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P. Concus R. Finn D. A. Hoffman
Editors

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Preface

This volume derives from a workshop on differential geometry, calculus of variations, and computer graphics at the Mathematical Sciences Research Institute in Berkeley, May 23–25, 1988. The meeting was structured around principal lectures given by F. Almgren, M. Callahan, J. Ericksen, G. Francis, R. Gulliver, P. Hanrahan, J. Kajiya, K. Polthier, J. Sethian, I. Sterling, E. L. Thomas, and T. Vogel. The divergent backgrounds of these and the many other participants, as reflected in their lectures at the meeting and in their papers presented here, testify to the unifying element of the workshop's central theme.

Any such meeting is ultimately dependent for its success on the interest and motivation of its participants. In this respect the present gathering was especially fortunate. The depth and range of the new developments presented in the lectures and also in informal discussion point to scientific and technological frontiers being crossed with impressive speed. The present volume is offered as a permanent record for those who were present, and also with a view toward making the material available to a wider audience than were able to attend.

We wish to express our appreciation to Irving Kaplansky, Director of MSRI, for his dedicated, personal role in making the workshop a reality, and to the MSRI and Lawrence Berkeley Laboratory staff for their expert assistance on the many details of the arrangements. The workshop received generous financial support from MSRI and LBL, and thereby from the National Science Foundation and the Department of Energy, which we gratefully acknowledge.

We are indebted to Alvy Ray Smith for joining us on the organizing committee.

Paul Concus (Chairman)
Robert Finn
David Hoffman

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Multi-functions Mod ν

FREDERICK J. ALMGREN, JR.

Abstract. We extend the theory of current valued multi-functions to multi-functions mod ν.

§1. Introduction.

This note is an introduction to and advertisement for current valued multi-functions. It is also an announcement and preliminary exposition of a corresponding new theory of multi-functions mod ν. Such multi-functions provide a novel perspective on various important problems in geometry and the calculus of variations in the context of geometric measure theory. They are indispensable in proofs of several basic theorems in geometric measure theory [**5**][**9**] and have provided a setting for both original and alternative proofs of other pivotal results [**2**][**15**]. They additionally are a device for computational geometry and its associated graphics [**3**][**7**] (see also [**8**]). The first use of such functions in the context of geometric measure theory was in [**1**]. The basic reference for current valued multi-functions is [**2**] (where they are called multiple-valued functions); expository accounts appear in [**3**][**7**]. Applications in complex analysis are set forth in [**4**][**12**][**16**]. One important example of a multi-function arises when one regards the roots of a (real or complex) polynomial as a function of its coefficients; see, for example, [**10**, 4.3.12]. The general theory of current slicing appears in [**10**, 4.3]; [**13**] is an illustrated introduction to geometric measure theory in general.

We here set forth multi-functions in the context of polyhedral chains with coefficients either in the integers or in the integers modulo ν. The extension to general rectifiable currents or flat chains mod ν is then relatively straightforward for those knowledgeable in the subtleties of geometric measure theory and perhaps irrelevant for the development of discrete algorithms in the geometric calculus of variations.

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Associated with piecewise affine mappings f, $g: \mathbf{R}^n \to \mathbf{R}^N$ are induced chain mappings

$$f_\sharp,\, g_\sharp: \mathbf{IP}_*(\mathbf{R}^n) \to \mathbf{IP}_*(\mathbf{R}^N)$$

of degree 0 between chain complexes of integral polyhedral chains. In particular these mappings respect the additive structure of $\mathbf{IP}_*(\mathbf{R}^n)$. Since $\mathbf{IP}_*(\mathbf{R}^N)$ is also a group under addition we can define an addition of such chain mappings, e.g.

