Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Geometric Topology and Shape Theory

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Geometric Topology and Shape Theory

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FOREWORD

From September 29 to October 10, 1986, the University of Zagreb sponsored a Postgraduate School and Conference on Geometric Topology and Shape Theory at the Interuniversity Centre of Postgraduate studies, Dubrovnik, Yugoslavia. This was the third such school and conference held at the Centre. The two previous ones were held from January 12 to January 30, 1976 and from January 19 to January 30, 1981.

The meeting was devoted to the interaction of Geometric Topology and Shape Theory. In particular, the aim was to cover the following topics: decomposition theory, cell-like mappings and CE-equivalent compacta, infinite-dimensional spaces, approximate fibrations and shape fibrations, fibered shape, ANR's and LC^n-compacta, manifolds and generalized manifolds, embeddings of continua into manifolds, complement theorems in shape theory, shape-theoretic methods in group theory, exact homologies and strong shape theory.

The articles in the Proceedings appear in alphabetic order by author.

The addresses of all participants and authors of contributed papers are given at the end of the volume.

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AN ALTERNATIVE PROOF OF M. BROWN'S THEOREM ON INVERSE SEQUENCES OF NEAR HOMEOMORPHISMS

by Fredric D. Ancel

Abstract. Theorem 4 of [B] is an interesting and useful result about inverse sequences of near homeomorphisms. We present a short alternative proof of this theorem. We thank Bob Daverman for a suggestion which has led to a slicker exposition.

Let (X, ρ) and (Y, σ) be compact metric spaces. Let $\mathfrak{M}(X, Y)$ denote the space of maps from X to Y endowed with the compact-open topology. A complete metric $\tilde{\sigma}$ on $\mathfrak{M}(X, Y)$ is defined by $\tilde{\sigma}(f, g) = \sup\{\sigma(f(x), g(x) : x \in X\}$. A map from X to Y is a *near homeomorphism* if it belongs to the closure of the set of homeomorphisms in $\mathfrak{M}(X, Y)$. Let $\varepsilon > 0$. A map $f : X \to Y$ is an ε-map if ρ-diam$(f^{-1}(y)) < \varepsilon$ for every $y \in Y$. Let $\mathfrak{M}_\varepsilon(X, Y)$ denote the set of all ε-maps in $\mathfrak{M}(X, Y)$. We shall use the following two basic facts.

Lemma 1. *Let X, Y and Z be compact metric spaces. Then composition $(f, g) \mapsto g \circ f : \mathfrak{M}(X, Y) \times \mathfrak{M}(Y, Z) \to \mathfrak{M}(X, Z)$ is continuous.*

This is easily proved using the uniform continuity of maps from Y to Z. One immediate consequence is that the composition of near homeomorphisms is a near homeomorphism.

Lemma 2. *Let (X, ρ) and (Y, σ) be compact metric spaces. For each $\varepsilon > 0$, $\mathfrak{M}_\varepsilon(X, Y)$ is an open subset of $\mathfrak{M}(X, Y)$.*

Proof. Let $f \in \mathfrak{M}_\varepsilon(X, Y)$. Set $\delta = (1/2) \inf\{\sigma(f(x), f(z)) : x, z \in X \text{ and } \rho(x, z) \geq \varepsilon\}$. Then $\delta > 0$. Let $g \in \mathfrak{M}(X, Y)$ such that $\tilde{\sigma}(f, g) < \delta$. We assert that $g \in \mathfrak{M}_\varepsilon(X, Y)$. If $x, z \in X$ and $g(x) = g(z)$, then $\sigma(f(x), f(z)) \leq \sigma(f(x), g(x)) + \sigma(g(z), f(z)) < 2\delta$. This makes $\rho(x, z) < \varepsilon$, and proves our assertion. □

A Theorem of M. Brown ([B], Theorem 4). *Suppose $X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \cdots$ is an inverse sequence of compact metric spaces and near homeomorphisms. If X_∞ is its inverse limit, then each $f_{\infty,k} : X_\infty \to X_k$ is a near homeomorphism.*

