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# Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics

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# Preface

This book is intended as an introductory text on the subject of Lie groups and algebras and their role in various fields of mathematics and physics. It is written by and for researchers who are primarily analysts or physicists, not algebraists or geometers. Not that we have eschewed the algebraic and geometric developments. But we wanted to present them in a concrete way and to show how the subject interacted with physics, geometry, and mechanics. These interactions are, of course, manifold; we have discussed many of them here—in particular, Riemannian geometry, elementary particle physics, symmetries of differential equations, completely integrable Hamiltonian systems, and spontaneous symmetry breaking.

Much of the material we have treated is standard and widely available; but we have tried to steer a course between the descriptive approach such as found in Gilmore and Wybourne, and the abstract mathematical approach of Helgason or Jacobson. Gilmore and Wybourne address themselves to the physics community whereas Helgason and Jacobson address themselves to the mathematical community. This book is an attempt to synthesize the two points of view and address both audiences simultaneously. We wanted to present the subject in a way which is at once intuitive, geometric, applications-oriented, mathematically rigorous, and accessible to students and researchers without an extensive background in physics, algebra, or geometry.

Our operating assumption is that our reader has a good founding in linear algebra and some familiarity with the basic ideas of group theory which are traditionally taught at the undergraduate level. For example, we assume he/she knows what a normal subgroup is and how a quotient group is formed. We also assume the reader is familiar with such elementary topological notions as continuity, compactness, and simple connectivity. We discuss in some detail the tensor calculus on manifolds, but we have avoided the technicalities of differentiable structures on manifolds. Some knowledge of differential geometry and tensor analysis would be helpful for a complete understanding of some of the specialized topics in geometry and mechanics.

Whether this book is a public success or not, it has certainly been a private one. In bridging the "jargon gap" between mathematician and physicist we have personally gained a deeper understanding of the material. One outcome of this process, for example, is our presentation of the material in Chapter 5 traditionally known as "tensor calculus on manifolds." We have presented that material here in the physicist's language of frame invariance, not the mathematician's notation of pull-backs ($\varphi^*$ and $\varphi_*$ and all that).

We have tried to introduce and use the language of differential forms in a way that will induce the physicist to learn the approach. The calculus of differential forms has had many promoters—the chief one being Cartan himself, and that alone should be ample recommendation. We discuss in some detail the Maurer–Cartan forms on a Lie group and their applications. For example, in §26 (Geometry "à la Cartan") we explain Cartan's derivation of the structure equations of Riemannian manifolds from the Maurer–Cartan equations of their isometry groups. This leads, for example, to the structure equations of surfaces of constant curvature, as well as to the structure equations of a surface embedded in $R^3$. It touches upon Cartan's brilliant theory of symmetric spaces, a topic which we have treated all too briefly.

The use of differential forms has not yet achieved widespread acceptance by the physics or applied mathematics community, but appears to be gaining ground. Differential forms, being dual to vector fields, are necessarily more abstract and less easily grasped as fundamental intuitive objects. Yet the exterior differential calculus is an excellent form of bookkeeping that takes into account orientation, provides the correct language for multidimensional integration, and is "frame independent"—that is, invariant under arbitrary coordinate transformations. It is perhaps worthwhile comparing the situation with the acceptance of Gibbs' vector notation, which (according to Struik in his text *Differential Geometry*) "after years of competition with other notations seems to have won the day. . . ." This in 1950! It appears that differential forms are still competing but have not yet won the day.

We have benefited from many discussions with our colleagues, in particular Leon Green, Robert Ellis, Paul Garrett, and Sid Webster at the University of Minnesota. We also thank the students who took the course at Minnesota, who asked so many good questions, and made so many valuable comments: Russell Brown, Susan Fischel, David Gregg, Gerald Warnecke, and Victor Zurkowski.