$$(zf_\sharp + wg_\sharp): \mathbf{IP}_*(\mathbf{R}^n) \to \mathbf{IP}_*(\mathbf{R}^N), \quad (zf_\sharp + wg_\sharp)(T) = z(f_\sharp T) + w(g_\sharp T),$$

whenever $z, w, \in \mathbf{Z}$ and $T \in \mathbf{IP}_k(\mathbf{R}^n)$. Caution: $(f_\sharp + g_\sharp)$ does not equal $(f+g)_\sharp$ because this latter addition uses addition in $\mathbf{R}^N$ rather than in $\mathbf{IP}_*(\mathbf{R}^N)$. It turns out that the mapping $(zf_\sharp + wg_\sharp)$ is determined by the its action on single point masses $[\![p]\!]$. The multi-function F associated with this sum is

$$F: \mathbf{R}^n \to \mathbf{IP}_0(\mathbf{R}^n), \quad F(p) = (zf_\sharp + wg_\sharp)\,[\![p]\!] = z[\![f(p)]\!] + w[\![g(p)]\!], \quad p \in \mathbf{R}^n.$$

In order to induce a chain mapping on $\mathbf{IP}_*(\mathbf{R}^n)$ it is not necessary that our F come from such f's and g's. Indeed, general Lipschitz continuous mappings $\mathbf{R}^n \to \mathbf{IP}_0(\mathbf{R}^n)$, locally of bounded mass, also naturally induce chain mappings on $\mathbf{IP}_*(\mathbf{R}^n)$. I do not know the extent to which general Lipschitz multi-functions can be approximated by sums of "single valued" maps. Multi-functions were studied extensively in [**2**]. As indicated above we will review some of these notions in the present context of polyhedral chains and piecewise affine multi-functions and show that they remain valid for polyhedral chains with coefficients in the integers modulo ν when the F's take values in zero dimensional polyhedral chains mod ν

§2. Simplicial decompositions and piecewise affine mappings.

(1) By a 0 *simplex* in $\mathbf{R}^n$ we mean a point in $\mathbf{R}^n$. By a k *simplex* in $\mathbf{R}^n$ we mean the convex hull Δ of some collection of $k+1$ points $p_0, p_1, \ldots, p_k$ in $\mathbf{R}^n$ which do not lie in any $(k-1)$ dimensional affine subspace of $\mathbf{R}^n$. The points $p_0, p_1, \ldots, p_k$ are called the *vertices* of Δ.

(2) When we say that $\mathbf{SX}_* = \mathbf{SX}_0 \cup \mathbf{SX}_1 \cup \ldots \cup \mathbf{SX}_n$ is a *simplicial decomposition* of $\mathbf{R}^n$ we mean that

(i) $\mathbf{SX}_0$ is a discrete set of points in $\mathbf{R}^n$ (necessarily countably infinite and unbounded).

(ii) $\mathbf{SX}_n$ is a family of n simplexes Δ in $\mathbf{R}^n$ whose union is $\mathbf{R}^n$ such that the vertices of each such simplex Δ are exactly those members of $\mathbf{SX}_0$ lying within Δ.

(iii) For $k = 1, \dots, n-1$ each $\mathbf{SX}_k$ is the family of all k simplexes Δ_k in $\mathbf{R}^n$ whose vertices are a subset of the vertices of some member Δ_n of $\mathbf{SX}_n$.

(3) $\mathcal{H}^k$ denotes Hausdorff's k dimensional measure in $\mathbf{R}^n$. $\mathcal{H}^k(\Delta_k)$ agrees with another other reasonable definition of the k dimensional area of a k simplex Δ_k.

(4) A function $f: \mathbf{R}^n \to \mathbf{R}^N$ is called *affine on A* provided the restriction of f to A is also the restriction to A of some affine mapping $\mathbf{R}^n \to \mathbf{R}^N$. A function $f: \mathbf{R}^n \to \mathbf{R}^N$ is called *piecewise affine* provided there is some simplicial decomposition $\mathbf{SX}_*$ of $\mathbf{R}^n$ such that f is affine on each n simplex in $\mathbf{SX}_n$. Piecewise affine functions are continuous.

