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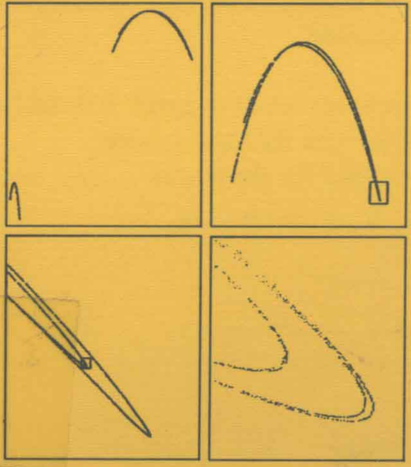
# Coexistence and Persistence of Strange Attractors



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## Preface

For dissipative dynamics, chaos is defined as the existence of strange attractors. Chaotic behaviour was often numerically observed, but the first mathematical proof of the existence, with positive probability (persistence), of a strange attractor was given by Benedicks and Carleson for the Hénon family, at the begining of 1990's. A short time later, Mora and Viana extended the proof of Benedicks and Carleson to the Hénon-like families in order to demonstrate that a strange attractor is also persistent in generic one-parameter families of surface diffeomorphisms unfolding a homoclinic tangency, as conjectured by Palis. In the present book, we prove the coexistence and persistence of any number of strange attractors in a simple three-dimensional scenario. Moreover, infinitely many of them exist simultaneously.

Besides proving this new non-hyperbolic phenomenon, another goal of this book is to show how the Benedicks-Carleson proof can be extended to families different from the Hénon-like ones.

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The authors

# Contents

# INTRODUCTION

This book deals with the existence and persistence of any number of coexisting strange attractors in three-dimensional flows. More precisely, we shall define a one-parameter family $X_\mu$ of piecewise regular vector fields on $\mathbf{R}^3$ and we shall prove that for each natural number $n$, there exists a positive Lebesgue measure set of parameter values for which $X_\mu$ has, at least, $n$ strange attractors. Moreover, $X_\mu$ exhibits an infinite number of strange attractors for some values of the parameter.

By an attractor we mean a compact invariant set $\Lambda$ having a dense orbit (transitive) and whose stable set $W^s(\Lambda)$ has a non-empty interior. For different notions of attractor see [14]. We call an attractor strange if it contains a dense orbit with a positive Liapunov exponent (sensitive dependence on initial conditions).

The term strange attractor was first used by D. Ruelle and F. Takens [20] to suggest that turbulent behaviour in fluids might be caused by the presence of attractors which are locally the product of a Cantor set and a piece of two-dimensional manifold. The notion of strange attractor associated to the sensitive dependence on initial conditions was needed to explain asymptotic dynamics which numerically or empirically manifest this kind of unpredictable behaviour. One of the most relevant dynamics of this type was earlier observed by Lorenz [11] on analysing the quadratic vector field

$$\begin{cases} x' = -10x + 10y \\ y' = 28x - y - xz \\ z' = -\frac{8}{3}z + xy \end{cases},$$

which follows from a truncation of the Navier-Stokes equations. Surprisingly, under small perturbations of the system, he seemed to get a persistent but not stable attractor, i.e. small perturbations of the original system give rise to nearby attractors but, in general, they are not topologically equivalent.

From a physical point of view, a certain degree of persistence is as relevant as the unpredictability of the dynamics resulting from the afore-mentioned sensitivity with respect to initial conditions. So, if a family $X_\mu$ of vector fields exhibits a strange attractor for the value of the parameter $\mu = \mu_0$, the dynamics of the attractor should only be considered if for every $\delta > 0$, strange attractors still exist for values of the parameter belonging to a positive Lebesgue measure set $E \subset B(\mu_0, \delta)$. In this case, the attractor is said to be persistent for the family $X_\mu$, and is said to be fully persistent if we can take $E = B(\mu_0, \delta)$ for some $\delta > 0$.