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# Contents

PART A

# LIE GROUPS AND ALGEBRAS

CHAPTER 1

# Lie Groups

## 1. Continuous Groups; Covering Groups

Sophus Lie (1842–1899) and Felix Klein (1849–1925) were students together in Berlin in 1869–70 when they conceived the notion of studying mathematical systems from the perspective of the transformation groups which left these systems invariant. Thus Klein, in his famous *Erlanger* program, pursued the role of finite groups in the studies of regular bodies and the theory of algebraic equations, while Lie developed his notion of continuous transformation groups and their role in the theory of differential equations. Lie's work was a *tour de force* of the 19th century, and today the theory of continuous groups is a fundamental tool in such diverse areas as analysis, differential geometry, number theory, differential equations, atomic structure, and high energy physics. This book is devoted to a careful exposition of the mathematical foundations of Lie groups and algebras and a sampling of their applications in differential equations, applied mathematics, and physics.

In this first chapter you will be introduced to a variety of important Lie groups, together with some of their properties.

A topological group is a group which is also a topological space (so that ideas such as continuity, connectedness and compactness apply) in which the group operations are continuous. A Lie group is a topological group which is also an analytic manifold on which the group operations are analytic. We will make this notion more precise later on.

The simplest example of a Lie group is the real line $\mathbb{R}^1$ with ordinary addition as the group operation. Similarly, $\mathbb{R}^n$ with the usual vector addition is a commutative (abelian) Lie group. Continuous matrix groups, or more generally, continuous groups of linear transformations of a vector space, are called *linear* Lie groups. For example, the set of all non-singular $n \times n$ matrices forms the group known as $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ depending on whether the entries are real or complex. The subset of all $n \times n$ matrices with determinant 1 forms a group called the *unimodular* or special linear group, which is denoted by $SL(n, \mathbb{R})$ or $SL(n, \mathbb{C})$. The orthogonal group, $O(n)$, is the group of $n \times n$ of matrices that satisfy $AA^t = 1$. These are a few examples of the so-called "classical" groups; we shall give a complete list at the end of this chapter.

The matrices

$$L_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \qquad a \in \mathbb{R}$$

form a linear group; and moreover,

$$L_a L_b = L_{a+b}.$$

This group is thus *isomorphic* to $\mathbb{R}^1$ and forms a *representation* of $\mathbb{R}^1$ by $2 \times 2$ matrices. In general, a *representation* of a group $\mathfrak{G}$ on a vector space $V$ is a homomorphism from $\mathfrak{G}$ into the invertible linear transformations of $V$. That is $a \to t_a$ in such a way that

$$a \circ b \to t_a t_b.$$

These representations need not always be matrix representations. For example we may represent $\mathbb{R}^1$ on the infinite dimensional vector space $C^\infty(\mathbb{R})$ (the infinitely differentiable functions on the line) by

$$(T_a f)(x) = f(x + a).$$

The idea of representation helps to clarify the subtle but sometimes important distinction between an abstract group and a variety of its realizations. Thus $\mathbb{R}^1$, the set of matrices $L_a$, the translations $T_a$, and the geometric translations along $\mathbb{R}^1$ itself are all distinct but isomorphic representations of the same abstract group.

An example of a non-abelian group is the group of upper triangular matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \qquad a, b, c \in \mathbb{R}.$$

This is isomorphic to the Heisenberg group which plays a fundamental role in quantum mechanics (see Chapter 4).

The matrices

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \theta \in \mathbb{R}$$

form a group, and $R(\theta)R(\gamma) = R(\theta + \gamma)$. These are the rotations of the plane, the group $SO(2)$. This group may also be realized as $S^1$, the unit circle in the complex plane with multiplication as the group operation.