Proof. Recall that $X_\infty = \{x \in \prod X_k : f_k(x_{k+1}) = x_k \text{ for each } k \geq 1\}$, and that each $f_{\infty,k} : X_\infty \to X_k$ is simply projection: $f_{\infty,k}(x) = x_k$ for $x \in X_\infty$. Let ρ_k be a metric on X_k such that ρ_k-diam$(X_k) < 1/k$. Then a metric ρ_∞ on X_∞ is defined by $\rho_\infty(x, z) = \sup\{\rho_k(x_k, z_k) : k \geq 1\}$ for

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This paper is in final form and no version of it will be submitted for publication elsewhere.
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$x, z \in X_\infty$. It follows that each $f_{\infty,k} : X_\infty$ is a $(1/k)$-map. Indeed, if $x, z \in X_\infty$ and $f_{\infty,k}(x) = f_{\infty,k}(z)$, then $x_j = z_j$ for $1 \le j \le k$; so $\rho_\infty(x, z) = \sup\{\rho_j(x_j, z_j) : j > k\} < 1/k$.

It suffices to prove that $f_{\infty,1} : X_\infty \to X_1$ is a near homeomorphism. To this end, let $\mathcal{F}$ denote the closure in $\mathfrak{M}(X_\infty, X_1)$ of the set of all maps of the form $h \circ f_{\infty,k}$ where $k \ge 1$ and $h : X_k \to X_1$ is a homeomorphism. Then $f_{\infty,1} \in \mathcal{F}$, and $\tilde{\rho}_1$ restricts to a complete metric on $\mathcal{F}$. We shall complete the proof by arguing that $\mathcal{F}$ has a dense subset consisting of homeomorphisms.

We remark that a map between compact metric spaces is onto, if it is a limit of onto maps. Hence, each of the near homeomorphisms $f_k : X_{k+1} \to X_k$ is onto. It follows that each $f_{\infty,k} : X_\infty \to X_k$ is onto. Consequently, each element of $\mathcal{F}$ is onto.

Let $\varepsilon > 0$. Set $\mathcal{F}(\varepsilon) = \mathcal{F} \cap \mathfrak{M}_\varepsilon(X, Y)$. Lemma 2 implies that $\mathcal{F}(\varepsilon)$ is an open subset of $\mathcal{F}$. We shall argue that $\mathcal{F}(\varepsilon)$ is a dense subset of $\mathcal{F}$. It suffices to show that if $k \ge 1$ and $h : X_k \to X_1$ is a homeomorphism then the δ-neighborhood of $h \circ f_{\infty,k}$ contains an element of $\mathcal{F}(\varepsilon)$ for every $\delta > 0$. Choose $j > k$ so that $1/j \le \varepsilon$. Notice that $h \circ f_{\infty,k} = h \circ f_k \circ \cdots \circ f_{j-1} \circ f_{\infty,j}$. We apply Lemma 1 twice: first to conclude that $f_k \circ \cdots f_{j-1}$ is a near homeomorphism; and second to see that we can choose a homeomorphism $g : X_j \to X_k$ so close to $f_k \circ \cdots \circ f_{j-1}$ that $\tilde{\rho}_1(h \circ f_{\infty,k}, h \circ g \circ f_{\infty,j}) < \delta$. Since $f_{\infty,j}$ is a $(1/j)$-map, then $h \circ g \circ f_{\infty,j} \in \mathcal{F}(\varepsilon)$.

The Baire Category Theorem implies that $\mathcal{H} = \cap\{\mathcal{F}(1/k) : k \ge 1\}$ is a dense subset of $\mathcal{F}$. Since each element of $\mathcal{H}$ is one-to-one, and each element of $\mathcal{F}$ is onto, then $\mathcal{H}$ is a dense set of homeomorphisms in $\mathcal{F}$. □

Reference

[B]. M. Brown, Some applications of an approximation theorem for inverse limits, *Proc. Amer. Math. Soc.* **11** (1960), 478–481.

