Little Mathematics Library

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LOBACHEVSKIAN GEOMETRY

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CONTENTS

AUTHOR'S NOTE

The aim of this book is to acquaint the reader with the fundamentals of Lobachevsky's non-Euclidean geometry.

The famous Russian mathematician N. I. Lobachevsky was an outstanding thinker, to whom is credited one of the greatest mathematical discoveries, the construction of an original geometric system distinct from Euclid's geometry. The reader will find a brief biography of N. I. Lobachevsky in Sec. 1.

Euclidean and Lobachevskian geometries have much in common, differing only in their definitions, theorems and formulas as regards the parallel-postulate. To clarify the reasons for these differences we must consider how the basic geometric concepts originated and developed, which is done in Sec. 2.

Apart from a knowledge of school plane geometry and trigonometry reading our pamphlet calls for a knowledge of the transformation known as inversion, the most important features of which are reviewed in Sec. 3. We hope that the reader will be able to grasp its principles with profit to himself and without great difficulty, since it, and Sec. 10, play very important, though ancillary, role in our exposition.

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На английском языке

Section 1. A Brief Essay on the Life and Work of N. I. Lobachevsky

Nikolai Ivanovich Lobachevsky was born on December 1, 1792 (November 20 by the old Julian style), the son of an ill-paid civil servant. Early in their life Nikolai Lobachevsky and his two brothers were left in the sole care of their mother, an energetic and clever woman, who, despite her extremely meager means, sent them all to the Kazan grammar school.

Lobachevsky studied there from 1802 to 1807, at Kazan University from 1807 to 1811. Possessing brilliant mathematical talents he successfully completed the course of studies and after graduating remained at the University to work for a professorship, which was conferred on him in 1816.

Lobachevsky's teaching left a deep impress on the memories of his students. His lectures were noted for their lucidity and completeness of their exposition. His knowledge of various branches of science was extensive and many-sided, which enabled him to lecture not only on mathematical subjects but also on mechanics, physics, astronomy, geodesy, and topography.

Lobachevsky was elected rector of Kazan University in 1827 and occupied this post for nearly twenty years. Being a talented and energetic administrator, with a good insight into the aims of higher education, he succeeded in making Kazan University a model higher educational institution of his time. On his initiative the university began publishing *Scientific Proceedings*. Under him construction of the university's buildings was broadly developed, an astronomical observatory founded.

But it was his scientific work that brought Lobachevsky world fame. He immortalized his name by creating the non-Euclidean geometry now called after him [1].

On 23 (11) February, 1826 Lobachevsky read a paper at a meeting of the Department of Physico-mathematical Sciences of Kazan University in which he first communicated the non-Euclidean geometry discovered by him. The first published presentation of its principles was his memoirs *On the Fundamentals of Geometry* published in 1829 and 1830 in the journal *Kazan Herald*.

Most of Lobachevsky's contemporaries did not understand his discovery, and his works on geometry had a hostile reception

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[1] Its other name – hyperbolic geometry – is due to the fact that in it a straight line like a hyperbola in Euclidean geometry has two infinitely removed points (see Sec. 4).

both in Russia and abroad. His ideas were too daring and departed too far from the notions that then dominated science so that much time had to pass before they won general recognition, which came only after his death.

Lobachevsky was not dissuaded of the correctness of his conclusions by his critics' attacks and continued, with his native energy and determination, to work on the development of his geometric system, publishing a number of works devoted to problems of non-Euclidean geometry. The last of them, completed by Lobachevsky not long before his death, had to be dictated as he himself was unable to write any more because of the blindness that affected him in his last years.

Lobachevsky's scientific activity was not restricted to investigation in geometry: he also made several fundamental contributions to algebra and calculus. The method of approximate solution of algebraic equations he worked out is very elegant and efficient.

Lobachevsky's philosophical views had a clearly expressed materialist slant. He considered experiment and practice the most reliable means of testing theoretical conclusions. He demanded teaching of mathematics such as would bring out the real phenomena behind mathematical operations.

In 1846 Lobachevsky was relieved of his duties at the University and appointed assistant trustee of the Kazan educational district.

He died on 24 (12) February, 1856. In 1896 a monument was erected in Kazan to honour his memory [1].

