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Preface

This book appears perhaps at the wrong moment, since it goes against the mathematical tide, which nowadays seems to be moving away from abstraction and conceptualization towards concreteness and specialization. Nevertheless, we have decided to publish it, rather than wait for the turn of the tide. Our purpose is to explain infinitesimals and infinitely large integers, as they were used before their elimination by the set-theoretic trend in mathematics. Our explanation doesn't go against this trend, but tries to give a consistent reinterpretation of infinitesimals in a set-theoretic context, through the use of sheaf theory.

A set-theoretic interpretation of infinitesimals appears to have been provided already by A. Robinson and his school, with the creation of non-standard analysis, and the reader may well wonder whether we are reformulating non-standard analysis in terms of sheaves. However, one should notice that *two* kinds of infinitesimals were used by geometers like S. Lie and E. Cartan, namely invertible infinitesimals and nilpotent ones. Non-standard analysis only takes the invertible ones into account, and the claims to the effect that non-standard analysis provides an axiomatization of *the* notion of infinitesimal is therefore incorrect. This is particularly astonishing when one realizes that notions like differential form, curvature, etc., were originally based upon the notion of nilpotent infinitesimal.

The use of sheaves to model nilpotent infinitesimals is not new. In fact, nilpotent infinitesimals are used in Grothendieck's theory of schemes to handle infinitesimal structures in the context of algebraic geometry. But the theory of schemes lacks an adequate language to deal directly with nilpotent infinitesimals, in the way that non-standard analysis provides such a language (and semantics) for invertible infinitesimals.

It was the discovery of Lawvere that a Grothendieck topos may be viewed as a universe of "variable" sets, and that consequently set-theoretic language can be interpreted directly in a topos. Therefore, working with the topos built from schemes, rather than with the schemes themselves, one obtains a model for this generalized set-theory with nilpotent infinitesimals.

Lawvere also discovered that by considering smooth versions of the toposes occurring in algebraic geometry (toposes built from rings of smooth functions, rather than polynomials) one obtains models for ordinary differential geometry. In these models, infinitesimal structures of the kind used by Cartan, for instance, can be interpreted directly, and in this context Cartan's arguments are literally valid.

These ideas, dating from 1967, remained unpublished, and were taken up only in the mid-seventies. This resulted in two main lines of development. On the one hand, there was the purely axiomatic development of differential geometry with nilpotent infinitesimals, or "synthetic differential geometry". On the other hand, smooth toposes were constructed, which showed not only the consistency of the axiomatic approach, but also provided a direct connection with the classical theory of manifolds.

The emphasis of our book is on this second line of development. Our main concern has been to show that synthetic differential geometry has a clear and direct relation to the classical theory. This relation is based on the fact that, unlike non-standard analysis, synthetic differential geometry has natural models built from smooth functions and their ideals. The main novelty of our approach, with regard to both non-standard analysis and synthetic differential geometry, is precisely the construction of such mathematically natural models containing nilpotent as well as invertible infinitesimals.

We started our collaboration at the end of 1982, when Reyes was spending his sabbatical year at the University of Utrecht. The actual writing of the book took place between the fall of 1983 and the spring of 1985. During this period, the authors were able to work in close contact. Besides several shorter visits, Reyes spent the summer of 1984 at the University of Amsterdam, and Moerdijk spent the academic year 1984–85 at McGill University.

We gave courses and seminars on parts of the contents of the book at the University of Utrecht in 82–83, at the University of Montreal in 83–84, and at McGill University in 84–85. Moreover, between 1983 and 1986, the material was presented in lectures at Aarhus during the workshop on categorical methods in geometry, at Bogotá during the seminario-taller de categorías, and at the universities of Paris, Lille, Cambridge, Columbia, Rome, Milano, Parma, Warsaw, Carnegie-Mellon, Maryland, Campinas, Saõ Paulo, La Rioja (Logrono), Zaragoza and Santiago de Compostela. We would like to thank our colleagues at these institutions who made these visits possible, for their hospitality and support.