STRONG HOMOLOGY THEORIES

Friedrich W. Bauer

0. Introduction:

Steenrod-Sitnikov homology theories are defined on the category of compact metrizable spaces by the *Milnor axioms*. There are very satisfactory characterization theorems for ordinary as well as for generalized Steenrod-Sitnikov homology theories (cf. [2],[9]).

In the meantime there appeared generalizations of Steenrod-Sitnikov homology theories for more general categories of topological spaces (cf. [8]). They are called *strong homology theories*, mostly because they allowed an extension over some strong shape categories. In the present paper an axiomatic characterization of a kind of generalized homology theory, called strong homology theory (rel. to some category of "good" spaces), is given. The axioms are the Milnor axioms with a continuity axiom (definition 1.3.) replacing the clusteraxiom (which is also some kind of weak continuity). In order to formulate this new axiom, we need the concept of a chain functor $\underline{C}_*$ *being related to a given homology theory* $\underline{h}_{*_o}$. (definition 5.3.)

This means that there exists an isomorphism between the derived homology theory $H_*(\underline{C}_*)$ and h_*. In [1] it is proved that each homology theory h_* admits a chain functor $\underline{C}_*$ which is related

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A detailed version of this paper will be submitted for publication elsewhere.

to h_*. Instead of requiring that h_* is continuous, one introduces the concept of a c-continuous chain functor $\underline{C}_*$. The essence of c-continuity for a homology theory lies in the statement that each $\underline{C}_*$, related to h_* on the subcategory $\underline{P}$ (mostly a category of ANRs or ANEs), allows an extension $\hat{\underline{C}}_*$ being related to h_* (on the given category of topological spaces where h_* is defined) which is c-continuous (as a chain functor). The fact that one has to go back to some kind of chain level, i.e. that one has to work with chains, cycles, boundaries instead of the original homology classes is not so surprising, recalling that this is a property of all "strong" constructions: Steenrod-Sitnikov homology in comparison to Cech homology, strong shape categories in comparison to ordinary shape and Boardman stabilization in comparison to the S-category.

In §1 we set up the axioms of a strong homology theory and deduce a uniqueness theorem (theorem 1.4.). In §2 we derive the existence of a strong homology theory (theorem 2.1.) from a construction which is performed in §3. We can only indicate the details for the case of a generalized homology theory. In order to accomplish this, ∞-categories and ∞-functors are needed and therefore briefly recorded in §6. In §4 we show, using the results of §3, that for compact metrizable spaces and $\underline{P}$ being the category of ANRs, strong and Steenrod-Sitnikov homology theories coincide (theorem 4.1.). So it turns out in particular, that the notion of c-continuity of a homology theory appears as the appropriate generalization of the clusteraxiom. In §5 we display the concept of a chain functor, referring to [1] for proofs.

1. Chain continuous homology theories:

Let $\underline{K} \subset \underline{Top}$ be a given full subcategory of the categories of topological spaces and $\underline{K}^2$ a category of pairs (X,A) in $\underline{K}$, $A \subset X$. In most cases one expects $\underline{K}^2$ to be the category of *all* pairs (X,A), $A \subset X \in \underline{K}$.

1.1. Definition: A family of funtors $h_* = \{h_n, \partial_n, n \in \mathbb{Z}\}$, with natural transformations $\partial_n: h_n(X,A) \to h_{n-1}(A)$ is called a *homology theory* on $\underline{K}$ (or on $\underline{K}^2$) whenever h_* is exact, homotopy invariant and satisfies a

strong excision axiom: Suppose (X,A), $(X/A,*) \in \underline{K}^2$, $A \subset X$, then the projection $p: (X,A) \to (X/A,*)$ induces an isomorphism $p_*: h_n(X,A) \xrightarrow{\approx} h_n(X/A,*)$.