Section 2. On the Origin of Axioms and Their Role in Geometry

To elucidate the role of axioms let us trace the general outline of the most important steps in the development of geometry from ancient times.

The birthplace of geometry was the countries of the Ancient East where important practical rules for measuring angles, areas of certain figures, and the volumes of the simplest solids were worked out thousands of years ago to meet the needs of land mensuration, architecture and astronomy. These rules were developed empirically (from experience) and appear to have been passed on by word of mouth: in the oldest texts that have come down to us we often come across applications of geometric rules but find no attempts to formulate them.

In the course of time the circle of the objects to which the geometric knowledge acquired was applied broadened, and a need arose to formulate the rules, in the most general form possible, which brought about a transition in geometry from concrete notions to abstract concepts. For example, the rule developed for measuring the area of a rectangular plot of land proved applicable to measuring the area of a carpet, the surface of a wall, etc., and as a result an abstract concept, a rectangle, arose.

So a system of knowledge was formed which came to be termed geometry. At its early stage it was an empirical science, i. e. one in which all the results were derived directly from experience.

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[1] The reader can find more details on Lobachevsky s life in

V. F. Kagan, *N. Lobachevsky and His Contribution to the World Science* by the Foreign Languages Publishing House, Moscow, 1957;

P. A. Shirokov and V. F. Kagan, *Structure of Non-Euclidean Geometry.* Issue 1 of the series *Lobachevsky's Geometry and Development of Its Ideas,* Moscow-Leningrad, 1950 (in Russian). One chapter of this book contains a brief and skilful presentation of Lobachevskian geometry understandable to a wide range of readers.

The development of geometry took a new direction when it was noticed that some of its propositions did not need empirical substantiation since they could be deduced from its other propositions as conclusions following from the laws of logic. The propositions of geometry were now divided into two classes: those established empirically (later called axioms) and those provable logically on the basis of axioms (theorems).

Because logical substantiation, which does not require either special instruments or numerous tiresome measurements, is technically much simpler than the empirical approach, the scientists of the antiquity were naturally faced with the problem of reducing to a minimum the number of propositions of the first type (axioms) so as to lighten the geometer's job by shifting its main load to the sphere of logical reasoning. This goal proved attainable since geometry was abstracted from all properties of bodies except extension, a most essential one but so simple that all possible geometric relationships can be deduced by the laws of logic from a limited number of premises or axioms.

Thus geometry was converted from an empirical science into a deductive one[1] with its present-day axiomatic presentation.

The earliest book that has come down to us with a systematic exposition of the main propositions of geometry was Euclid's *Elements* written around 300 B.C. This work has the following structure: after definitions and axioms come the proofs of theorems and solutions of problems; each new theorem being proved on the basis of axioms and previously proved theorems. The axioms are not proved but simply stated.

For two thousand years Euclid's *Elements* enjoyed undisputed authority among scholars. But one point in it did not seem quite justified. That was the parallel-postulate stated as follows:

If a straight line falling on two straight lines makes the two interior angles on the same side of it taken together less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles together less than two right angles[2].

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[1] A deduction is the deriving of a conclusion. A science is called deductive when its new statements are deduced from preceding ones by way of logic.

[2] In school textbooks Euclid's parallel-postulate is replaced by the following equivalent proposition: *Only one straight line can be drawn parallel to a given straight line through a point not on this line.*

Two axioms of Euclidean or any other geometry are considered equivalent when the same conclusions follow from both, provided all the other axioms of the geometry remain valid.

The validity of Euclid's axiom of parallels aroused no doubts. The uncertainty regarding it lay in something else: was it justifiable to place it among the axioms? could it not be proved from the axioms of his *Elements* and so transferred to the category of a theorem?

Initially the attempts to prove the parallel-postulate reflected the tendency mentioned above to reduce the number of geometric propositions requiring empirical substantiation. In the course of time the situation has changed: the empirical origin of the axioms was forgotten and they came to be treated as self-evident truths, independent of any experience or experiment whatsoever[1]. This view gave rise to the conviction that the parallel-postulate, which it is difficult to recognize as self-evident because of its complexity, was not in fact an axiom and so the statement affirmed in it could be proved. But the many efforts in this direction did not produce positive results; like an enchanted treasure, the parallel-postulate would not yield up its secrets to investigators. The attempts to prove it, which consumed a tremendous amount of mental effort by generations of scholars, failed as the price of idealistic interpretation of the essence of axioms.