We gratefully acknowledge the almost continuous financial support by the Netherlands Organization for Pure Research (ZWO), le Conseil de recherches en sciences naturelles et en génie du Canada, and le Ministère de l'éducation du Gouvernement du Québec. In particular, Reyes' visits to Utrecht and Amsterdam were partly supported by ZWO, and we are grateful to Dirk and Dook van Dalen, and Anne Troelstra for making these visits possible and pleasant. Moerdijk's year in Montréal was made possible by an invitation of the Groupe interuniversitaire en études catégoriques, and we would like to express our thanks to all the members of the Groupe for creating such pleasant working conditions.

During these years, those who have helped us are too numerous to be mentioned here individually. But we are specially indebted to Dana Scott. It was he who suggested the possibility of writing a monograph on models of synthetic differential geometry, who gave us advice on the organization of the book, presented it to Springer-Verlag, and provided the facilities to prepare the final text. A special word of thanks also goes to Bill Lawvere, without whose constant support we would never have been able to write this book, and to Ngo Van Quê, who had the patience to explain some analysis and differential geometry to us ignorant logicians. Moreover, we would like to thank Oscar Bruno, Marta Bunge, Eduardo Dubuc, Iole Druck, Luis Español, Alfred Frölicher, André Joyal, Anders Kock, Michael Makkai, Colin McLarty, Peter Michor, María del Carmen Mínguez, Wil van Est, and Gavin Wraith for valuable conversations and comments on parts of the manuscript.

We would also like to thank Yvonne Voorn and Lise Perreault, who have typed endless earlier versions, Roberto Minio for valuable advice on matters connected with the editing of this book, and Staci Quackenbush, who prepared this final text that you have before you.

Finally, the second author probably would have not survived this experience, had it not been for the encouragement of Marie. Not only did she give advice on a variety of matters connected with the book, but her unreasonable conviction that this project could be brought to an end, proved to be contagious.

Montréal, I.M., G.E.R.

Contents

Introduction

The theory of manifolds goes back to Riemann's lecture "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen" ("On the hypotheses which lie at the foundations of geometry"), which was delivered on June 10, 1854, to the faculty of the University of Göttingen. Since there were members of the faculty who knew little mathematics, Riemann chose a rather informal style of exposition to make his lecture intelligible. In part one of this lecture, he set himself the task of "constructing the concept of a multiply extended quantity from general notions of quantity", a task he regards as being of a "philosophical nature, where difficulties lie more in the concepts than in the construction ...."

On the basis of this lecture alone, it seems nearly impossible to determine what form such a construction should take, and hence we cannot know how far Riemann had advanced towards the accomplishment of his task. For a modern reader, however, it is very tempting to regard his efforts as an endeavor to define a "manifold", and it is precisely the clarification of Riemann's ideas, as understood by his successors, which led gradually to the notions of manifold and Riemannian space as we know them today.

In this context it is important to notice that Riemann himself pointed out in his lecture the existence of "manifolds in which the fixing of positions requires not a finite number, but either an infinite sequence or a continuous manifold of numerical measurements. Such a manifold form, for instance, the possibilities for a function in a given region, the possible shapes of a solid figure, etc." This quotation reveals already a first limitation of the theory of manifolds in its modern guise:

The category M *of C^∞ -manifolds and C^∞ -maps between them is not cartesian closed. In particular, the space of C^∞ -maps between two manifolds is not necessarily a manifold.*

The need for a cartesian closed category of smooth spaces and smooth maps has repeatedly been pointed out in connection to physics. We mention the following considerations, due to Lawvere (1980): The motion of a certain body B (for example, a 0-dimensional system of particles, a 1-dimensional elastic cord, a 2-dimensional flexible shell, a 3-dimensional solid) is often represented by a map

$$q: T \times B \to E,$$

where T is (the 1-dimensional space to measure) time, and E is the ordinary flat 3-dimensional space. Thus, the motion may be thought of as assigning to a couple (time, particle of B) the corresponding place in E during the motion.

