metric will be called a **cusp of rank one**. In §3 we study the fundamental solution of the heat equation for certain locally invariant differential operators on the cusp Y. For the same kind of locally invariant differential operators on Y we investigate in §4 the Neumann problem on the submanifolds Y_b, $b > 0$. These results are basic for §§5 and 6. In §5 we consider manifolds with cusps of rank one. Such a manifold is a complete Riemannian manifold X which is the union of a compact manifold with boundary and a finite number of cusps of rank one. For simplicity we shall assume throughout this book that X has a single cusp. Thus $X = X_0 \cup Y_1$ where X_0 is a compact manifold with boundary, Y_1 is a cusp of rank one and $X_0 \cap Y_1 = \partial X_0 = \partial Y_1$. The extension of our results to manifolds with several cusps requires nothing which is essentially new. On X we shall consider differential operators D which are locally invariant at infinity, i.e., there exists a locally invariant differential operator on the cusp Y whose restriction to Y_1 coincides with the restriction of D to Y_1. To be able to apply harmonic analysis at infinity we consider a restricted class of differential operators which we call **generalized Dirac operators**. Let E be a complex vector bundle over X whose restriction to Y_1 coincides with the restriction to Y_1 of a certain locally homogeneous vector bundle over Y. A **generalized Dirac operator** is a first-order formally selfadjoint elliptic differential operator D on $C^\infty(X,E)$ which is locally invariant at infinity and such that D^2 coincides, up to a zero-order operator, with the operator induced by the Casimir operator of G on $C^\infty(Y_1,E|Y_1)$. Geometrically interesting operators are of this form. In §5 we shall prove that D^2 acting in $L^2(X,E)$ with domain $C_c^\infty(X,E)$ is essentially selfadjoint. Let H be the unique selfadjoint extension of D^2. In §6 we shall investigate the spectral resolution of H. For this purpose we introduce an auxiliary operator H_0 whose continuous spectrum can be explicitly described such that $(H + I)^{-1}$ is a compact perturbation of $(H_0 + I)^{-1}$. H_0 is obtained from H by imposing Neumann boundary conditions on the hypersurface ∂Y_2, $Y_2 \subset Y$ as above. Employing the results of §4 we determine the continuous spectrum of H_0. Then we prove that the wave operators $W_\pm(H,H_0)$ exist and are complete which implies that the absolutely continuous parts of H and H_0 are unitarily equivalent. To establish the existence and completeness of the wave operators we employ the method introduced by Enss [31] in quantum scattering theory. Actually we shall apply an abstract version of this method introduced by Amrein, Pearson and Wollenberg [2], [16]. This method gives even more - the absence of the singularly continuous spectrum of H. This is Theorem 6.17. Then we continue with the study of the eigenvalues of H. We employ the method of Donnelly [28], which he used in the case of $\mathbb{Q}$-rank one locally symmetric spaces. There is no problem to extend this method to our case. The result is that the number of eigenvalues of H which are less than λ, $\lambda > 0$, is bounded by a constant multiple of λ^d for a certain $d \in \mathbb{N}$. Let H_d be the restriction of H to the subspace spanned by the eigenfunctions of H. Then our estimate on the growth of the number of eigenvalues implies that, for each $t > 0$, $\exp(-tH_d)$ is of the trace class. Another consequence is that $\ker D \cap L^2$ is finite-dimensional. Therefore each generalized chiral Dirac operator $D\colon C^\infty(X,E^+) \longrightarrow C^\infty(X,E^-)$ has a finite L^2-index, denoted $L^2\text{-Ind}\, D$. Let H^+ (resp. H^-) be the unique selfadjoint extensions of D^*D (resp. DD^*). Then the results of §6 imply

$$L^2\text{-Ind}\, D = \operatorname{Tr}(\exp(-tH_d^+)) - \operatorname{Tr}(\exp(-tH_d^-)) \ . \qquad (0.1)$$