The groups $\mathbb{R}^1$ and $S^1$ are intimately related, for the mapping $\rho$: $\mathbb{R}^1 \to S^1$ defined by $\rho(x) = e^{2\pi i x}$ is a continuous homomorphism from $\mathbb{R}^1$ onto $S^1$. The transformation is locally one-to-one; (see Fig. 1.1) but globally it is infinite-to-one, for all points $x + n$ (with $n$ an integer) map onto the same point $e^{2\pi i x}$ in $S^1$. The kernel of $\rho$ is $Z$, the discrete group of integers, and $S^1$ is isomorphic with the quotient group $\mathbb{R}^1/Z$. We say that each $z$ *lifts* to an infinity of points in $\mathbb{R}^1$ (see Fig. 1.1). Moreover, the unclosed path $0 \leq t \leq 1$ in $\mathbb{R}^1$ is mapped onto the closed path $z(t) = e^{2\pi i t}$ in $S^1$. The latter cannot be shrunk to a point while remaining in $S^1$. Yet any closed path in $\mathbb{R}^1$ can be shrunk down to a single point. We say that $\mathbb{R}^1$ is simply connected while $S^1$ is not.

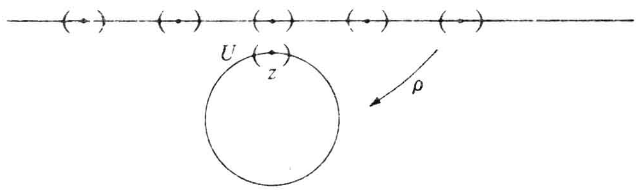



Figure 1.1. The neighborhood $U$ of $z \in S^1$ lifts to an infinite collection of disjoint intervals in $\mathbb{R}^1$, each of which is mapped 1–1 onto $U$ by $\rho$. We say $\rho$ is locally one-to-one.

In this example, $\mathbb{R}^1$ is the *universal covering group* of $S^1$. The universal covering group of a connected topological group $G$ is a simply connected topological group $\hat{G}$ together with a continuous homomorphism $\rho$ from $\hat{G}$ onto $G$ which is locally one-to-one. Such a universal covering group exists for every connected topological group, and in particular for every Lie group, and is unique up to isomorphism. Moreover $\hat{G}/\mathrm{Ker}\,\rho \simeq G$.

Universal covering groups, their representations, and their homomorphisms into non-simply connected groups, are fundamental to our understanding of symmetry in quantum mechanics. (See Section 1 of Wigner's 1939 paper for a discussion of this point, Wigner [1].) For example, particles with half-integer spin—such as electrons, quarks, $^3$He nuclei—transform according to *spinor representations* of the rotation and Lorentz groups. These are double valued representations of the geometrical symmetry groups, but are single-valued representations of the covering groups.

In this chapter we shall construct the universal covering groups and the covering homomorphisms for the rotation and Lorentz groups. Readers interested in a more careful treatment of covering groups can profitably consult Pontryagin, Chevalley, or Singer and Thorpe.

## Exercises

1. Construct the covering homomorphism and the covering groups of $S^1$ having the form $R^1/pZ$ where $pZ$ is the set of integral multiples of the integer $p$. $\mathbb{R}^1$ is the universal covering group of these groups. Construct a five valued representation of $S^1$.

2. What is the universal covering group of the torus, $T^n = S^1 \times S^1 \times \cdots \times S^1$, the group whose elements are $(z_1, z_2, \ldots, z_n)$ with $|z_i| = 1$?

3. A matrix $U$ is called unitary if $UU^* = I$. Show that the $n \times n$ unitary matrices form a group, called $U(n)$. Show that the unitary matrices of determinant 1 form a subgroup, called $SU(n)$. Show that $SU(2)$ consists of all matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 = 1.$$

## 2. The Rotation Group in $R^3$

The group of rotations of three dimensional Euclidean space is called $SO(3)$. Every rotation $R \in SO(3)$ can be parameterized by an axis of rotation $\hat{n}$ and the angle $\theta$ of rotation about this axis: $R = (\hat{n}, \theta)$. The axis requires two angles $(\alpha, \beta)$ for its specification, so three parameters are needed to specify a general rotation: $SO(3)$ is a three parameter group. A useful visualization of the elements of $SO(3)$ is to picture a solid ball of radius $\pi$. A point $P$ inside the ball a distance $\theta$ from the origin, 0, represents the counterclockwise rotation about the axis $\overrightarrow{0P}$ by an angle $\theta$. Since the parameters range over a compact set, $SO(3)$ is a *compact group*; the origin 0 corresponds to the identity transformation.