For other purposes, however, it is useful to consider motion as a map

$$\overline{q}: B \to E^T$$

which assigns to each particle of B its path through E, where E^T is the space of (smooth) paths. The action of the vector space V of translations of the flat space E allows us to define a map

$$(\dot{\ }): E^T \to V^T$$

using Newton's notation. By composing with $\overline{q}$ we obtain a new map which, in turn, gives us (by adjunction) the velocity map

$$v: T \times B \to V$$

of the motion q.

Still another way of considering motion is necessary for some purposes, namely as a map

$$\overline{\overline{q}}: T \to E^B$$

which assigns to a time the (smooth) placement of the body in space at that time. Letting μ be the mass distribution of B, we obtain a map (by convexity of E)

$$\frac{1}{\mu(B)} \int_B (\ ) d\mu: E^B \to E,$$

which assigns to each placement of B the corresponding position of the center of mass. Once again, composing this map with $\overline{\overline{q}}$, we obtain a new map

$$T \longrightarrow E$$

giving for each time the center of mass of the systems in motion at that time.

The various connections between these ways of regarding motion should be expressed precisely by the adjunctions available from the cartesian closed structure of a category of smooth spaces and smooth maps. In the words of Lawvere: "The E,B,T transforms (i.e., the adjunctions) are *more* (at least as) fundamental as any particular determination of the objects as "consisting" of points, opens, paths, etc., and indeed any such determination which does not admit these transformations is ultimately of only specialized interest".

The second limitation of the theory of manifolds may briefly be formulated as follows:

The category M *of* C^∞ *-manifolds lacks finite inverse limits. In particular, pullbacks of manifolds are generally not manifolds.*

This implies that curves and algebraic varieties, of the kind already studied by Descartes, are not manifolds. The trouble here is that algebraic varieties may have (and usually do have) singularities, whereas manifolds cannot. As a consequence of this exclusion, one finds that, despite many interactions, differential geometry and algebraic geometry follow their separate ways, and the methods of one cannot, without violence, be applied to the other.

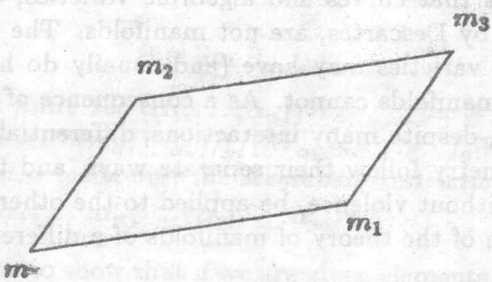
A limitation of the theory of manifolds of a different nature is:

The absence of a convenient language to deal explicitly and directly with structures in the "infinitely small".

We mention here a rather technical example, a theorem due to Ambrose, Palais and Singer, which will be discussed in Chapter V. This theorem asserts the equivalence between symmetric connections and sprays on a manifold. Connections and sprays are operations on infinitesimal structures, and one would like to show their equivalence directly. However, an appropriate language to make a direct comparison is lacking, and one first has to transform these "infinitesimal" structures into "local" ones by integration. The comparison is then possible, since it is at this "local" level that the language of the classical theory of manifolds is adequate. Finally, one returns to the original "infinitesimal" structures by some limit process, inverse to integration.

This detour should not be necessary if one had a convenient language for infinitesimals at one's disposal.

Despite the absence of such a language for infinitesimals, geometers like G. Darboux, S. Lie and E. Cartan often used "synthetic" reasoning in their work. We shall illustrate this style of reasoning with an example taken from E. Cartan (1928). After stating the formulas of Green, Stokes and Ostrogradsky, Cartan (loc. cit, p. 207) continues: "The operation which allows us to construct such formulas may be described in a very simple way. Let us first consider the case of a simple integral $\overline{\omega}(d)$ [where $\overline{\omega}(d)$ is a differential 1-form, and d is a symbol of differentiation] taken along a closed circuit (C). Let (S) be a (part of a) surface limited by (C), in n-dimensional space. Let us introduce in (S) two symbols of differentiation d_1, d_2 which are interchangeable [i.e., they commute], and let us divide (S) into the corresponding network of infinitely small parallelograms. If m is the vertex of one of these parallelograms (cf. figure)