(5) It is sometimes useful to factor a mapping through its graph space. With this in mind, we here fix

$$\Sigma: \mathbf{R}^n \times \mathbf{R}^N \to \mathbf{R}^n, \qquad \Pi: \mathbf{R}^n \times \mathbf{R}^N \to \mathbf{R}^N$$

as projections on the factors indicated and define

$$1_{\mathbf{R}^n} \bowtie f: \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^N, \qquad (1_{\mathbf{R}^n} \bowtie f)(x) = (x, f(x))$$

whenever f maps $\mathbf{R}^n$ to $\mathbf{R}^N$. Clearly

$$\Sigma \circ (1_{\mathbf{R}^n} \bowtie f) = 1_{\mathbf{R}^n} \quad \text{and} \quad \Pi \circ (1_{\mathbf{R}^n} \bowtie f) = f.$$

The point of such factorizations is that $(1_{\mathbf{R}^n} \bowtie f)$ is one-to-one while Π is infinitely differentiable.

§3. Common refinements of simplicial decompositions.

The following proposition is useful when one wishes to make definitions based on simplicial decompositions.

Proposition.

(1) *Suppose $\mathcal{S}$ is any finite collection of simplexes in $\mathbf{R}^n$ of various dimensions. Then there is some simplicial decomposition $\mathbf{SX}_*$ of $\mathbf{R}^n$ such that each k dimensional member of $\mathcal{S}$ is the union of (finitely many) members of $\mathbf{SX}_k$.*

(2) *Corresponding to each collection $\mathbf{SX}_*^{(1)}, \mathbf{SX}_*^{(2)}, \ldots, \mathbf{SX}_*^{(M)}$ of simplicial decompositons of $\mathbf{R}^n$ there is some simplicial decomposition $\mathbf{SX}_*$ of $\mathbf{R}^n$ such that each member of any $\mathbf{SX}_k^{(j)}$ is a union of (necessarily finitely many) members of $\mathbf{SX}_k$.*

§4. General currents.

For our purposes a k (dimensional) *current* in $\mathbf{R}^n$ is a continuous real valued linear function on the real vector space of infinitely differentiable k forms $\varphi: \mathbf{R}^n \to \wedge^k \mathbf{R}^n$ having compact support. The Euclidean current $\mathbf{E}^n$ assigns to each n form φ the number $\int_{\mathbf{R}^n} \langle \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n, \varphi \rangle \, d\mathcal{L}^n$. With several exceptions (e.g. Euclidean currents), the currents T we will consider have compact support, denoted spt$T$. The boundary of a general k current T is the $k-1$ current ∂T defined by setting $\partial T(\omega) = T(d\omega)$, e.g. in geometrically reasonable cases, Stokes's theorem becomes a definition. For proper smooth mappings $f: \mathbf{R}^n \to \mathbf{R}^N$ the image current $f_\sharp T$ is defined by setting $(f_\sharp T)(\psi) = T(f^\sharp \psi)$; this implies, in particular, that $f_\sharp \circ \partial = \partial \circ f_\sharp$ since $f^\sharp \circ d = d \circ f^\sharp$.

Integration of differential forms over Lipschitz singular chains (in the sense of algebraic topology) enables one to regard such chains as currents; one obtains the same current if a chain is subdivided or if it is changed by an orientation preserving domain reparametrization. The so called *integral currents* $\mathbf{I}_*(\mathbf{R}^n)$ are a strong closure (in the space of general currents) of the currents obtained by integration over Lipschitz singular chains (see the original definition [**11**, 3.7]) and, as such, have similar combinatorial and homological properties.

§5. Oriented simplexes and polyhedral chains.

A 0 current T in $\mathbf{R}^n$ is called an *oriented* 0 *simplex* provided $T = [\![p]\!]$ for some point p in $\mathbf{R}^n$; this means $T(\varphi) = [\![p]\!](\varphi) = \varphi(p)$ for each smooth function φ (with compact support). A k current T ($k > 0$) is called an *oriented* *k simplex* provided there is a k simplex Δ having vertices $p_0, p_1, \ldots, p_k$ such that

$$T = [\![p_0, p_1, \ldots, p_k]\!] = \mathbf{t}(\Delta, 1, \xi).$$

This notation means that, for each smooth differential k form φ having compact support,