A non-periodic hyperbolic attractor is strange, fully persistent and even stable. From numerical analysis, Lorenz's attractor seems to be strange, fully persistent but not stable. M. Hénon [7] found a possible persistent (but not fully persistent) strange attractor for the family

$$H_{a,b}(x, y) = (1 - ax^2 + y, bx)$$

with $a = 1.4$ and $b = 0.3$. At the begining of 1990's, in a historical and very complex paper [3], M. Benedicks and L. Carleson proved mathematically that the Hénon family has persistent strange attractors for values of the parameters close to $a = 2$ and $b = 0$. A short time later, L. Mora and M. Viana [15] proved that, such as J. Palis had conjectured, generic one-parameter families of diffeomorphisms on a surface which unfold a homoclinic tangency have strange attractors or repellers (negative attractors) with positive probability in the parameter space. For a proof of this result in higher dimensions, see [25].

Homoclinic orbits were discovered by H. Poincaré a century ago. In his famous essay on the stability of the solar system, Poincaré showed that the invariant manifolds of a hyperbolic fixed point could cut each other at points, called homoclinics, which yield the existence of more and more points of this type and consequently, a very complicated configuration of the manifolds, [18]. Many years later, G. Birkhoff [4] showed that in general, near a homoclinic point there exists an extremely intrincated set of periodic orbits, mostly with a very high period. By the mid-1960's, S. Smale [21] placed his geometrical device, the Smale horseshoe, in a neighbourhood of a tranversal homoclinic orbit, thus explaining Birkhoff's result and arranged the complicated dynamics that occur near a homoclinic orbit by means of a conjugation to the shift of Bernouilli.

The strange attractors found in [15] arise from the creation or destruction of Smale horseshoes associated to the transversal homoclinic points which appear as the result of the bifurcation of a tangential homoclinic point. Roughly speaking, homoclinic bifurcations mean the creation of transversal homoclinic orbits resulting from small perturbations of the dynamical system. Though one case of homoclinic bifurcation is that of homoclinic tangency, there are, however, interesting examples of homoclinic bifurcations that do not correspond to homoclinic tangencies. For an extensive study of the phenomena which accompany homoclinic bifurcations, see the book by J. Palis and F. Takens, [17]. We also quote this reference as a suitable complement to this introduction. In Chapter 7 of [17] the authors propose homoclinic bifurcations as the doorways (the only ones in dimension two) to non-hyperbolic dynamics: coexistence of infinitely many sinks, persistence of Hénon-like attractors, etc. In the present book we place ourselves in one of these doorways from which we have access to an infinite number of strange attractors and to any finite number of persistent strange attractors, within a three-dimensional vector field framework.

In order to place ourselves within this framework, we evoke the following result proved by P. Sil'nikov [22]: In every neighbourhood of a homoclinic orbit of a hyperbolic fixed point of an analytical vector field on $\mathbf{R}^3$ with eigenvalues $\lambda$ and $-\rho \pm i\omega$ such that $0 < \rho < \lambda$, there exists a countable set of periodic orbits. This result is similar to Birkhoff's for diffeomorphisms and thus, it should be completed by proceeding as Smale did. So, C. Tresser [24] proved that in every neighbourhood of such a homoclinic orbit, an infinity of linked horseshoes can be defined in such a way that the dynamics is conjugated to a subshift of finite type on an infinite number of symbols. If, on the contrary, $0 < \lambda < \rho$, then the dynamics is trivial: The $\omega$-limit set of any point in a neighbourhood of the homoclinic orbit is contained in the closure of this orbit. In the case $\lambda = \rho$, we shall prove our main result:

**Theorem A.** *In the set of three-dimensional vector fields having a homoclinic orbit to a fixed point with eigenvalues* $\lambda > 0$ *and* $-\lambda \pm i\omega$ *satisfying* $\left|\frac{\lambda}{\omega}\right| < 0.3319$, *there exists a one-parameter family* $X_a$ *of piecewise regular vector fields such that for every neighbourhood* $V$ *of the homoclinic orbit, for each* $k \in \mathbf{N}$ *and for every value of the parameter* $a$ *in a set of positive Lebesgue measure depending on* $k$, *at least* $k$ *strange attractors coexist in* $V$. *Moreover, for some value of the parameter* $a$, *there exist infinitely many strange attractors contained in* $V$.