The most common type of erroneous proof of Euclid's parallel-postulate was to replace it by an equivalent proposition, for instance: *a perpendicular and an oblique line on one and the same straight line intersect*; or, *there is a triangle similar to a given triangle but not equal to it*; or: *the locus of points equidistant from a given straight line and located on one side of it, is a straight line*; or, *any three points are either collinear or cocyclic*. Later we shall demonstrate that all these propositions are fallacious if Euclid's axiom of parallels does not hold. Consequently, by taking any of them as an axiom we thereby assume the validity of Euclid's parallel-postulate, i, e. assume to be correct what we want to prove.

Lobachevsky took a different path in his investigations in the theory of parallels. Having started with attempts to prove the axiom of parallels he soon noticed that one of them led to quite unexpected results. This attempt consisted in using proof by contradiction (reductio ad absurdum) and was based on the following argument: if Euclid's parallel-postulate is the consequence

[1] It is known that persons who were born blind but who have had their eyesight restored surgically, cannot distinguish a cube from a sphere for some time after the operation without first touching them. This demonstrates a need of experience for the correct perception of geometrical images, without which geometric concepts cannot be formed.

of the other postulates of *Elements*, and if one assumes, in spite of it, that *at least two straight lines not intersecting a given straight line can be drawn through a point located outside of that line in the same plane as it*, then this assumption should lead sooner or later, in its immediate or remote consequences, to a contradiction. But in considering more and more new consequences of this assumption, Lobachevsky became convinced that, no matter how paradoxical they seemed from the standpoint of Euclidean geometry, they formed a consistent system of theorems that could form the basis of a new scientific theory.

Thus the foundation of non-Euclidean geometry[1] was laid; its axiom of parallels differs from the Euclidean and coincides with the assumption given above which we shall refer to henceafter as Lobachevsky's parallel-postulate.

But it still remained obscure whether it could confidently be stated that not one of the infinite set of possible consequences of Lobachevsky's parallel-postulate would lead to a contradiction. Lobachevsky outlined a way to solve this problem, pointing out that the consistency of the geometry discovered by him should follow from the possibility of arithmetizing it, i. e. from the possibility to reduce the solution of any geometric problem to arithmetic calculations and analytic transformations by using the formulas of the hyperbolic trigonometry derived by himself. Other scientists later found rigorous proofs of the consistency of his geometry.

Lobachevsky's investigations in the domain of the hyperbolic geometry were very wide covering its elements, trigonometry, analytical geometry and differential geometry. Using the methods of his geometry he derived more than 200 new formulas for calculating definite integrals.

Lobachevsky's discovery was considered by his contemporaries and even by his pupils as monstrous nonsense, insolent defiance of logic and common sense[2]. Such an attitude toward a great

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[1] It has since been found that, apart from the geometry discovered by Lobachevsky, many other non-Euclidean geometries can be constructed.

[2] One cannot, of course, groundlessly suspect Lobachevsky's contemporaries of being unable to understand his discovery: many did not express any opinion, possibly because the range of their scientific interests did not include the sphere of Lobachevsky's investigations; we also know that the famous German mathematician Karl Gauss and the outstanding Hungarian geometer János Bolyai, who, independently of Lobachevsky, came to the idea of the possibility of constructing a non-Euclidean geometry, shared his views. But Gauss, fearing not being understood and ridiculed, did not publish any support of Lobachevsky's

idea demolishing hallowed conceptions is not surprising. Copernicus's heliocentric theory, which denied what seemed completely obvious and asserted what seemed unthinkable, had just as hostile a reception. It needed very profound considerations to understand the admissibility of two different geometries. Let us now turn to presentation of some of the most easily understood arguments.

The section on plane geometry in school textbooks studies a plane independently of the surrounding space; in other words, planimetry is the geometry of a Euclidean plane. Geometries of certain curvilinear surfaces are also well known; an example is spherical geometry, which is widely used in astronomy and other branches of knowledge.