The concept of a *strong homology theory* depends on a given full subcategory $\underline{P}^2 \subset \underline{Top}^2$. We do not assume that $\underline{P}^2$ is a subcategory of $\underline{K}^2$. However this can of course be easily achieved by introducing the full subcategory $\underline{K}'^2 \subset \underline{Top}^2$ which is determined by $\underline{K}^2$ and $\underline{P}^2$. In our main example $\underline{K}$ is the category of compact metrizable spaces $\underline{Com}$ and $\underline{P}^2$ the category of all ANR pairs having the homotopy type of a compact ANR pair. We cannot confine ourselves merely to compact ANRs because we must have that with each $P \in \underline{P}$ the function space P^I is also contained in $\underline{P}$. This motivates the following agreement (cf.[1o] Theorem 1):

In case of $\underline{P}^2$ we assume throughout that to each $(P,Q) \in \underline{P}^2$ there exists a homotopy equivalent $(P',Q') \in \underline{P}^2$ such that $(P'/Q',*) \in \underline{P}^2$.

Before we are giving the definition of a strong homology theory (rel.$\underline{P}^2$), let us consider the case $\underline{K} = \underline{Com}$ and recall

the clusteraxiom: Let $(X,x_o) = \overset{\infty}{\underset{i=1}{\mathrm{Cl}}}(X_i,x_{io})$ be the cluster (or strong wedge) of the based spaces $X_i = (X_i,x_{io})$, then the natural mapping

$$\pi\colon h_*(X,x_o) \to \prod_{i=1}^{\infty} h_*(X_i,x_{io})$$

is an isomorphism.

This is weaker than the requirement of *continuity*, stating that for each inverse system of pairs $(\underline{X,A}) = \{(X_\alpha, A_\alpha),\pi^\alpha_\beta\}$ in $\underline{K}^2$ on has

$$h_*(\lim(\underline{X,A})) \approx \lim h_*(X_\alpha,A_\alpha).$$

There are homology theories on Com satisfying the clusteraxiom (the so-called Steenrod-Sitnikov homology theories) which are not continuous. An ordinary homology theory satisfying the clusteraxiom is according to a famous result by J.Milnor [9], characterized by its coefficient group. There is a related result for generalized homology theories (being the subject of [2]).

So it is reasonable to be looking for a substitute of the clusteraxiom for categories $\underline{K}$ larger than Com. Here the clusteraxiom can be formulated but there is no hope that it gives rise to a characterization of homology theories.

Let to this end $\underline{C}_*\colon \underline{K} \to \underline{ch}$ be chain functor being related to h_* (cf. §5, definition 5.3.), $(X,A) \in \underline{K}^2$ and let $\underline{P}_{(X,A)}$ be the category of pairs in $\underline{P}^2$ over (X,A): The objects are continuous mappings $(g\colon (X,A) \to (P,Q))$, $(P,Q) \in \underline{P}^2$, and the morphisms are commutative triangles: $r\colon g_1 \to g_2$,
$g_i\colon (X,A) \to (P_i,Q_i)$, $r\colon (P_1,Q_1) \to (P_2,Q_2)$, $rg_1 = g_2$.
Let $\underline{c} = \{c_g\}$ be an assignment $g \in \underline{P}_{(X,A)} \mapsto c_g \in C_n(P,Q)$ satisfying

$$r_\# c_{g_1} = c_{g_2}$$

for any morphism $r\colon g_1 \to g_2$.

We call $\bar{C}_n(X,A) = \{\underline{c}\}$ the set of all such families.

1.2. Definition: A D-functor $\underline{C}_*: \underline{K} \to \underline{ch}$ is chain-continuous (c-continuous) rel. $\underline{P}^2$, whenever:

1) To each $\underline{c} \in \bar{C}_*(X,A)$, $(X,A) \in \underline{K}^2$ there exists a unique $c \in C_*(X,A)$ satisfying

$$g_{\#}c = c_g \ , \ g \in \underline{P}_{(X,A)}$$

2) If $c_g \in C^!_*(P,Q)$, for all $g \in \underline{P}_{(X,A)}$, then we have $c \in C^!_*(X,A)$.