Piecewise regular vector fields with a Sil'nikov homoclinic orbit were constructed in [24]. In fact, these orbits arise in families of analytical vector fields as a codimension-one phenomenon, [19]. Recently, in [9] the authors proved the existence of vector fields verifying the hypotheses of Sil'nikov's theorem in generic unfoldings of codimensionfour singularities in $\mathbf{R}^3$ (Sil'nikov bifurcation). This homoclinic bifurcation occurs when the parameters take values on a manifold of codimension one. Just off this manifold the homoclinic orbit disappears and an infinite number of horseshoes given in [24] are destroyed. Then, as a consequence of [15], a method of constructing families of quadratic vector fields on $\mathbf{R}^3$ which display strange attractors is obtained.

Unlike the Sil'nikov bifurcation mentioned above, the homoclinic orbit in Theorem A endures for each vector field $X_a$. Therefore, in a neighbourhood of this orbit we can choose a suitable transversal section $\Pi_0$ and define the transformation $T : \Pi_0 \to \Pi_0$ associated to the flow. After splitting $\Pi_0$ into a countable union of rectangles, $R_m$, and carrying out adequate changes of variable, we get the following sequence of families of diffeomorphisms

$$T_{\lambda,a,b}(x,y) = \left(f_{\lambda,a}(x) + \frac{1}{\lambda}\log\left(1+\sqrt{b}y\right), \sqrt{b}\left(1+\sqrt{b}y\right)e^{\lambda x}\sin x\right),$$

with $b = e^{-2\pi\lambda m}$ and $m \in \mathbf{N}$. For a large enough $m$, each $T_{\lambda,a,b}$ is a small perturbation of $\Psi_{\lambda,a}(x,y) = (f_{\lambda,a}(x), 0)$, where $f_{\lambda,a}(x) = \lambda^{-1}\log a + x + \lambda^{-1}\log\cos x$. Thus, Theorem A is an immediate consequence of the following one:

**Theorem B.** *Fixed $0 < \lambda < 0.3319$, for every $m_0 \in \mathbf{N}$ and for each $k \in \mathbf{N}$ there exists a positive Lebesgue measure set $E = E(k)$ of values of the parameter $a$ such that, for every $a \in E$, there exist, at least, $k$ transformations $T_{\lambda,a,b}$, with $b < e^{-2\pi\lambda m_0}$, having a strange attractor. Moreover, there exist values of the parameter $a$ for which infinitely many $T_{\lambda,a,b}$ simultaneously have a strange attractor.*

Mora and Viana defined a renormalization in a neighbourhood of a homoclinic point to transform a generic family of diffeomorphisms unfolding a homoclinic bifurcation into a Hénon-like family. These families are defined in Proposition 2.1 of [15] so as to be suitable small perturbations of $H_a(x,y) = (1 - ax^2, 0)$ just as the Hénon family $H_{a,b}$ is for small values of $b$. The changes of variable which we have to carry out to obtain $T_{\lambda,a,b}$, play the same role as does the renormalization in [15]. We shall prove in Proposition 1.3 the conditions which make $T_{\lambda,a,b}$ a good perturbation of $\Psi_{\lambda,a}$. Then, we shall say that $T_{\lambda,a,b}$ is an adequate unfolding of $\Psi_{\lambda,a}$.

From this stage and in spite of $f_{\lambda,a}$ not being the quadratic map, the proof of Theorem B can be developed by means of a cautious adaptation of the ideas and the arguments in [3] and [15]. Nevertheless, since the density of these references makes them hard to read, we feel it is both useful and necessary to give a proof in detail in order to facilitate the understanding of the intricate inductive method and the control of the numerous estimates required. We also try, thereby, to show how the ideas for the Hénon-like families can be applied to adequate unfoldings of unimodal maps which are distinct from the quadratic one. Maybe these unfoldings take part in many other cases where possible strange attractors have also been observed numerically. See, for instance, [5].