In every science the simplest concepts are most important. In Euclidean geometry these are the concepts of point, straight line, and plane. These terms are retained in non-Euclidean geometries, so that by a "straight line" is meant a line along which the shortest distance is measured between two points; a "plane" is a surface possessing the property such that if two points of a "straight line" belong to the surface, then all the points of that "straight line" belong to the surface. In spherical geometry, for instance, a sphere and its great circles are referred to, respectively, as a "plane" and "straight lines". This terminology is quite appropriate since, in any geometry, a "straight line" is the simplest of all lines and a "plane" is the simplest of all surfaces, the former possessing the most important property of the Euclidean straight line and the latter of the Euclidean plane [1].

Let us note certain features of spherical geometry. For illustrative purposes we shall consider it as the geometry of the surface of a globe. It is not difficult to grasp that two "straight lines" in this geometry (e. g. two meridians) always intersect at two diametrically opposite points on the globe. Furthermore, the sum of the angles of a spherical triangle is greater than π; for example, in a triangle bounded by a quarter of the equator and by the arcs of two meridians (Fig. 1) all three angles are right angles [2].

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ideas, and Bolyai, seeing that his own investigations in non-Euclidean geometry (published in 1832) had not received recognition, abandoned his mathematical studies. Thus Lobachevsky was left alone to struggle for the correctness of his ideas.

[1] It should be noted that in projective geometry there is no concept of the distance between two points; the interpretation of the concepts of a "straight line" and of a "plane" does not apply in such geometries.

[2] The angle between two lines at their point of intersection is defined as the angle between their tangents at this point.

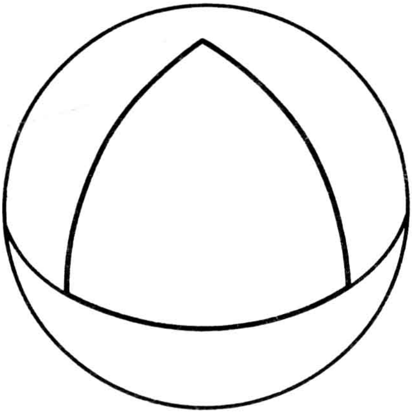

Fig. 1

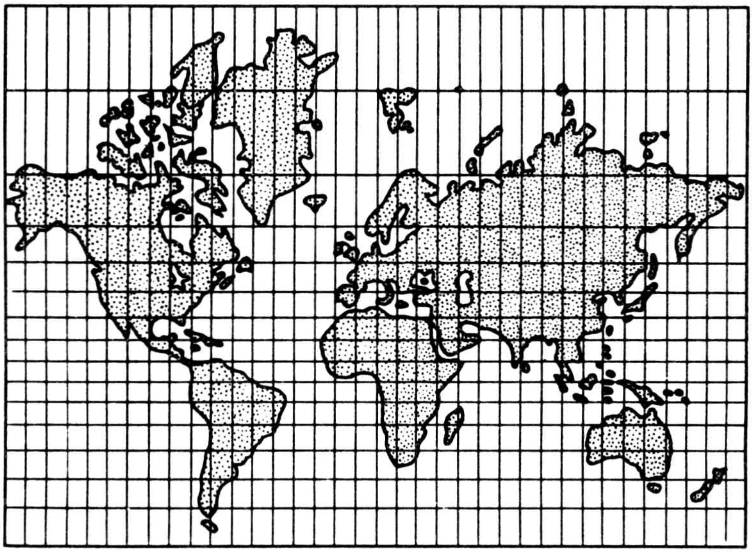

Fig. 2

Apart from globes, of course, maps of the terrestrial surface are used in geography. This is equivalent to studying spherical geometry by considering maps of a sphere, which is quite possible provided that it is indicated how to measure the actual lines and the actual angles between them from their representations on the map, for the latter are distorted and the character of the distortion is not uniform over the whole map. On maps of the Earth's surface employing Mercator's projection[1] (Fig. 2). for example, the meridians are projected as parallel lines, and the perpendiculars, which correspond to the geographic parallels, are such that a segment representing 1° of a parallel has the same length irrespective of the latitude; but in reality the length of one degree of a parallel is the shorter the higher the latitude.

Since a surface has two dimensions, the geometry studying figures lying on a certain surface is usually called two-dimensional, and the surface itself a two-dimensional space. Two types of two-dimensional geometry have been known since antiquity, Euclidean (for a plane) and spherical. Mathematicians did not assign special importance to the existence of a two-dimensional non-Euclidean geometry, namely spherical geometry, for the simple reason that the sphere was studied in three-dimensional Euclidean space, which made them disregard the non-Euclidean properties of the sphere as such.