In §7 we construct the kernel $e(z,z',t)$ of the heat operator $\exp(-tH)$. For this purpose we employ a variant of the usual parametrix method as in [62]. To construct the parametrix at infinity we apply the results of §3. In §8 we construct a system of generalized eigenfunctions for the operator H. In the locally symmetric case such a system of generalized eigenfunctions is given by the **Eisenstein series**. We call the generalized eigenfunctions in the general case **Eisenstein functions**. The proof of the existence of an analytic continuation of the Eisenstein functions is due to L.Guillopé [38]. Using the Eisenstein functions we get an explicit description of the wave operators $W_\pm(H,H_0)$. Together with the results of §6 we recover in this way all facts known about the spectral resolution of the Casimir operator acting on sections of a locally homogeneous vector bundle over a $\mathbb{Q}$-rank one locally symmetric space. In §9 we investigate the spectral shift function $\xi(\lambda;H,H_0)$ associated to H and H_0. The main result is Theorem 9.25 which gives an explicit expression for the spectral shift function $\xi^c(\lambda;H,H_0)$ associated to the absolutely continuous parts H_{ac} and $H_{0,ac}$ of H and H_0, respectively. These results are used in §10 to derive a preliminary version of our index formula for a generalized chiral Dirac operator

$$D\colon C^\infty(X,E^+) \longrightarrow C^\infty(X,E^-) \ .$$

We work within the supersymmetric framework developed in [82]. Thus we regard $\begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$ as an operator in $L^2(X,E^+) \oplus L^2(X,E^-)$ with domain given by $C_c^\infty(X,E^+) \oplus C_c^\infty(X,E^-)$ and we denote by Q its unique selfadjoint extension. Set $H = Q^2$. Then we introduce a free Hamiltonian $\tilde{H}_o$ which is a modification of the free Hamiltonian H_o considered in §6. To define it recall that the boundary of the cusp Y_1 is a fibre bundle over a compact locally symmetric space $\Gamma_M\backslash X_M$. If we restrict D to sections of $E^+|Y_1$ which are constant along the fibres then we get an operator

$$\mathcal{D}_o : C^\infty(\mathbb{R}^+ \times \Gamma_M\backslash X_M, E_M^+) \longrightarrow C^\infty(\mathbb{R}^+ \times \Gamma_M\backslash X_M, E_M^-)$$

which is of the form

$$\mathcal{D}_o = \eta(\frac{\partial}{\partial u} + \tilde{D}_M) \quad .$$

Here η is a bundle isomorphism and $\tilde{D}_M$ is a selfadjoint operator in $C^\infty(\Gamma_M\backslash X_M, E_M^+)$. Let Q_o^+ be the closure in L^2 of $\partial/\partial u + \tilde{D}_M$ with respect to the non local boundary conditions of [7] and let Q_o^- be its Hilbert space adjoint. This is the closure in L^2 of $-\partial/\partial u + \tilde{D}_M$ with respect to the adjoint boundary conditions. Then $Q_o = \begin{pmatrix} 0 & Q_o^- \\ Q_o^+ & 0 \end{pmatrix}$ is a selfadjoint operator in $L^2(\mathbb{R}^+ \times \Gamma_M\backslash X_M, E_M^+ \oplus E_M^+)$ and our free Hamiltonian is given by $\tilde{H}_o = Q_o^2$. Then (Q,Q_o) define a supersymmetric scattering theory in $L^2(X,E^+ \oplus E^-)$ (c.f. Ch.I). Therefore, we may split the spectral shift function $\xi^c(\lambda;H,H_o)$ into a bosonic part $\xi_+^c(\lambda;H,H_o)$ and a fermionic part $\xi_-^c(\lambda;H,H_o)$. Further, let $e_+(z,z',t)$ (resp. $e_-(z,z',t)$) be the bosonic (resp. fermionic) part of the heat kernel $e(z,z',t)$ for the Hamiltonian H. Then our first result is

$$\begin{aligned} L^2\text{-Ind}\, D = & \int_X \{\operatorname{tr} e_+(z,z,t) - \operatorname{tr} e_-(z,z,t)\}dz \; + \\ & + \sum_\omega \frac{\text{sign}}{2} \operatorname{erfc}(|\omega|\sqrt{t}) \; + \\ & + t\int_0^\infty (\xi_+^c(\lambda;H,H_o) - \xi_-^c(\lambda;H,H_o))e^{-t\lambda}d\lambda \quad . \end{aligned} \tag{0.2}$$

Here ω runs over the eigenvalues of $\tilde{D}_M$ and erfc is the complementary error function. Using Theorem 9.25 and results of [82], it follows that $\xi_+^c(\lambda;H,H_o) - \xi_-^c(\lambda;H,H_o)$ is constant on $\mathbb{R}^+$. This constant is zero if the lower bound of the essential spectrum of H is positive. According to [7], the second term on the right hand side of (0.2) has an asymptotic expansion (as $t \longrightarrow 0$) whose constant term is $\frac{1}{2}(\eta(0)+h)$ where $\eta(0)$ is the Eta invariant of $\tilde{D}_M$ and

h = dim ker $\tilde{D}_M$. Let $\alpha^{\pm}(z)$ be the constant term in the asymptotic expansion of $\operatorname{tr} e_{\pm}(z,z,t)$ as $t \to 0$. Then the local index theorem [5], [34], leads to our first version of an index formula

$$L^2\text{-Ind}\, D = \int_X (\alpha^+(z) - \alpha^-(z))dz + U + \frac{1}{2}\eta(0) - \frac{1}{2}(h_\infty^+ - h_\infty^-) \ . \qquad (0.3)$$