This book is organized as follows:

In Chapter 1 we introduce the afore-mentioned changes of variable for defining the transformations $T_{\lambda,a,b}$. In fact, it is shown that $T_{\lambda,a,b}$ is an adequate unfolding of $f_{\lambda,a}$. Next, we prove that, for every positive $\lambda$, there exists a value of the parameter $a$, $a(\lambda)$, such that $f_{\lambda,a(\lambda)}$ has a homoclinic orbit and that, for a sufficiently small $\lambda$, for instance $\lambda < 0.3319$, the Schwarzian derivative of the map $f_{\lambda,a(\lambda)}$ is negative. This means that $f_{\lambda,a(\lambda)}$ has no periodic attractors, [23].

In Chapter 2 we study the unimodal family $f_{\lambda,a}$ for $0 < \lambda < 0.3319$. It is shown that there exists a constant $c_0 > 0$ such that, for every $0 < c < \min\{c_0, \log(1+\lambda)\}$, there is a value of the parameter $a_0 = a_0(\lambda, c) < a(\lambda)$ close to $a(\lambda)$ and a positive Lebesgue measure set $E = E(\lambda, c) \subset [a_0, a(\lambda)]$ such that every $a \in E$ satisfies the exponential growth condition for every $n \in \mathbf{N}$, i. e.,

$$|D_n(a)| = \left|\left(f_{\lambda,a}^n\right)'(f_{\lambda,a}(c_\lambda))\right| \geq e^{cn} \text{ for every } n \in \mathbf{N}.$$

This result, which is stated in Theorem 2.1, is a consequence of Theorem 6.1 in [13], that is, of the Benedicks and Carleson theorem for unimodal maps distinct from the quadratic one. However, since comprehension of the unidimensional case will be necessary to understand the bidimensional dynamics, which is studied in successive chapters, and since many of the specific ideas used in the study of $f_{\lambda,a}$ will be evoked in the study of $T_{\lambda,a,b}$, we have to develop a different proof from the one given in [13].

To construct a positive Lebesgue measure set $E$ such that the exponential growth condition holds for every $a \in E$, we proceed by induction on the length $n$ of the orbit of $f_{\lambda,a}(c_\lambda)$. Clearly, whenever this orbit remains far from the critical point (and this is easily obtained for a number $N$ of initial iterates and for the values of the parameter belonging to an interval $[a_N(\lambda), a(\lambda)]$), the orbit of $f_{\lambda,a}(c_\lambda)$ will be $e^c$-expansive, where $c$ depends on the distance between the initial orbit and $c_\lambda$. This remark allows us to start the inductive process, but, since the length of the interval $[a_N(\lambda), a(\lambda)]$ tends to zero as $N$ tends to infinity, we have to let the orbit of $f_{\lambda,a}(c_\lambda)$

accede to any sufficiently small neighbourhood $(c_\lambda - \delta, c_\lambda + \delta)$ of the critical point, at iterates which will be called returns. In this case, since the derivative of the unimodal map tends to zero as the distance between the return and the critical point tends to zero, we have to control such distances. To this end, it seems to be natural to permit this distance to decrease as the return iterate increases, because the small derivative may be distributed in a larger exponent in the definition of expansiveness. Hence, if $\Omega_{n-1} \subset [a_N(\lambda), a(\lambda)]$ denotes the set of values of the parameter $a$ for which $f_{\lambda,a}(c_\lambda)$ is $e^c$-expanding up to time $n-1$, we remove, from $\Omega_{n-1}$, those parameters for which the following basic assumption does not hold:

$$\left|f^n_{\lambda,a}(c_\lambda) - c_\lambda\right| \geq e^{-\alpha n},$$

where $\alpha > 0$ is a small positive constant. In this way, a set $\Omega'_n$ is constructed in a correct, but unfinished, posing of the problem. In fact, we also have to control the rate of previous iterates to the return whose expansiveness has been annihilated by the small derivative at the return. Here is where the reason for the inductive method becomes patent:

Since the orbit at the return is close to the critical point, their successive iterates, and consequently the derivatives at these iterates, are close each other. In this context, the binding period $[n+1, n+p]$ is defined by taking the largest natural number $p$ such that

$$\left|f^{n+j}_{\lambda,a}(c_\lambda) - f^j_{\lambda,a}(c_\lambda)\right| \leq e^{-\beta j} \text{ for } 1 \leq j \leq p,$$

where $\beta > 0$ is a small constant. By taking $\alpha < \beta$ small enough it is shown that the length of the binding period is smaller than $n$. Then, by using the inductive hypothesis for the orbit of the critical point and bearing in mind the closeness between its iterates and the respective iterates of the return, the small derivative at the return is proved to be compensated during the binding period. The remainder of the iterates outside the binding periods are called free iterates and they will be used to recover the exponential growth of the orbit. Therefore, the rate of these iterates has to be sufficiently large, for which we have to remove from $\Omega'_n$ the parameters not satisfying the following free assumption:

$$F_n(a) \geq (1-\alpha)n,$$

where $F_n(a)$ denotes the number of free iterates in $[1, n]$.

In this way, the sets $\Omega_n$ are inductively constructed so that if, in each step, the measure of the excluded set exponentially decreases with respect to $n$, then the set $E$ announced in Theorem 2.1 can be obtained by intersecting all the sets $\Omega_n$. The detailed development of the whole process requires a large number of estimates. We finish this advance of Chapter 2 by calling attention to the relationship between the different constants taking part in the process and to their adequate and orderly selection:

First, we consider an arbitrary $\lambda < 0.3319$. Once $\lambda$ is chosen, the constant $c_0$ of Theorem 2.1 depends on $\lambda$ and is given in Proposition 2.2. Once $c$ is fixed with $0 < c < \min\{c_0, \log(1+\lambda)\}$, in the definition of binding period we take $\beta = \beta(\lambda, c)$ small enough. With respect to the constant $\alpha$ taking part in basic and free assumptions, this will depend on $\lambda$, $c$ and $\beta$ and will be taken sufficiently small with respect to them. In order to establish the concept of return, a constant $\delta = \delta(\lambda, c, \beta, \alpha)$ is chosen which is related to the natural number $\Delta$ ($\delta \approx e^{-\Delta}$) given in Proposition 2.2. In accordance with this proposition, $\Delta$ has to be large enough. Hence, $\delta$ will be taken small enough and, in particular, $\delta$ is always said to be sufficiently less than $\lambda$, $c$, $\beta$ and $\alpha$. Schematically, we write

$$\lambda \to c_0(\lambda) > c >> \beta(\lambda, c) >> \alpha(\lambda, c, \beta) >> \delta(\lambda, c, \beta, \alpha).$$

Finally, the inductive process will be started in an iterate $N = N(\lambda, c, \beta, \alpha, \delta) >> \Delta$. Then, for fixed $N$, a set $\Omega_0 = [a_0, a(\lambda)]$ is constructed, where the inductive process starts. Lastly, $a_0$ only depends on $\lambda$ and $c$.

In the remaining chapters we prove Theorem B. From Chapter 1, we know that the closure of the unstable manifold of the saddle-point $P_{a,m}$ is an attracting set. Therefore, this set will be a strange attractor whenever the existence of a dense orbit with a positive Liapunov exponent is stated.

Though $T_{\lambda,a,b}$ is close to $\Psi_{\lambda,a}$ for small values of $b$, the expansiveness, with positive probability, along the orbit of the critical point of $\Psi_{\lambda,a}$ does not easily extend to $T_{\lambda,a,b}$. The hardness of this extension begins to appear in the definition of critical points for the bidimensional map. In fact, the role of critical points is now played by points on $W^u(P_{a,m})$ such that the differential map of $T_{\lambda,a,b}$ sends the tangent vector to $W^u(P_{a,m})$ at these points into a contractive direction, that is, into a direction which is exponentially contracted by all the iterates of the differential map. These concepts will be accurately stated in the inductive process framework which takes part in the proof of Theorem B.

In Chapter 3 we study, for each $n \in \mathbf{N}$, the maximally contracting and maximally expanding directions for the $n$-th iterate of the differential map. Under inductive hypotheses of expansiveness, remarkable properties of these directions are established. Of course, the known expansiveness on the first iterates allows the inductive process to start.