As a result of Lobachevsky's investigations it was realized that not only are surfaces with non-Euclidean properties imaginable but also three-dimensional non-Euclidean spaces.

The introduction of the concept of three-dimensional non-Euclidean geometries may puzzle unless we give the following explanation.

It is sometimes convenient to represent the results of studying a certain class of phenomena in a geometrical form. Data on the growth of labour productivity, for example, are often shown in the form of graphs and diagrams. This demonstrates that various real processes and states with no direct connection with geometry can be depicted by means of geometric images.

If a graph is considered as a line on a Euclidean plane it becomes clear that images of the two-dimensional Euclidean geometry are employed in our example. In more complicated cases we may have to resort to three-dimensional and even

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[1] Gerhard Mercator (1512–1594) was an outstanding Flemish cartographer. The projection proposed by him in 1569 became universally accepted and the nautical charts have been compiled by it ever since.

multi-dimensional Euclidean and non-Euclidean geometries. But it does not follow that they all describe the relations in extension; there are theories that employ geometric terms in their formulations and these terms, generally speaking, are assigned meanings that are not related to spatial concepts. Thus by adding time as a fourth dimension to the three dimensions of real space we introduce the concept of four-dimensional space in which a given time interval is considered as a "segment of a straight line". In most cases this approach only creates an appearance of vizualization, nevertheless it facilitates the analysis of phenomena to a certain extent when they are studied by this method.

So the construction of non-Euclidean geometries is justified by the possibility of applying their conclusions to actually existing objects. The fact that these conclusions are expressed in terms of geometry is of no real consequence; it is not hard to modify the geometric formulations so that they correspond to the properties of the objects and phenomena in question.

The substitution of certain concepts for others, let us note, is a common practice in applied mathematics when a theory describes qualitatively different objects governed by the same mathematical laws [1].

Three-dimensional geometries call for special attention. Irrespective of their other applications they can be regarded as hypotheses claiming to describe the properties of real space. Which one corresponds most closely to reality is a problem that can only be solved by experimental testing.

But let us note the following fact, important for our further exposition: a map of a Lobachevskian plane can be constructed on a Euclidean plane, and in more than one way, just as is done for a sphere. We shall use analysis of such a map as the basis for our study here of hyperbolic geometry.

Lobachevsky's geomerty received general recognition in the following circumstances. In 1868 the Italian geometer Eugenio (1835-1900) Beltrami has discovered that there was a surface in the Euclidean space that possessed the properties of a Lobachevskian plane, or rather of a certain segment of this plane (if the shortest lines on the surface are considered as "straight lines"). This discovery, which soon led to the

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[1] As to the practical application of this principle see the section on simulation in V. G. Boltyansky's *Differentiation Explained* (Mir Publishers, Moscow).

construction of various maps of the Lobachevskian plane, convinced scientists of the correctness of the Russian geometer's ideas, gave an impetus to a deeper study of his work, and stimulated the starting of many investigations in the field of non-Euclidean geometries.

The discovery of non-Euclidean geometries posed an extremely complicated problem to physics, that of explaining whether real physical space was Euclidean as had earlier been believed, and, if it was not, to what type of non-Euclidean spaces it belonged[1]. To answer it, it is necessary to check the validity of the axioms experimentally, it being clear that, with the improvement in measuring instruments, the reliability of the experimental data obtained will increase and with it the possibilities of penetrating into details that earlier escaped investigators' attention.

Thus Lobachevsky brought geometry back to a materialist interpretation of its axioms as propositions postulating the basic geometric properties of space, perceived by humanity as the result of experience.

We still cannot consider the problem of the geometric structure of real physical space completely resolved. Nevertheless we may note that in the modern theory of relativity real space is considered on the basis of numerous data to be non-Euclidean, and to have geometric properties more complex than those of Lobachevskian space. One of the heaviest blows to belief in the Euclidean structure of real space was dealt by the discovery of the physical law that there can be no velocity exceeding the velocity of light.

Now we can answer the question one hears fairly often, namely, which of the two geometries, Euclid's or Lobachevsky's is the true one.