1.3. Definition: A homology theory $h_* = \{h_n, \partial_n\}$ on $\underline{K}^2$ is called *chain-continuous* (c-continuous) or a *strong homology theory* rel. $\underline{P}^2$ whenever to each chain functor $\underline{C}_*$ related to $h_* | \underline{P}^2$ (i.e. there exists an isomorphism $\mu: h_* | \underline{P}^2 \approx H_*(\underline{C}_*)$ on $\underline{P}^2$), there exists a c-continuous chain functor $\hat{\underline{C}}_*: \underline{K} \to \underline{ch}$ being related to h_* (now on $\underline{K}^2$) and an inclusion $\nu: \underline{C}_* \subset \hat{\underline{C}}_* | \underline{P}$ of chain functors, inducing an isomorphism of homology theories on $\underline{P}^2$.

Remarks and examples:

One is tempted to require merely the *existence* of a s-continuous chain functor being related to h_*. However this is a condition which does not give the kind of homology which we expect. Suppose for example that $\underline{K} = \underline{Top}$ and $\underline{P}$ = category of compact polyhedra, then singular homology H_*(with coefficients in any abelian group G) has this property: Let $C_*: \underline{K} \to \underline{ch}$ be the functor assigning to each space X the singular chain complex $C_*(X)$ and define $C_*(X,A)$ in the classical way by forming the quotient $C_*(X,A) = C_*(X)/\text{im } i_{\#}$, $i: A \subset X$. Endow $\underline{C}_*$, $\underline{C}_*(X) = C_*(X,X)$ with the structure of a chain functor in a trivial way ($C^!_*(X,A) = C_*(X)$; ψ, φ are identities). Then $\underline{C}_*$ is easily seen to be c-continuous. This provides us also with an example of a chain functor related to singular homology.

Let on the other hand $\underline{h}_* = H_*(\ ;G)$ on $\underline{P}$ = finite polyhedra, be again simplicial homology with coefficients in G. There exists on $\underline{Com} = \underline{K}$ a D-functor ${}^s\underline{C}_*$ yielding Steenrod-Sitnikov homology theory ${}^sH_*(\ ;G)$ as related homology theory (cf.[3] for the construction of Sitnikov chains $c = \{c_i^n, x_i^{n+1}\}$ and for further references). Again ${}^s\underline{C}_*$ turns out to be c-continuous and ${}^sH_*(\ ;G) \approx H_*(\ ;G)$ on $\underline{P}^2$. But ${}^sH_*$ is certainly not isomorphic to singular homology on $\underline{Com}$.

On the other hand wer are able to deduce from definition 1.2. the following uniqueness tneorem:

<u>1.4. Theorem:</u> Let ${}^1h_*$, ${}^2h_*: \underline{K}^2 \to \underline{Ab}^2$ be two c-continuous rel. $\underline{P}^2$ homology theories on $\underline{K}$ being isomorphic on $\underline{P}^2$, then ${}^1h_*$ and ${}^2h_*$ are isomorphic as homology theories on $\underline{K}^2$.

Proof: Assume for the sake of simplicity that ${}^1h_*$ and ${}^2h_*$ are equal on $\underline{P}$ and call ${}^ih_*|\underline{P}^2 = h_*$.

Let $\underline{C}_*$ be a chain-functor related to h_*, then we find a c-continuous chain functor ${}^1\underline{C}_*$ related to ${}^1h_*$ together with an embedding

$$\nu_1: \underline{C}_* \subset {}^1\underline{C}_*|\underline{P},$$

inducing an isomorphism of homology theories on $\underline{P}^2$. Moreover there exists a c-continuous chain functor ${}^2\underline{C}_*$ being related to ${}^2h_*$ and an embedding

$$\nu_2: {}^1\underline{C}_*|\underline{P} \subset {}^2\underline{C}_*|P,$$

inducing an isomorphism of homology theories on $\underline{P}^2$. Now we define λ: ${}^1h_* \to {}^2h_*$ as follows:

Let $z \in \zeta \in {}^1h_n(X,A)$, $z \in Z_n({}^1C_*(X,A))$ be a cycle and $\underline{z} = \{\nu_2(g_\#(z)\} \in {}^2\bar{C}_*(X,A)$, $g \in \underline{P}_{(X,A)}$. Due to the c-continuity of ${}^2\underline{C}_*$ there exists a $z_2 \in Z_n({}^2C_*(X,A))$ such that $g_\#(z_2) = \nu_2(g_\#(z))$, allowing us to set

(1) $\lambda[z] = [z_2].$

In fact we can do a little more and use this procedure to define a transformation of chain functors (cf. definition 5.5.)

$$\lambda_\#: {}^1\underline{C}_* \to {}^2\underline{C}_*$$

inducing the above mentioned transformation (1) (cf. proposition 5.6.) of homology theories.

There exists an inverse ω: ${}^2h_* \to {}^2h_*$ to λ: Let ${}^3\underline{C}_*$ be a c-continuous chain functor related to ${}^1h_*$, ν_3: ${}^2\underline{C}_*|\underline{P} \subset {}^3\underline{C}_*|\underline{P}$ an embedding, inducing an isomorphism of homology theories on $\underline{P}^2$.Then ω is constructed in the same way as λ (after replacing $\underline{C}_*$ by ${}^2\underline{C}_*|\underline{P}$ and ${}^1\underline{C}_*$ by ${}^3\underline{C}_*$). Suppose $z_2 = \lambda_\#(z) \in Z_n({}^2\underline{C}_*(X,A))$ then $\omega_\#(z_2)$ is immediately recognized as z: We have

$$\{\nu_3 g_\#(z_2)\} = \{\nu_3\nu_2 g_\#(z)\} \in {}^3\bar{C}_*(X,A)$$

to which corresponds by c-continuity a $z_3 \in {}^3\underline{C}_n(X)$. However by uniqueness we conclude that $z = z_3$.

So we have

$$\omega\lambda = 1: {}^1h_* \to {}^1h_*.$$

In the same way we find a $\lambda': {}^1h_* \to {}^2h_*$ such that $\lambda'\omega = 1$, so that $\lambda = \lambda'$ follows. -

2. The existence theorem:

The central step in the proof of the existence of a strong homology theory h_* rel. $\underline{P}^2$, extending a given homology theory on the subcategory $\underline{P}^2$, is embodied in establishing an assignment $\underline{C}_* \mapsto {}^s\underline{C}_*$, assigning to each chain functor on $\underline{P}$ a c-continuous chain functor ${}^s\underline{C}_*$ on the larger category $\underline{K}'$ together with an embedding

(1) $\qquad \nu: \underline{C}_* \subset {}^s\underline{C}_* | \underline{P}$

inducing an isomorphism of homology theories on the subcategory $\underline{P}^2$. We require the following additional properties of the assignment $\underline{C}_* \mapsto {}^s\underline{C}_*$:

s1) The derived homology $H_*({}^s\underline{C}_*)(\quad)$ satisfies a strong excision axiom (hence $H_*({}^s\underline{C}_*)(\quad)$ is a homology theory on $\underline{K}^2$).

s2) Let ${}^1\underline{C}_*$, ${}^2\underline{C}_*$ be two chain functors on $\underline{P}$, $\gamma: H_*({}^1\underline{C}_*) \approx H_*({}^2\underline{C}_*)$ a natural isomorphism between the derived homology theories (on $\underline{P}^2$). Then there exists a natural isomorphism

$$ {}^s\gamma: H_*({}^{s1}\underline{C}_*) \approx H_*({}^{s2}\underline{C}_*). $$

We will deal with the problem of establishing such an assignment in §3. In order to be able to accomplish this, we have to impose the following restrictive conditions upon the relationship between $\underline{P}$ and $\underline{K}$:

p1) Let $P \in \underline{P}$, then the function space P^I is contained in $\underline{P}$.