A moment's reflection shows that we really do only require a ball of radius $\pi$, not $2\pi$, but that antipodal points on the surface represent the same rotation. This enables us to show that $SO(3)$ is not a simply connected space—see path #2 in Fig. 1.2: it is not contractible to a point. We will find the simply connected covering group of $SO(3)$ later in this chapter.

Another realization of $SO(3)$ follows from the observation that rotations are linear transformations of $\mathbb{R}^3$ that preserve the inner product

$$(a, b) = \sum_{i=1}^{3} a^i b^i.$$

If $R$ is the matrix of the linear transformation then

$$(Ra, Rb) = (a, b),$$

so

$$R^t R = \mathbb{1}.$$

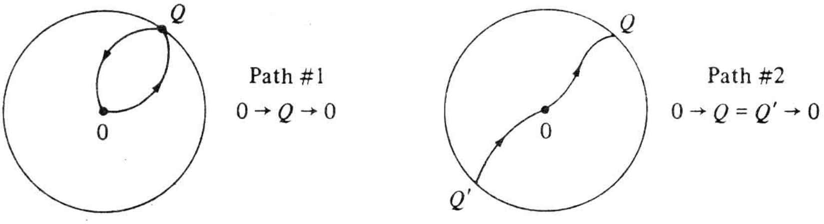



Figure 1.2

This relation shows that $R$ is non-singular, so

$$R^t = R^{-1}.$$

These are the *orthogonal* matrices. We have actually gotten a bit more than $SO(3)$ this way, for $R = -\mathbb{1}$ is orthogonal but is an *inversion* rather than a rotation: it reverses orientation of space. The group of orthogonal $3 \times 3$ matrices is denoted by $O(3)$, while the rotations are $SO(3)$, the special orthogonal matrices with determinant $+1$. The group $O(3)$ is not connected but is the union of the sets $\{R \in SO(3)\}$ and $\{-R, R \in SO(3)\}$. The identity, $\mathbb{1}$, is in $SO(3)$, and since $SO(3)$ is connected (although not simply connected) it is called the *connected component of the identity* in $O(3)$.

For later reference we display the orthogonal matrices corresponding to counter clockwise rotations of $\mathbb{R}^3$ about the coordinate axes:

$$R_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix},$$

$$R_2(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}, \tag{1.1}$$

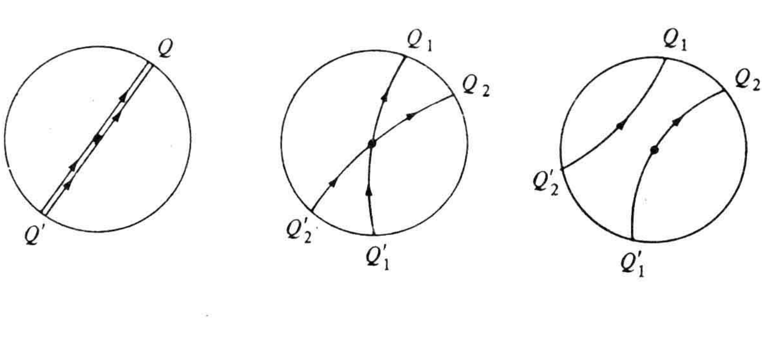
$$R_3(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercises on $O(3)$

1. Show that $A \to \det A$ is a homomorphism of $O(3)$ into the group $Z_2 = \{+1, -1\}$ with multiplication as the group operation. What is the kernel of this homomorphism? What are the cosets of $O(3)$ modulo that kernel? Show that $O(3)$ is not connected.

2. Given $A \in SO(3)$ show that 1 is always an eigenvalue. What is the geometrical significance of the eigenvector?

3. Show that $SO(3)$ is "doubly connected" in the sense that path #2 (Fig. 1.2) traversed twice can be shrunk to a point. Begin with the sequence of deformations shown below.