and if m_1 and m_2 are the vertices obtained from the operations d_1 and d_2, we have

$$\int_m^{m_1} \overline{\omega} = \overline{\omega}(d_1), \quad \int_m^{m_2} \overline{\omega} = \overline{\omega}(d_2)$$

$$\int_{m_1}^{m_3} \overline{\omega} = \int_m^{m_2} \overline{\omega} + d_1 \int_m^{m_2} \overline{\omega} = \overline{\omega}(d_2) + d_1\overline{\omega}(d_2)$$

$$\int_{m_2}^{m_3} \overline{\omega} = \overline{\omega}(d_1) + d_2\overline{\omega}(d_1);$$

therefore the integral $\overline{\omega}$ taken along the boundary of the parallelogram is equal to $\overline{\omega}(d_1) + (\overline{\omega}(d_2) + d_1\overline{\omega}(d_2)) - (\overline{\omega}(d_1) + d_2\overline{\omega}(d_1)) - \overline{\omega}(d_2) = d_1\overline{\omega}(d_2) - d_2\overline{\omega}(d_1)$. The expression in the second member is nothing else but the bilinear covariant of $\overline{\omega}$ [i.e., in modern language, the exterior derivative]. For instance, if Pdx is a term of $\overline{\omega}$,

$$d_1(Pd_2x) - d_2(Pd_1x) = d_1Pd_2x - d_2Pd_1x = (dPdx)$$

[in modern notation: $dP \wedge dx$]. We obtain, thus, the Stokes' formula

$$\int Pdx + Qdy + Rdz = \int\int dPdx + dQdy + dRdz,$$

which may be extended to any number of variables".

Let us remark that this way of defining the exterior derivative by circulation along an infinitesimal parallelogram, obtaining Stokes' theorem as a byproduct, is quite popular (and rightly so!) among physicists and engineers, who keep on using this kind of reasoning.

The "symbol of differentiation" occurring in this quotation may seem rather mysterious. Let us quote again from Cartan (loc. cit, p. 179), where the sense of this notion is elucidated for Riemann spaces: "Let us consider two different systems of differentiation d and δ. The quantities du^i may be considered as products of an "infinitely small" constant parameter α by functions $\xi^i(u^1, \ldots u^n)$ (which are either determined or left undetermined):

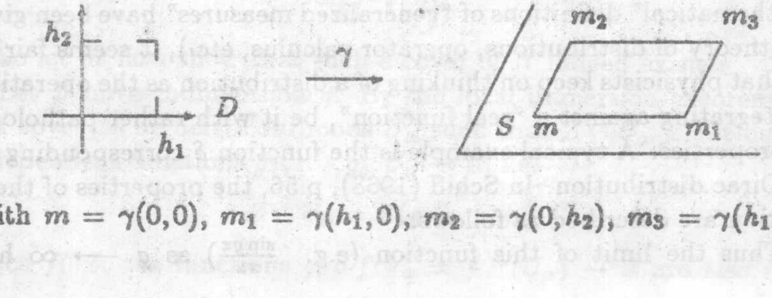
$$du^i = \alpha\xi^i(u^1, \ldots, u^n).$$

Similarly,

$$\delta u^i = \beta\eta^i(u^1, \ldots, u^n).$$

Let m be an arbitrary point with coordinates (u^i) of the Riemann space; let m_1 be the point with coordinates $(u^i + du^i)$ and m_2 the point with coordinates $(u^i + \delta u^i)$. The vector $\overrightarrow{mm_1}$ defines an elementary displacement d; the vector $\overrightarrow{mm_2}$ an elementary displacement δ."