p2) Let $\bar{h}_1, \bar{h}_2 \colon \underline{P}_{(Y,B)} \to \underline{P}_{(X,A)}$ be two functors satisfying $\bar{h}_i\,(g\colon (Y,B) \to (P,Q))\colon (X,A) \to (P,Q)$ and $i\colon (Y,B) \subset (Y',B')$ an inclusion such that for all $g \in \underline{P}_{(Y',B')}$ one has

$$\bar{h}_1(gi) = \bar{h}_2(gi),$$

then we have

$$\bar{h}_1 = \bar{h}_2.$$

If $\underline{P}$ is the category of ANR pairs having the homotopy type of a compact ANR pair and $\underline{K}$ the category $\underline{\text{Com}}$ of compact metrizable spaces, then p1), p2) are both fulfilled: p1) is well-known while p2) follows from the following observation: Assume that $g\colon (Y,B) \to (P,Q) \in \underline{P}_{(Y,B)}$ and $g = g'j$, $j\colon (Y,B) \subset (\bar{P}_1,\bar{Q}_1) \in \underline{P}$ then we find a commutative triangle of inclusions

$$\begin{array}{ccc} (Y,B) & \xrightarrow{\ i\ } & (Y',B') \\ j \big\downarrow & & \cap\, k \\ (\bar{P}_1,\bar{Q}_1) & \overset{l}{\subset} & (P',Q') \in \underline{P} \end{array}$$

for suitably choosen k. We have

$$\bar{h}_i(g) = g'\bar{h}_i(j)$$

$$l\,\bar{h}_1(j) = \bar{h}_1(ki) = \bar{h}_2(ki) = l\,\bar{h}_2(j).$$

Since l is an inclusion, we get

$$\bar{h}_1(j) = \bar{h}_2(j)$$

whence

$$\bar{h}_1 = \bar{h}_2$$

follows.

Suppose now p1), p2) are fulfilled and we have already obtained an assignment $\underline{C}_* \mapsto {}^s\underline{C}_*$ such that s1), s2) holds, then we deduce:

2.1. Theorem: Let $h_* = \{h_n, \partial_n\}$ be a homology theory on $\underline{P}^2$, then there exists a strong homology theory rel. $\underline{P}^2$ on $\underline{K}^2$ such that $h_* = {}^s\underline{h}_* | \underline{P}^2$.

Proof: According to theorem 5.4. (or [1] theorem 8.1.) we find a chain functor $\underline{C}_*$ which is related to h_*. We set ${}^s h_* = H_*({}^s\underline{C}_*)$ and claim that ${}^s h_*$ is a homology theory on $\underline{K}^2$: The homotopy axiom is a consequence of p1) and condition 1) in definition 5.3., while the excision follows from s1). The exactness of ${}^s h_*$ is implied by proposition 5.2.. The transformation (1) induces an isomorphism of homology theories. Let ${}^1\underline{C}_*$ be any chain functor related to h_* on $\underline{P}^2$, then s2) guarantees the existence of an isomorphism

$${}^s h_* \approx H_*({}^s\underline{C}_*) \approx H_*({}^{s1}\underline{C}_*).$$

So ${}^{s1}\underline{C}_*$ is also related to ${}^s h_*$ thereby revealing itself as a strong homology theory rel. $\underline{P}^2$.-

Remark: Observe that there may exist pairs $(P,Q) \in \underline{K}^2$, $Q,P \in \underline{P}$, such that ${}^s h_*(P,Q)$ is not isomorphic to an eventually given $h_*(P,Q)$ (which happens if h_* has been already defined on all pairs $(P,Q) \in \underline{K}^2$, $P,Q \in \underline{P}$).

3. The chain functor ${}^s\underline{C}_*$:

We are now indicating a construction of the assignment $\underline{C}_* \mapsto {}^s\underline{C}_*$. Here we assume for the sake of simplicity that h_* is an ordinary homology theory. In a subsequent remark we point out some of the technical details which are needed for the general case