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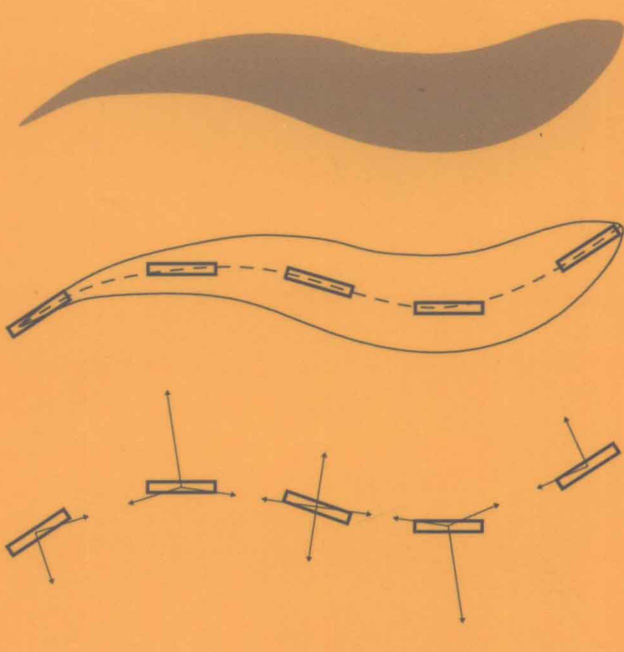
Alexander Y. Khapalov

Controllability of Partial Differential Equations Governed by Multiplicative Controls



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To Irina, Elena and Dasha

Foreword

In a typical mathematical model of a controlled distributed parameter process one usually finds either boundary or internal locally distributed controls to serve as the means to describe the effect of external actuators on the process at hand. However, these classical controls, entering the model equations as *additive* terms, are not suitable to deal with a vast array of processes that can change their principal intrinsic properties due to the control actions. Important examples here include (but not limited to) the *chain reaction*-type processes in biomedical, nuclear, chemical and financial applications, which can change their (reaction) rate when certain "catalysts" are applied, and the so-called "smart materials", which can, for instance, alter their frequency response.

The goal of this monograph is to address the issue of global controllability of partial differential equations in the context of *multiplicative (or bilinear) controls*, which enter the model equations as coefficients. The mathematical models of our interest include the linear and nonlinear parabolic and hyperbolic PDE's, the Schrödinger equation, and coupled hybrid nonlinear distributed parameter systems associated with the swimming phenomenon.

Pullman, WA, USA
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Alexander Khapalov

Preface

This monograph developed from the research conducted in 2001–2009 in the area of controllability theory of partial differential equations. The concept of controllability is a principal component of Control Theory which was brought to life in the 1950's by numerous applications in engineering, and has received the most significant attention both from the engineering and the mathematical communities since then.

A typical control problem deals with an evolution process which can be affected by a certain parameter, called control. Normally, the goal of a control problem is to steer this process from the given initial state to the desirable target state by selecting a suitable control among available options. If this is indeed possible, then one usually desires to achieve this steering while optimizing a certain criterion, solving what is called an optimal control problem.

Controllability theory studies the first part of the above-described control process. Namely, given any initial state, it studies the richness of the range of the mapping: *control* $\longrightarrow$ *state of the process* (at some moment of time).

Controllability theory was originally developed in the 1960's for the linear ordinary differential equations, governed by the additive controls. Later, since the 1970's it became the subject of keen interest for the researchers working in the area of partial differential equations as well. As a result, nowadays there exists a quite comprehensive controllability theory for the linear pde's governed by the additive controls which can act inside the system's space domain (locally distributed or point controls) or on its boundary (boundary controls). In such context, the respective mathematical methods are essentially the methods of the theory of linear operators, particularly, of the duality theory.

In this monograph, however, the subjects of interest are the multiplicative controls that enter the system equations as coefficients. Therefore, the aforementioned *control* $\longrightarrow$ *state of the process* mapping becomes highly nonlinear, even if the original pde is linear. This gives rise to the necessity of developing a different methodology for this type of controllability problems.

In this monograph we address this issue in the context of linear and semilinear parabolic and hyperbolic equations, as well as the Schrödinger equation. Particular attention is given to nonlinear swimming models. In the introduction we discuss the motivation for the use of multiplicative controls as opposed to the classical additive ones, and compare the mathematical methods involved in the respective studies.

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Pullman, WA, USA *Alexander Khapalov*
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Chapter 1
Introduction

1.1 Controlling PDE's: Why Multiplicative Controls?

In the mathematical models associated with controlled distributed parameter systems evolving in bounded domains two types of controls – boundary and internal locally distributed – are typically used. These controls enter the model as *additive* terms (having in mind that the boundary controls can be modeled by making use of suitable additive Dirac's functions) and have localized support. The latter is either a part of the boundary or a set within the system's space domain. Such control can, for example, be a source in a heat/mass-transfer process or a piezoceramic actuator placed on a beam. Publications in this area are so numerous that it is simply impossible to mention them all here – allow us just to refer the reader to our very limited bibliography below, associated mostly with the immediate content of this monograph and to the references therein.