4. Projective 3-space, $P_3(\mathbb{R})$, is the set of lines through the origin in $\mathbb{R}^4$. Open sets are open cones of lines. Show that $P_3(\mathbb{R})$ is homeomorphic to the three sphere $S^3$ in $\mathbb{R}^4$ with antipodal points identified. Then show that this sphere is homeomorphic to the solid ball in $R^3$ with antipodal points on the surface identified, that is, to $SO(3)$.

## 3. The Möbius Group

The linear fractional transformations, or Möbius transformations, $M$, form the group of conformal transformations which map the extended complex plane one-to-one onto itself. An element of this group is a transformation $m: C \to C$ with

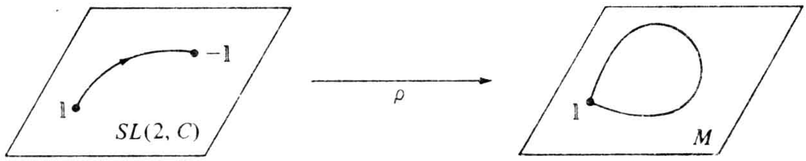
$$m(z) = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0.$$

The condition $ad - bc \neq 0$ assures that the transformation is invertible. There is a homomorphism $\rho$ from $GL(2, C)$ into $M$ given by

$$\rho: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to m(z) = \frac{az + b}{cz + d}.$$

The reader may check that $\rho$ indeed is a homomorphism. For all $\lambda$ the matrices $\lambda\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $GL(2, C)$ go into the *same* Möbius transformation. Since $\det\left[\lambda\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \lambda^2 \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we may always chose $\lambda$ in two ways so that the determinant of $\lambda\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $+1$. Thus each Möbius transformation is covered by two matrices of determinant one, i.e. by two elements of $SL(2, C)$.

Our homomorphism shows that $M$ cannot be simply connected, for a path from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in $SL(2, C)$ is mapped by $\rho$ onto a closed path in $M$ which cannot be shrunk to a point. $SL(2, C)$ is a covering group of $M$, and $SL(2, C)$ is simply connected (see exercise 6, p. 18), so $SL(2, C)$ is the universal covering group of $M$.



The homomorphism from $SL(2, C)$ to the Möbius transformations was obtained by Klein and Cayley in the following way. Let $u$ and $v$ be complex numbers and suppose

$$u' = au + bv$$

$$v' = cu + dv.$$

Putting $z = u/v$ and $w = u'/v'$ we obtain

$$w = \frac{az + b}{cz + d}.$$

The two component vectors $\begin{pmatrix} u \\ v \end{pmatrix}$ have come to be called *spinors* in physics and are used to describe particles of spin 1/2.

The set of Möbius transformations with $a, b, c, d$ *real* form a subgroup of Möbius transformations which preserve the upper half plane: if

$$w = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{R},$$

then $\operatorname{Im} w > 0$ whenever $\operatorname{Im} z > 0$. The associated matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ form the subgroup $SL(2, R)$ of $SL(2, C)$.

The upper half plane with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{dz\, d\bar{z}}{(\operatorname{Im} z)^2}$$

is known as the Poincaré half plane. It is a two dimensional Riemannian manifold with constant Gaussian curvature $K = -1$ (see Singer and Thorpe, Chapter 7). The Möbius transformations we discussed above, i.e. those which come from $SL(2, R)$, are in fact isometries of this manifold. That is, they preserve the metric tensor: if

$$w = \frac{az + b}{cz + d} \qquad a, b, c, d \in \mathbb{R}, \qquad ad - bc = 1$$

then

$$\frac{dw\, d\bar{w}}{(\operatorname{Im} w)^2} = \frac{dz\, d\bar{z}}{(\operatorname{Im} z)^2}.$$

Exercises

1. Prove that $SL(2, R)$ preserves the upper half plane, and the metric given above.
2. Which $SL(2, R)$ transformations preserve the norm $|u|^2 + |v|^2$ of a spinor $\begin{pmatrix} u \\ v \end{pmatrix}$?
3. Why isn't $GL(2, C)$ the universal covering group of $M$?