The term U is determined by the asymptotic expansion of the heat kernel at infinity. We call U the **unipotent contribution** to the L^2-index of D. $h_\infty^{\pm}$ are the dimensions of spaces of extended L^2-solutions of D and D*, respectively, with limiting values in $\ker \tilde{D}_M$. In particular, the last term vanishes if the lower bound of the essential spectrum of H is positive or, equivalently, if the continuous extension $\bar{D}: H^1(X,E^+) \longrightarrow L^2(X,E^-)$ of D is a Fredholm operator. Of course the index formula (0.3) is not of much use unless the unipotent contribution U has been made more explicit. We deal with this problem in §11. In the case of Hilbert modular varieties we proved in [63] that the unipotent contribution to the L^2-index of the signature operator is given by the value at s=1 of a certain L-series associated to the cusp (c.f. [63,(5.57)]). We shall show that in the general case one has in principle the same picture. In the first part of §11 we reduce the computation of the unipotent contribution to the study of the asymptotic expansion (as $t \longrightarrow 0$) of the integral (11.39). Using ideas of a forthcoming paper of W.Hoffmann [51] the integral (11.39) is then converted into a finite sum of unipotent orbital integrals and a certain non-invariant integral (c.f. (11.56)). In this book we shall not investigate the unipotent orbital integrals and the non-invariant integral occurring in (11.56) in general. This requires the knowledge of a Fourier inversion formula for the corresponding distributions on G. A Fourier inversion formula for unipotent orbital integrals for groups of real rank one was established by Barbasch [14]. If $G = G_1 \times \cdots \times G_r$ where each G_i is a connected semisimple Lie group of real rank one then one can employ the results of Barbasch to compute the unipotent orbital integrals in our case. Under the same assumption one can also deal with the non-invariant integral in (11.56). This leads to our final index formula for the case where $G = G_1 \times \cdots \times G_r$ with G_i as above (c.f. Theorem 11.77). If $\operatorname{rank} G > \operatorname{rank} K$, then the unipotent contribution U vanishes. If $\operatorname{rank} G = \operatorname{rank} K$, then there is an L-series associated to the cusp and the differential operator D such that the unipotent contribution U to the L^2-index of D is given by the value at $s=0$ of this L-series. This generalizes our results concerning the L^2-index of the signature operator on Hilbert modular varieties obtained in [63]. There is no doubt that a similar result will be true without any restriction on G. Moreover, if the group Γ arises from an arithmetic situation then the L-series introduced by (11.67b) should have an arithmetic meaning.

In this book we shall discuss only one application of our index formula. This is the proof of the Hirzebruch conjecture which will be derived from our index formula in §12. It is clear that this method can be generalized to other locally symmetric spaces. This will lead to a generalization of Hirzebruch's conjecture which can be proved along the same lines.

I thank L.Guillopé for pointing out mistakes in the first draft of this book.

It is a pleasure for me to thank Mme. Breiner (I.H.E.S.) and Mme. Wüst (Berlin) who typed parts of the manuscript.

TABLE OF CONTENTS

CHAPTER I
PRELIMINARIES

For the convenience of the reader we shall collect here some auxiliary results from the theory of linear operators in Hilbert space.

Let $\mathcal{H}$ be a separable Hilbert space. The domain of any operator T in $\mathcal{H}$ will be denoted by $\mathbb{D}(T)$. Let T be a symmetric positive operator in $\mathcal{H}$ with dense domain $\mathbb{D}(T)$. On $\mathbb{D}(T)$ we define a new scalar product by

$$(f,g)_T = (f,g) + (Tf,g) \quad , \quad f,g \in \mathbb{D}(T) \quad .$$

Following Friedrichs [33] , introduce the subspace $\mathbb{D}[T] \subset \mathcal{H}$ by

$$\mathbb{D}[T] = \{ f \in \mathcal{H} \mid \exists \{f_n\}_{n \in \mathbb{N}} \subset \mathbb{D}(T) \text{ such that } \|f_n - f\| \to 0 \text{ as } n \to \infty \text{ and } \|f_n - f_m\|_T \longrightarrow 0 \text{ as } n,m \longrightarrow \infty \} .$$