The algorithms used for constructing critical approximations of order $n$ from critical approximations of order $n-1$ are introduced in Chapter 4. A point $z$ belonging to $W^u(P_{a,m})$ is said to be a critical approximation of order $n$ if the image under the differential map of the tangent vector to $W^u(P_{a,m})$ at $z$ lies on the maximally contracting direction for the $n$-th iterate of the differential map. Critical approximations of order $n$ play, in the respective step of the inductive process, the same role as that of the critical point in the unimodal case. In order to prove expansiveness on the orbit of every critical approximation $z$, we have to control, at its returns, the distance between the respective iterate and the critical approximations of order equal to or lower than the order of $z$. In fact, it will be sufficient to control the distance to a certain critical approximation placed in a determined situation (tangential position). As in the unidimensional case, it will be possible to compare the exponential growth in the successive iterates of returns with the growth in the respective iterates of a critical approximation in tangential position (binding point) during a period of time which will also be called binding period.

To guarantee the existence of binding points, the afore-mentioned algorithms will need to adduce sufficient critical approximations and these approximations will have to be distributed in a suitable way, as their orders increase, on the different branches that $W^u(P_{a,m})$ defines in its continuous folding process. This adequate distribution is obtained by ordering the branches of $W^u(P_{a,m})$ by means of the concept of generation. In the fourth chapter it is also proved that from old critical approximations close new ones are constructed in such a way that each critical approximation generates a convergent sequence of critical approximations. By definition, critical points are the limits of these sequences. Expansiveness along the orbits of the image of every critical approximation yields expansiveness of the orbit of the respective critical point.

We argue by induction in Chapter 5 in order to rigorously define the recurrent process for constructing the critical set $C_n$ (set of critical approximations of order

$n-1$). In this chapter the concepts of returns, binding points and binding periods associated to each point bound to $C_n$ are introduced. Next, we deal with a new difficulty which arises in the treatment of the bidimensional problem: the folding phenomenon. When a return $\mu$ of a critical approximation $z_0$ takes place, that is, when $z_\mu = T^\mu_{\lambda,a,b}(z_0)$ is bound to $C_\mu$, the slope of the vector $\omega_\mu = DT^\mu_{\lambda,a,b}(z_1)(1,0)$ may be high. In this case, the study of the behaviour of $\omega_\mu$, which coincides with the study of the expansiveness of the orbit of $z_1$, is far from being an unidimensional problem. Nevertheless, it will be proved that, after a number $l$ of iterates, the slope of $\omega_{\mu+l} = DT^{\mu+l}_{\lambda,a,b}(z_1)(1,0)$ is very small again. The period $[\mu+1, \mu+l]$ will be called the folding period associated to the return $\mu$. During this period we only have knowledge about the evolution of the vector $h_\mu$, which corresponds to the horizontal component of $\omega_\mu$. The choice of iterates, on which the study of the behaviour of the vectors $\omega_j$ is replaced by the study of the behaviour of $h_j$, is called the splitting algorithm. Chapter 5 ends by establishing the set of inductive hypotheses which allow us to state the expansiveness, up to time $n$, of every point bound to $C_n$. In Chapters 6 and 7 these inductive hypotheses are proved at time $n$.

The main objective of Chapter 6 is to find, for every free return $n$ of a critical approximation $z_0 \in C_n$, a binding point $\zeta_0 \in C_n$ in tangential position. The loss of exponential growth at each return is estimated throughout the chapter. In Chapter 7 it is shown that these losses are compensated by the exponential growth, in the first iterates, of the vectors $h_j(\zeta_1)$, where $\zeta_0$ is the binding point associated to the considered return, $\zeta_1 = T_{\lambda,a,b}(\zeta_0)$ and $h_j(\zeta_1)$ is the respective vector given by the splitting algorithm related to the orbit of $\zeta_1$. Finally, an upper bound for the binding period associated to each return of every critical approximation is obtained.

Chapter 8 is the longest of this book and many references to previous chapters, especially to Chapter 2, are made there. The process of exclusion of parameters needed to deduce the $e^c$-expansiveness of the image of every critical point for a positive Lebesgue measure set is developed. The starting point of this chapter is the existence, with certain properties, of analytic continuations of the critical approximations. These properties are also inductively proved and they permit us to assume that the binding point is independent of $a$ (for small changes of $a$) as occurs in dimension one. To procede as in Chapter 2, it is necessary to redefine the sets $C_n$, taking new critical sets with, perhaps, less elements but still sufficient ones so as to ensure the existence of binding points in every return. On the other hand, the cardinal of $C_n$ is small enough so that, after removing the parameters for which the orbit of some critical approximation fails to be expansive, a positive Lebesgue measure set $E$ remains.