Following this explanation, we shall interpret these notions as follows (see Chapter IV): A symbol of differentiation (on S) is a map $d: D \to S$, where D is the set of first-order infinitesimals, i.e. $D = \{h \in R | h^2 = 0\}$. Two such symbols d, δ commute if there is some $\gamma: D \times D \to S$ such that $\gamma(h, 0) = d(h)$ for all $h \in D$ and $\gamma(0, h) = \delta(h)$ for all $h \in D$. A differential 1-form (on S) is a map $\omega: S^D \times D \to R$. Cartan defines the exterior derivative of ω, i.e., a differential 2-form $d\omega: S^{D\times D} \times D \times D \to R$ via the circulation along the infinitisimal parallelogram $(\gamma, h_1, h_2) \in S^{D\times D} \times D \times D$. In fact, (γ, h_1, h_2) may be pictured as follows:



with $m = \gamma(0,0)$, $m_1 = \gamma(h_1, 0)$, $m_2 = \gamma(0, h_2)$, $m_3 = \gamma(h_1, h_2)$.

The circulation is simply

$$\int_m^{m_1} \omega + \int_{m_1}^{m_3} \omega - \int_{m_2}^{m_3} \omega - \int_m^{m_2} \omega$$

where these infinitesimal integrals are defined by

$$\int_m^{m_1} \omega = \omega([h \mapsto \gamma(h,0)], h_1) = \omega(d_1, h_1)$$

$$\int_m^{m_2} \omega = \omega([h \mapsto \gamma(0,h)], h_2) = \omega(d_2, h_2)$$

$$\int_{m_1}^{m_3} \omega = \omega([h \mapsto \delta(h_1, h), h_2)$$

$$\int_{m_2}^{m_3} \omega = \omega([h \mapsto \gamma(h_1, h_2), h_1).$$

To continue Cartan's argument, we make the blunt assumption (which is a consequence of the so-called Kock-Lawvere axiom, see II.2.4) that any function $f\colon D \to R$ may be developed in Taylor series to obtain $f(h) = f(0) + hf'(0)$.

By applying this formula to $f(h_1) = \omega([h \mapsto \gamma(h_1, h)], h_2)$, we obtain $\int_{m_1}^{m_3} \omega = \int_m^{m_2} \omega + h_1 \cdot f'(0)$, which is Cartan's formula, but for the notation. To complete the definition of the exterior derivative, we let $d\omega(\gamma, h_1, h_2) =$ circulation of ω along (γ, h_1, h_2). Using infinitesimal integrals, we may thus write, letting $\partial(\gamma, h_1, h_2)$ be the circuit (C),

$$\int_{(\gamma,h_1,h_2)} d\omega = \int_{\partial(\gamma,h_1,h_2)} \omega$$

which is the infinitesimal version of Stokes' theorem. From here, as shown in detail in Chapter IV, we can derive the usual, finite version of Stokes' theorem. How this theorem relates to the classical one will be explained later on in this introduction (and more extensively in Chapter IV).

Rather than multiplying the examples of this kind, we now give a different one coming from theoretical physics. Although several "mathematical" definitions of "generalized measures" have been given (the theory of distributions, operator calculus, etc.), it seems fair to say that physicists keep on thinking of a distribution as the operation of integrating against a "real function", be it with rather pathological properties. A typical example is the function δ corresponding to the Dirac distribution. In Schiff (1968), p.56, the properties of the δ function are described as follows:

"Thus the limit of this function (e.g. $\frac{\sin gx}{\pi x}$) as $g \longrightarrow \infty$ has all the properties of the δ function: it becomes infinitely large at $x = 0$, it has unit integral, and the infinitely rapid oscillations as $|x|$ increases mean that the entire contribution to an integral containing this function comes from the infinitesimal neighborhood of $x = 0$."