The norm $\|\cdot\|_T$ can be extended to a norm on $\mathbb{D}[T]$. $\mathbb{D}[T]$ equipped with $\|\cdot\|_T$ is a Hilbert space and $\mathbb{D}(T)$ is a dense subspace. By $\tilde{T}_0$ we shall denote Friedrichs' extension of T [33]. $\tilde{T}_0$ is a positive selfadjoint operator in $\mathcal{H}$. Its domain is given by

$$\mathbb{D}(T_0) = \mathbb{D}[T] \cap \mathbb{D}(T^*)$$

and $\tilde{T}_0 = T^* \mid \mathbb{D}(\tilde{T}_0)$. $\tilde{T}_0$ is the unique positive selfadjoint extension of T which satisfies $\mathbb{D}(\tilde{T}_0) \subset \mathbb{D}[T]$. Moreover, one has $\mathbb{D}[\tilde{T}_0] = \mathbb{D}[T]$, [1, No 109,Theorem 2], [30,XII,§5] . Another way to define Friedrichs' extension is the following one. Consider the quadratic form

$$q(f) = (Tf,f) \quad , \quad f \in \mathbb{D}(T) \quad ,$$

and let $\bar{q}$ be its closure in $\mathcal{H}$. There exists a salfadjoint operator T' in $\mathcal{H}$ which represents the quadratic form $\bar{q}$. This operator coincides with Friedrichs' extension $\tilde{T}_0$ defined above [52,VI,§2] .

Now we recall briefly some results of the Krein-Birman theory of the spectral shift function [18], [86] . Let H, H_0 be bounded self-adjoint operators in the Hilbert space $\mathcal{H}$ and assume that $H - H_0$ is of the trace class. Then the function

$$\xi(\lambda) = \xi(\lambda;H,H_o) = \pi^{-1}\lim_{\varepsilon\downarrow 0} \arg\det[1 + (H - H_o)(H_o - \lambda - i\varepsilon)^{-1}]$$

exists for a.e. $\lambda\in\mathbb{R}$. This is the spectral shift function associated to H and H_o. It has the following properties:

i) $\xi\in L^1(\mathbb{R})$ with $\|\xi\|_{L^1} \leq \|H - H_o\|_1$ where $\|\cdot\|_1$ is the trace norm.

ii) $Tr(H - H_o) = \int_{-\infty}^{\infty} \xi(\lambda)d\lambda$.

iii) $\xi(\lambda) = 0$ outside the smallest interval containing the spectra of of H and H_o.

iv) Let $\Phi\in C_o^\infty(\mathbb{R})$. Then $\Phi(H) - \Phi(H_o)$ is of the trace class and

$$Tr(\Phi(H) - \Phi(H_o)) = \int_{-\infty}^{\infty} \Phi'(\lambda)\xi(\lambda)d\lambda .$$

Let $\mathcal{H}^{ac}$ and $\mathcal{H}_o^{ac}$ be the absolutely continuous subspaces of H and H_o , respectively, and denote by P^{ac} and P_o^{ac} the corresponding orthogonal projections of $\mathcal{H}$ onto $\mathcal{H}^{ac}$ and $\mathcal{H}_o^{ac}$, respectively. Since $H - H_o$ is of the trace class, it follows that the wave operators

$$W_\pm(H,H_o) = \operatorname{s-lim}_{t\to\pm\infty} e^{itH}e^{-itH_o}P_o^{ac}$$

exist and are complete, i.e., $W_\pm(H,H_o)$ is an isometry of $\mathcal{H}_o^{ac}$ onto $\mathcal{H}^{ac}$ which intertwines the absolutely continuous parts $H_{o,ac} = H_o|\mathcal{H}_o^{ac}$ and $H_{ac} = H|\mathcal{H}^{ac}$ of H_o and H, respectively (c.f. [52,X,§4]). Then the scattering operator S is defined by

$$S = W_+^*W_- .$$

S is a unitary operator on $\mathcal{H}_o^{ac}$ which commutes with $H_{o,ac}$. Let $dE_{ac}(\lambda)$ be the spectral measure of $H_{o,ac}$. Then there is a corresponding spectral decomposition for S:

$$S = \int S(\lambda)dE_{ac}(\lambda)$$

where $S(\lambda)$ is a bounded operator in the Hilbert space $\mathcal{H}(\lambda)=\frac{dE_{ac}(\lambda)}{d\lambda}\mathcal{H}$. $S(\lambda)$ is the scattering matrix of H,H_o. It is related to the spectral shift function $\xi(\lambda)$ by

$$\exp(-2\pi i\xi(\lambda)) = \det S(\lambda)$$

for a.e. $\lambda \in \sigma_{ac}(H_o)$ (c.f.[16,V,19.1.5]).