The global interpretation of the proof of Theorem B, developed throughout the final six chapters of the book, is not simple but a much simpler treatment does not seem to exist. The inductive method is used so frequently that the reader will have to control which step is applied each time. Furthermore, some concepts have to be redefined and, therefore, the validity of many arguments already proved have to be supervised later on. As in Chapter 2, special attention has to be paid to the relationship and order of choice of the different constants. Here, two new constants are needed: The constant $K$ introduced in Chapter 1 when the adequate unfolding of $f_{\lambda,a}$ is stated and the constant $b$. $K$ only depends on $\lambda$ and almost every constant arising from Chapter 2 depends on it. The constant $b$ depends on the remaining constants, is the last one to be selected and is chosen sufficiently small in each argument.

Once the expansiveness of the orbit of the critical points is achieved, which corresponds to the longest part of the proof of Theorem B, the density of the orbit of the critical point of generation zero is demonstrated for a set of parameters with positive Lebesgue measure, say $\epsilon$. Since $\epsilon$ does not depend on $m$, provided that $m$ is sufficiently large, we deduce the coexistence and persitence of any number of strange attractors.

The book ends with the exposition of some numerical experiments.

# Chapter 1

# SADDLE-FOCUS CONNECTIONS

In this chapter, we consider autonomous differential equations in $\mathbf{R}^3$

$$\begin{cases} x' = -\rho x + \omega y + P(x,y,z) \\ y' = -\omega x - \rho y + Q(x,y,z) \\ z' = \lambda z + R(x,y,z) \end{cases} \tag{1.1}$$

where $\rho, \lambda$ and $\omega$ are positive real numbers and $P$,$Q$ and $R$ are sufficiently smooth maps, vanishing together with their first order derivatives at the origin. Then, the origin $\boldsymbol{\theta}$ is a fixed point of the saddle-focus type, with eigenvalues $\lambda$ and $-\rho \pm \omega i$. Under linearizing assumptions, the flow in a neighbourhood of $\boldsymbol{\theta}$ is given by

$$\begin{cases} x(t) = e^{-\rho t}(x_0 \cos \omega t + y_0 \sin \omega t) \\ y(t) = e^{-\rho t}(-x_0 \sin \omega t + y_0 \cos \omega t) \\ z(t) = z_0 e^{\lambda t} \end{cases} \tag{1.2}$$

In a more general framework, we now consider a sufficiently smooth family of vector fields

$$f : (\mu, \mathbf{x}) \in I \times \mathbf{R}^3 \longrightarrow f(\mu, \mathbf{x}) \in \mathbf{R}^3,$$

where $I$ is an interval of parameters and, for each $\mu \in I$, $f(\mu, \mathbf{x})$ is a vector field of type (1.1). We also assume that for every $\mu \in I$, $f(\mu, \mathbf{x})$ is topologically conjugated to its linear part in a neighbourhood $U$ of $\boldsymbol{\theta}$, [6].

These families unfold interesting dynamic behaviours when there exists a homoclinic orbit to $\boldsymbol{\theta}$ for some value of the parameter $\mu$. Thus, we choose $P$,$Q$ and $R$ in such a way that $f(0, \mathbf{x})$ has a solution $\mathbf{p} : t \in \mathbf{R} \to \mathbf{p}(t) \in \mathbf{R}^3$ satisfying that $\mathbf{p}(t) \to \boldsymbol{\theta}$ as $t \to \pm\infty$. This solution defines a homoclinic orbit $\Gamma_0 = \{\mathbf{p}(t) : t \in \mathbf{R}\}$.

In order to describe the dynamics near $\Gamma_0$, a return map can be defined on a certain rectangle $\Pi_0$ contained in the set $\{(x,y,z) \in U : x = 0, y > 0, z > 0\}$. This