We wish to point out the following features of these arguments. First of all, in the first argument no mention is made of atlases and coordinates, although manifolds are mentioned; in other words, this argument is intrinsic, and proceeds by directly manipulating geometric objects, namely differential forms and infinitesimal parallelograms. Secondly, infinitesimals are freely used, making the notion of limit unnecessary (for the particular purpose at hand). Notice, however, that in the first argument the infinitesimals must be *nilpotent*. Obviously, such infinitesimals will not do to define the δ function–for this one needs the notion of *invertible* infinitesimals, and correspondingly, of *infinitely large* reals, to make the integral of δ add up to 1.

As a final illustration of the need for an adequate language to deal with infinitesimal structures, we would like to mention the following quotation taken from the preface to S. Lie's article (1876) (also quoted in Kock (1981)): "The reason why I have postponed for so long these investigations, which are basic to my other work in this field, is essentially the following. I found these theories originally by synthetic considerations. But I soon realized that, as appropriate [zweckmässig] the synthetic method is for discovery, as difficult it is to give a clear exposition on synthetic investigations, which deal with objects that till now have almost exclusively been considered analytically. After long vacillations, I have decided to use a half-synthetic, half-analytic form. I hope my work will serve to bring justification to the synthetic method besides the analytic one."

In this book, we will describe an approach to analysis and differential geometry, *smooth infinitesimal analysis*, which avoids the three limitations of the category of manifolds discussed above. The basic ideas of this approach are mainly due to F. W. Lawvere, and can be seen to originate from the work of C. Ehresmann, A. Weil, and A. Grothendieck. The aim is to construct categories of spaces, the so-called *smooth toposes*, which contain the category of manifolds (or more precisely, there is a full and faithful embedding of the category of C^∞-manifolds into each of these smooth toposes). Moreover, in each of these smooth toposes *inverse limits* of spaces and *function spaces* can be adequately constructed, in particular *infinitesimal spaces* like the ones needed in (our interpretation of)

Cartan's arguments, e.g., the space D of first-order infinitesimals.

The construction of these smooth toposes proceeds in two steps: one first embeds the category of manifolds $\mathbb{M}$ in the category $\mathbb{L}$ of "loci", a category of formal varieties. This new category has finite inverse limits and contains infinitesimal spaces, but function spaces can generally not be constructed in $\mathbb{L}$. As a second step, therefore, $\mathbb{L}$ is endowed with a natural Grothendieck topology, and the resulting topos $\mathrm{Sh}(\mathbb{L})$ of sheaves on $\mathbb{L}$ for this topology is the required extension of $\mathbb{L}$ in which function spaces with good properties can be constructed,

$$\mathbb{M} \subset \mathbb{L} \subset \mathrm{Sh}(\mathbb{L}).$$

This construction and variants thereof will be discussed in detail in this book, and at this stage we just sketch the idea of the extension of the category $\mathbb{M}$ of manifolds to the category $\mathbb{L}$ of loci.

To motivate the definitions, let us recall the functorial approach to algebraic geometry, as exposed in Demazure & Gabriel (1970), for example. An (algebraic) locus such as $S^1 = \{(x,y)|x^2+y^2=1\}$ is identified with a functor $S^1: C \to \mathit{Sets}$, where C is the category of commutative rings; S^1 associates with a ring A the set $S^1(A) = \{(a,b) \in A^2 | a^2+b^2=1\}$, and with a ring homomorphism $A \xrightarrow{f} B$ the obvious restriction $S^1(f): S^1(A) \to S^1(B)$, sending (a,b) to $(f(a), f(b))$. As morphisms between one such locus, i.e. a functor $C \to \mathit{Sets}$, and another, one takes simply the natural transformations. Besides the usual "spaces" such as the sphere S^1, the line R given by $R(A) =$ the underlying set of A, etc., one also has "infinitesimal loci". For example, the locus $D = \{x \in R | x^2 = 0\}$, i.e. $D(A) = \{a \in A | a^2 = 0\}$, plays the rôle of the space of first-order infinitesimals. In fact, the category of algebraic loci is simply the dual (or opposite) of the category of finitely generated commutative rings: a ring $A = \mathbb{Z}[X_1, ..., X_n]/(p_1, ..., p_k)$ corresponds to the locus $\ell(A) = \{x \in R^n | p_1(x) = .. = p_k(x) = 0\}$, i.e. to the functor

$$B \mapsto \mathrm{Hom}(A,B) \cong \{\underline{b} \in B^n | p_1(\underline{b}) = .. = p_k(\underline{b}) = 0\}.$$