Finally, we shall discuss some results from [82] on supersymmetric scattering theory. Assume that τ is a unitary involution in the Hilbert space $\mathcal{H}$. The ±1 eigenspaces $\mathcal{H}_\pm$ of $\mathcal{H}$ are called the bosonic and fermionic subspaces, respectively. A selfadjoint operator Q in $\mathcal{H}$ is called a **supercharge** with respect to τ if

$$\tau Q = - Q\tau \quad \text{on} \quad \mathbb{D}(Q) .$$

The selfadjoint operator $H = Q^2 \geq 0$ is called the associated Hamiltonian. Any operator H of this form for some Q and τ is called a Hamiltonian with supersymmetry.

A supersymmetric scattering theory in a Hilbert space $\mathcal{H}$ with a unitary involution τ is given by a pair (Q,Q_o) with the following properties:

i) Q and Q_o are supercharges with respect to τ.

ii) Let $H = Q^2$ and $H_o = Q_o^{\,2}$ be the associated Hamiltonians. Then the wave operators

$$W_\pm(H,H_o) = \text{s-}\lim_{t \to \pm\infty} e^{itH} e^{-itH_o} P_o^{ac}$$

exist and are complete. Here P_o^{ac} denotes the orthogonal projection of $\mathcal{H}$ onto the absolutely continuous subspace $\mathcal{H}_{o,ac}$ of H_o.

iii) $W_\pm(H,H_o)$ are intertwining operators for Q and Q_o:

$$QW_\pm(H,H_o) = W_\pm(H,H_o)Q_o \quad \text{on} \quad \mathbb{D}(Q_o) \cap \mathcal{H}_{o,ac} .$$

A sufficient condition for the existence of a supersymmetric scattering theory is given by

<u>LEMMA 1.1</u>. Assume that Q and Q_o are supercharges in $\mathcal{H}$ with respect to τ and

$$Q\exp(-tQ^2) - Q_o\exp(-tQ_o^{\,2})$$

is of the trace class for all $t > 0$. Then Q and Q_o define a supersymmetric scattering theory in $\mathcal{H}$.

For the proof see [82].

Assume that (Q,Q_o) define a supersymmetric scattering theory with respect to τ. Let

$$Q_o | H_o^{ac} = \int Q_o(\lambda) dE_{ac}(\lambda)$$

be the spectral decomposition of $Q_o | H_o^{ac}$ with respect to the spectral measure $dE_{ac}(\lambda)$ of $H_{o,ac}$. Since H_o commutes with τ it follows that

$$\tau = \int \tau(\lambda) dE_{ac}(\lambda)$$

and the Hilbert space $H(\lambda) = \frac{dE_{ac}(\lambda)}{d\lambda} H$ admits a decomposition

$$H(\lambda) = H_+(\lambda) \oplus H_-(\lambda) \tag{1.2}$$

into the ± 1 eigenspaces of $\tau(\lambda)$. With respect to this decomposition we may write

$$Q_o(\lambda) = \begin{pmatrix} 0 & q_-(\lambda) \\ q_+(\lambda) & 0 \end{pmatrix} .$$

Now observe that S commutes with τ too, and, by iii), S commutes also with Q_o. This implies that with respect to the decomposition (1.2) we have

$$S(\lambda) = \begin{pmatrix} S_+(\lambda) & 0 \\ 0 & S_-(\lambda) \end{pmatrix}$$

and

$$q_+(\lambda) S_+(\lambda) = S_-(\lambda) q_+(\lambda) .$$

CHAPTER II
CUSPS OF RANK ONE

The manifolds we shall consider in this book are locally symmetric near infinity with cusp-like ends. In this chapter we shall describe the locally symmetric spaces which occur as ends of our manifolds and we establish the assumption and notation.