In terms of applications it appears that the above-mentioned classical additive controls can adequately model only those controlled processes which do not change their principal physical characteristics due to the control actions. They rather describe the affect of various externally added "alien" sources or forces on the process at hand. This limitation, however, excludes a vast array of new and not quite new technologies, such as, for example, "smart materials" and numerous biomedical, chemical and nuclear chain reactions, which are able to change their principal parameters (e.g., the frequency response or the reaction rate) under certain purposefully induced conditions ("catalysts").

The intent of this monograph is to address the just-outlined issues in the context of global controllability of partial differential equations through the introduction and study of *multiplicative* (also known as *bilinear*) controls. These controls enter the system equations as *coefficients*. Accordingly they can change at least some of the principal parameters of the process at hand, such as, for example, a natural frequency response of a beam or the rate of a chemical reaction. In the former case this can be caused, e.g., by embedded "smart" alloys and in the latter case by various catalysts and/or by the speed at which the reaction ingredients are mechanically mixed. Our *main goal in this monograph is to introduce a new controllability methodology suitable for the study of linear and nonlinear pde's in the framework of multiplicative controls.*

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It is also important to notice that currently there are only very few publications available in the area of controllability of distributed parameter systems by means of multiplicative controls. This is in a very sharp contrast with the corresponding research in the framework of ordinary differential equations where many interesting results were obtained for the period of several decades now (see, e.g., the survey [5] and remarks on bibliography in the end of this introduction).

Let us give now several examples of important applications motivating the use of multiplicative controls in the framework of pde's. As the reader will see later, different types of pde's give rise to different concepts of controllability. (In other words: "What is the "right question" to ask here?)

Example 1.1 (The nuclear chain reaction). This chain reaction is characterized by the fact that the number of particles of the diffusing material increases by the reaction with the surrounding medium. For example, a nuclear fission results from the collision of neutrons with active uranium nuclei, which leads to the occurrence of new neutrons whose number is greater than one. These neutrons, in turn, react with active nuclei in the same way and hence the number of neutrons increases. If this process is treated approximately as a linear diffusion process, we arrive to the following (simplified) equation:

$$u_t = a^2 \Delta u + v(x,t)u, \tag{1.1}$$

where $u(x,t) \geq 0$ is the neutron density at point x at time t and $v > 0$, since the chain reaction is equivalent to the existence of sources of diffusing materials (neutrons) proportional to their concentration (neutron density).

In a nuclear plant the chain reaction is typically controlled by means of so-called "control rods," which in turn can absorb neutrons. In equation (1.1) this can be associated with the change of the value and sign of the coefficient v, which thus can be regarded as a multiplicative control.

Note that, if one wants to use the traditional additive control to describe the above fission model, then this would lead one to an equation like

$$u_t = a^2 \Delta u + v(x,t),$$

where $v(x,t)$ is the additive locally distributed control. In terms of applications, this type of control would amount to controlling the chain reaction by somehow adding into or withdrawing out of the chamber at will a certain amount of neutrons, which is not realistic.

A similar modeling approach applies to numerous biomedical, chemical and heat- and mass-transfer reactions and other processes (e.g., arising in the population dynamics and financial mathematics), involving various types of "catalysts." The corresponding models can also be nonlinear.

Example 1.2 (Biomedical applications). The following system of nonlinear equations models the interaction of leukocytes and "invading" bacteria in a cell (see [3] and, e.g., [60], p. 499).

Denote by u, y and z respectively the leukocyte, bacterial, and attractant concentrations. Then we have the following:

1. Bacteria diffuse, reproduce, and are destroyed when they come in contact with leukocytes:

$$y_t = \mu y_{xx} + (k_1 - k_2 u)y. \tag{1.2}$$

2. The chemoattractant is produced by bacterial metabolism and diffuses:

$$z_t = D z_{xx} + k_3 y. \tag{1.3}$$

3. The leukocytes are chemotactically attracted to the attractant and they die as they digest the bacteria, so that

$$u_t = J_x + (k_4 - k_5 y)u, \tag{1.4}$$

where $J = k_6 u + k_7 u z_x$ is the leukocyte flux.

In the above $k_i, i = 1, \ldots, 7$ and D are various coefficients, and equations (1.2)–(1.4) are complemented by a set of boundary conditions – we refer to [60] for details. The question of interest here is – *if and when the leukocytes can successfully fight against a bacterial invasion* (again see [60] and the references therein for different ways to approach this issue). One can try to analyze this very challenging nonlinear problem as a controllability one with the goal to achieve the steering to a suitable equilibrium by means of (some of) the coefficients k_i's treated as bilinear controls. The corresponding control actions can be interpreted, e.g., as the use of a drug to create the conditions (that is, to change the reaction rate) such that bacteria will die. (To the contrary, the use of traditional additive controls would mean that one has an option to add into or withdraw out of the given cell some leukocytes or bacteria, which does not seem realistic.)