In our case, the category of commutative rings is replaced by that of C^∞*-rings*. A C^∞-ring is a ring A in which we can interpret every C^∞-function $\mathbb{R}^m \longrightarrow \mathbb{R}$ as an operation $A^m \longrightarrow A$ (and not just polynomial functions, as in the case of commutative rings), and a map between two such C^∞-rings is a ring homomorphism which preserves this additional structure, a "C^∞*-homomorphism*". The category $\mathbb{L}$ is simply the dual of the category of finitely generated C^∞-rings, and for a given such C^∞-ring A, the corresponding locus-an object of $\mathbb{L}$–is denoted by $\ell(A)$.

Any manifold M is represented as an object of $\mathbb{L}$ via the C^∞-ring of smooth functions on M, $C^\infty(M)$. Furthermore, we have infinitesimal spaces such as $D = \ell(C^\infty(\mathbb{R})/(x^2))$, and $\Delta = \ell(C_0^\infty(\mathbb{R}))$ where $C_0^\infty(\mathbb{R})$ is the C^∞-ring of germs at 0 of smooth functions on $\mathbb{R}$, which will play the rôle of ***first-order infinitesimals*** and ***infinitesimals*** respectively, as will be shown later on in detail. An important space of infinitesimals is the locus $\mathbb{I} = \ell(C_0^\infty(\mathbb{R} - \{0\})$, the ring of restrictions to $\mathbb{R} - \{0\}$ of the germs at 0; $\mathbb{I}$ plays the rôle of the set of ***invertible infinitesimals.*** We also have such loci as $\ell(C^\infty(\mathbb{N})/K)$, where $C^\infty(\mathbb{N}) = \mathbb{R}^\mathbb{N}$ is the ring of smooth functions on the natural numbers, and K is the ideal of eventually vanishing functions; this locus will act as the set of ***infinitely large natural numbers.***

When a smooth topos like $\mathrm{Sh}(\mathbb{L})$ is described in this way, namely as a category of "spaces" which extends the usual category of manifolds, its close relation to the classical theory is clear. But the structure of these spaces, being sheaves on $\mathbb{L}$, is rather complicated, and the synthetic arguments described earlier can only be interpreted in a very round-about way.

However, and this is a crucial aspect of our whole approach, a smooth topos can also be regarded as a "universe of sets", inside which one can describe constructions and give arguments in a purely set-theoretical language, so that much of the complexity of the structures used is no longer explicitly there. There is one limitation, however, to the use of set-theoretical arguments and constructions when applied in this new context: they should be constructive, and no use of the axiom of choice or the law of the excluded middle can be made.

Regarding the topos $\mathrm{Sh}(\mathbb{L})$ in this way, synthetic arguments like Cartan's can be carried out almost word by word in $\mathrm{Sh}(\mathbb{L})$. Furthermore, this point of view enables us to apply many of the classical definitions, constructions and (constructive!) proofs literally to this more general category of spaces, without ever making explicit that we are really dealing, ***not*** just with ***sets of points***, but with ***sheaves*** on $\mathbb{L}$.

To give a simple example, Cartan's argument for Stokes' theorem is constructively valid, and–working in $\mathrm{Sh}(\mathbb{L})$ as a universe of sets–it applies to an arbitrary "set", i.e., to any object of $\mathrm{Sh}(\mathbb{L})$. When one now "decodes" this set-theoretic way of looking at the sheaves on $\mathbb{L}$, one obtains the usual form of Stokes' theorem for manifolds, as we will explain in detail in Chapter IV. (In fact, one obtains a more general result, including a form of Stokes' theorem for spaces