Let G be a connected noncompact real semisimple Lie group with finite center. G is admissible in the sense of [66, Ch.2]. Let K be a maximal compact subgroup of G and set $\tilde{Y} = G/K$. The Lie algebras of G and K are denoted by $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Let $B(.,.)$ be the Killing form of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition, Θ the Cartan involution of $\mathfrak{g}$ (or of G) with respect to K and $(.,.)_\Theta$ the scalar product on $\mathfrak{g}$ defined by $(X_1,X_2) = -B(X_1,\Theta(X_2))$, $X_1,X_2 \in \mathfrak{g}$

Now let (P,S) be a split parabolic subgroup of G with split component A. For all details concerning parabolic subgroups the reader is referred to [66,Ch.2], [78,I]. Among the split components of P there is a unique one which is Θ-stable. This is the special split component of (P,S). Throughout this book we shall assume that the split component of (P,S) is Θ-stable. Let U be the unipotent radical of P and let L be the centralizer of A in G. L is a Levi subgroup of P and $P = UL$. Since A is Θ-stable the same is true for L. As usual, introduce the associated admissible closed reductive subgroup M of G such that $L = UM$ with $M \cap A = \{1\}$. Then

$$P = UAM$$

is the Langlands decomposition of P with respect to the split component A and $S = UM$. Let $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{u}$ be the Lie algebras of M, A and U, respectively. Since $U\backslash S = M$, we get a canonical homomorphism

$$\pi_P|_M : S \longrightarrow M \qquad (2.1)$$

The rank of (P,S) is the dimension of the split component A. Throughout this paper we only shall consider split parabolic subgroups (P,S) of rank one.

<u>DEFINITION 2.2</u>. Let (P,S) be a split parabolic subgroup of G of rank one and let $\Gamma \subset S$ be a discrete uniform subgroup without torsion. The manifold $Y = \Gamma\backslash\tilde{Y}$ is called a cusp of rank one associated to (P,S) and Γ.

<u>EXAMPLE 2.3</u>. Let $G = (SL(2,\mathbb{R}))^n$ and $K = (SO(2))^n$. Then $G/K = H^n$, where H is the upper half-plane. Let F be a totally real number field of degree n over $\mathbb{Q}$ and let $\mathbf{M}$ be a complete $\mathbb{Z}$-module of F, i.e. $\mathbf{M}$ is an additive subgroup of F which is free abelian of rank n. Denote by $U_{\mathbf{M}}^{+}$ the subgroup of those units ε of $\mathcal{O}_F$ (the ring of integers of F) which are totally positive and satisfy $\varepsilon\mathbf{M} = \mathbf{M}$. $U_{\mathbf{M}}^{+}$ is free abelian of rank $n-1$ [50,p.200]. Let $\mathbf{V} \subset U_{\mathbf{M}}^{+}$ be a subgroup of finite index and set

$$\Gamma_{\mathbf{M},\mathbf{V}} = \left\{ \begin{pmatrix} \varepsilon^{1/2} & \varepsilon^{-1/2}\mu \\ 0 & \varepsilon^{-1/2} \end{pmatrix} \;\middle|\; \varepsilon \in \mathbf{V}\ ,\ \mu \in \mathbf{M} \right\}$$

$\Gamma_{\mathbf{M},\mathbf{V}}$ is a subgroup of $SL(2,F)$. Now observe that there are n different embeddings of F into the real numbers. These embeddings will be denoted by $x \in F \longmapsto x^{(j)} \in \mathbb{R}$, $j=1,\ldots,n$. Using the different embeddings of F into $\mathbb{R}$ we get a map $SL(2,F) \longrightarrow (SL(2,\mathbb{R}))^n$ by sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,F)$ to

$$\left\{ \begin{pmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)} \end{pmatrix}, \ldots, \begin{pmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{pmatrix} \right\} \in (SL(2,\mathbb{R}))^n .$$

This is obviously an embedding. In particular, $\Gamma_{\mathbf{M},\mathbf{V}}$ can be considered as a subgroup of $(SL(2,\mathbb{R}))^n$. This subgroup is discrete. Let B be the subgroup of $SL(2,\mathbb{R})$ consisting of upper triangular matrices and set $P = B^n$. P is a parabolic subgroup of G. The unipotent radical U of P is given by

$$U = \left\{ \left(\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix} \right) \;\middle|\; x_1,\ldots,x_n \in \mathbb{R} \right\} .$$

Let

$$M = \left\{ \left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \ldots, \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n^{-1} \end{pmatrix} \right) \;\middle|\; \lambda_i \in \mathbb{R}^{x}\ ,\ \lambda_1 \cdots \lambda_n = \pm 1 \right\}$$

and set $S = UM$. Then (P,S) is a split parabolic subgroup of G of rank one. The split component A is given by