No similar question arises regarding the two-dimensional Euclidean and spherical geometries; both are obviously true, but each has its own sphere of application. The formulas of spherical geometry cannot be used for plane figures nor those of two-dimensional Euclidean geometry for figures on a sphere. The same is true of the different three-dimensional geometries: each of them, being logically consistent, has its application in a certain field, not necessarily geometrical in character; but it would be invalidated if we ascribed it a universal character.

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[1] When this problem is considered, the possibility of real space being non-uniform must not be neglected, that is to say, the possibility that its geometric structure may be different at different points.

As to the geometric structure of real space, the problem, as we have indicated, comes within the domain of physics and cannot be resolved by means of pure geometry. Its specific feature, by the way, is that no geometry represents spatial relations with absolute accuracy; the molecular structure of matter, for example, precludes the existence of solids of dimensions perceivable by touch that would have the geometric properties of an ideal sphere. Therefore the application of geometric rules to the solution of concrete problems inevitably produces only approximate results. So our concept of the geometric structure of real space boils down to a scientifically justified conviction that one geometry provides a better description of actual spatial relations than others.

Though the theory of relativity uses the formulas of non-Euclidean geometry, it does not follow that Euclid's geometry must be discarded, as happened to astrology, alchemy and to pseudo-sciences like them. Both geometries are tools for investigating spatial forms but the non-Euclidean enables finer studies to be made while Euclid's is adequate for solving most practically important problems with a very high degree of accuracy; and since it is, at the same time, characterized by great simplicity, its wide application is always permanently guaranteed.

To conclude this brief outline let us note the new ideas introduced by Lobachevsky into the development of geometry.

The scientific contributions of this outstanding thinker were not restricted to his unveiling of the thousand-year-old mystery of the axiom of parallels; the significance of his work was immeasurably greater.

By subjecting one of Euclid's axioms to critical analysis, Lobachevsky laid the basis for reconsideration of some of initial propositions of the Euclidean system, which subsequently led to the development of rigorous scientific principles for the axiomatic construction of geometry and other branches of mathematics.

Lobachevsky's discovery of hyperbolic geometry freed the science of spatial forms from the narrow framework of the Euclidean system. His geometry found direct application in the theory of definite integrals and in other spheres of mathematics.

Lobachevsky initiated the treatment of problems that could not have arisen in the former state of mathematics, including that of the geometric structure of real space. Without it the theory of relativity, one of the greatest achievements of modern physics, could not have been developed. Taking Lobachevsky's investigation as a start, scientists have built a theory that makes it possible to analyse the processes taking place inside the atomic nucleus.

In conclusion let us note the gnoseological significance[1] of the ideas of this great Russian mathematician. Before Lobachevsky geometry was dominated for centuries by the idealistic view originating with the Greek philosopher Plato. By ascribing the axioms of the Euclidean system an absolute character, Plato denied their empirical origin. Lobachevsky decisively shattered this outlook and returned geometry to a materialist position.

Section 3. Inversion

Suppose there to be a rule that allows the transition from any given figure to another in such a way that the second is completely defined once the first is fixed, and vice versa. The transition so made is called a geometric transformation. The most commonly used geometric transformations are parallel translation, similarity transformation, rotation of a figure, projection, and inversion. Inversion is used extensively in mathematics, for example, as a method of solving problems of construction, in the theory of functions of a complex variable, and in studying maps of a Lobachevskian plane.

In this section we will define inversion and its related concepts, and consider a number of its basic properties.

Let a circle k be drawn in a plane α, with a radius r and a centre O, and a point A, not identical with the point O. Let us take a point A' on the ray OA in such a way that the product of the segments OA and OA' equals the square of the radius of the circle k:

$$OA \cdot OA' = r^2 \qquad (1)$$

Let us agree to call points A and A' **symmetrical** with respect to the circle k.

If either of the points A and A' is located outside the circle k, then the second point lies within it, and vice versa; for example, from the inequality $OA > r$ we conclude, taking condition (1) into account, that $OA' < r$. But if either A or A' lies on the circle k then A and A' coincide.

Consider Fig. 3 in which AB is a tangent to the circle k, and BA' is a perpendicular to OA. Since OA' is the projection of

[1] Gnoseology – the science of cognition.