Example 1.3 (Non-homogeneous bilinear system). Denote by $u(x,t)$ the temperature of a rod of unit length at point x at time t. Then the following model describes the heat-transfer in this rod in accordance with Newton's Law (e.g., [142]):

$$\begin{aligned} u_t &= u_{xx} + v(u - \theta(x,t)) \text{ in } Q_T = (0,1) \times (0,T), \\ u &= 0 \quad \text{in } \Sigma_T, \quad u\,|_{t=0} = u_0 \in L^2(\Omega). \end{aligned} \tag{1.5}$$

Here the term $v(u - \theta(x,t))$ describes the heat exchange between the rod and the surrounding medium of temperature $\theta(x,t)$. We can regard v as a bilinear control. It is known that v is proportional to the heat-transfer coefficient, which depends on the environment, the substance at hand, and its surface area. Note that in this example the "mathematical boundary" in the corresponding initial and boundary value problem is not the same as the actual physical boundary of the body.

Non-homogeneous models as in (1.5) arise also in many other applications dealing with the heat- and mass-transfer. If the heat (mass)-transfer involves fluids (air), the corresponding bilinear control v also depends on the speed of the fluid. The latter can be controlled in some applications by the induced magnetic field. Alternatively, the surface area can be changed when the substance at hand is a polymer (e.g., a planar array of gel fibers can be controlled to maximize the surface area exposed to the surrounding fluid). Also, we refer to the so-called "extended" surface applications ("smartly" added/controlled fins, pins, studs, etc.) when one wishes to increase/decrease the exchange between the source and ambient fluid.

Example 1.4 (Variable vibration response). An important practical example here is the SMA-composite beam containing NiTi fibres that can change its vibration response when heated by an electrical current (this can be interpreted as a variable load).

We found only two early references related to the area of bilinear controllability for the linear wave and beam equations. Namely, in the pioneering work [8] by J.M. Ball, J.E. Marsden, and M. Slemrod the approximate controllability of the rod equation

$$u_{tt} + u_{xxxx} + v(t)u_{xx} = 0$$

with hinged ends and of the wave equation

$$u_{tt} - u_{xx} + v(t)u = 0$$

with Dirichlet boundary conditions, where v is control (the axial load), was shown making use of the nonharmonic Fourier series approach under the additional "non-traditional" assumption that all the modes in the initial data are active. The results of [8] are discussed in detail in Chapter 9 below.

We also refer to [90] further exploring the ideas of [8] in the context of simultaneous control of the rod equation and Schrödinger equation.

Example 1.5 (Multiplicative controllability of the Schrödinger equation). This equation arises in such modern technologies as nuclear magnetic resonance, laser spectroscopy and quantum information science.

Let us give an example of a possible setup of multiplicative controllability problem for the Schrödinger equation due to Rouchon [131], Beauchard [11], and Beauchard and Coron [16].

Consider a quantum particle of mass m with potential V in a non-Galilean frame of absolute position $D(t)$ in R^1. It can be represented by a complex-valued wave function $\phi(t,z)$ which solves the following Schrödinger equation:

$$i\hbar\frac{\partial\phi}{\partial t}(t,z) = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial z^2}(t,z) + V(z-D(t))\phi(t,z). \tag{1.6}$$

With a suitable change of variables (see [16]) and assuming that $m = 1$ and $\hbar = 1$, equation (1.6) can be re-written as follows:

$$i\frac{\partial \psi}{\partial t}(t,x) \;=\; -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2}(t,x) \;+\; (V(x) - u(t)x)\psi(t,x), \tag{1.7}$$

where $u(t) = -\ddot{D}(t)$. Equation (1.7) describes the nonrelativistic motion of particle with potential V in a uniform electrical field $t \to u(t) \in R$, which can be viewed as a *multiplicative control*.

Recently, a substantial progress has been made in the study of controllability properties of (1.7), see [11, 12, 14, 16, 18, 26, 35, 124] and the references therein. We discuss these results in Part IV below.

Example 1.6 (Swimming phenomenon). The swimming phenomenon is undoubtedly among the most interesting mathematical problems arising in the fluid mechanics. We discuss this very intriguing phenomenon from the multiplicative controllability viewpoint in Part III.

1.2 Additive Controls vs Multiplicative Controls: Methodology

Let us try to highlight, in an informal setting, some of the principal differences between additive and multiplicative controls in terms of approach to the concept of controllability itself and of the mathematical methods which are typically used to study the respective properties.

1.2.1 Additive Controls

Consider the following abstract evolution equation governed by an *additive control*:

$$\frac{dy(t)}{dt} = Ay(t) + Bv(t), \quad t \in (0,T),\, T > 0, \tag{1.8}$$
$$y(0) = y_0 \in H.$$

Here A is a self-adjoint operator with dense domain in the real Hilbert space H, generating a C^o semigroup of bounded linear operators on H, $v \in L^2((0,T);V)$ is control with values in the Hilbert space V and $B : V \to H$ is a linear bounded operator. Let us assume that we are interested in the study of controllability properties of solutions to (1.8) in $C([0,t_0];H)$.