$$T(\varphi) = [\![p_0, p_1, \ldots, p_k]\!](\varphi) = \mathbf{t}(\Delta, 1, \xi)(\varphi) = \int_{x\in\Delta} \langle \xi(x), \varphi(x)\rangle \, d\mathcal{H}^k x;$$

here ξ is the unit simple k vector valued orientation function on Δ which assigns to each point x in Δ the k vector $\xi(x) = \eta/|\eta|$ where $\eta = (p_1 - p_0) \wedge (p_2 - p_0) \wedge \ldots \wedge (p_k - p_0)$. If $\Delta(1), \ldots, \Delta(M)$ are distinct members of some **SX** with $\bigcup_i \Delta(i) = \Delta$ then clearly

$$\mathbf{t}(\Delta, 1, \xi)(\varphi) = \sum_{i=1}^{M} \mathbf{t}(\Delta(i), 1, \xi)(\varphi),$$

i.e. as currents oriented simplexes are identified with the sum of subdivisions. The abelian group generated by all oriented k simplexes within the vector space of general k currents is called the group of *integral polyhedral k chains* in $\mathbf{R}^n$ and is here denoted $\mathbf{IP}_k(\mathbf{R}^n)$. If $T = [\![p_0, p_1, \ldots, p_k]\!] = \mathbf{t}(\Delta, 1, \xi)$ as above, then our notational conventions are illustrated by the requirements that

$$3T = 3[\![p_0, p_1, \ldots, p_k]\!] = \mathbf{t}(\Delta, 3, \xi)$$

and

$$-7T = -7[\![p_0, p_1, \ldots, p_k]\!] = 7[\![p_1, p_0, p_2, \ldots, p_k]\!] = \mathbf{t}(\Delta, 7, -\xi).$$

The middle entries in the right most expressions above, e.g. 3 and 7, are always positive since they represent a surface density. General *rectifiable k currents* are of the form $T = \mathbf{t}(S, \theta, \xi)$ which means

$$T(\varphi) = \int_{x\in S} \langle \xi(x), \varphi(x)\rangle \, \theta(x) \, d\mathcal{H}^k x$$

for each φ; here S is a k rectifiable subset of $\mathbf{R}^n$ oriented by ξ and having integer valued density function θ. The *mass* of T is the number

$$\mathbf{M}(T) = \sup\{T(\varphi) \colon \|\varphi\| \le 1\} = \int_S \theta \, d\mathcal{H}^k.$$

It turns out that the integral currents mentioned above are precisely those rectifiable currents whose current boundaries are also rectifiable.

According to Stokes's theorem

$$\partial[\![p_0, p_1, \dots, p_k]\!] = \sum_{j-0}^{k} (-1)^j [\![p_0, \dots, p_{j-1}, p_{j+1}, \dots, p_k]\!]$$

for each oriented k simplex $[\![p_0, p_1, \dots, p_k]\!]$. This current boundary operator is a homomorphism $\partial\colon \mathbf{IP}_k(\mathbf{R}^n) \to \mathbf{IP}_{k-1}(\mathbf{R}^n)$ $(k > 0)$. The collection of polyhedral chains of all dimensions forms a chain complex in the obvious way. It is useful to extend this chain complex by adding a homomorphism

$$\partial\colon \mathbf{IP}_0(\mathbf{R}^n) \to \mathbf{Z}, \quad \partial T = T(1), \quad \text{e.g. } \partial\left(\sum_i z_i[\![p_i]\!]\right) = \sum_i z_i.$$

It follows readily from Proposition 3 that for each fixed integral polyhedral k chain T in $\mathbf{IP}_k(\mathbf{R}^n)$ there exists a simplicial decomposition $\mathbf{SX}_* = \bigcup_\alpha \mathbf{SX}_\alpha$ of $\mathbf{R}^n$ such that

$$T = \sum_{i=1}^{N} \mathbf{t}(\Delta_k(i), \theta(i), \xi(i)) \qquad (\text{some } N, \Delta_k(i), \theta(i), \xi(i))$$

where $\Delta_k(1), \dots, \Delta_k(N)$ are distinct members of $\mathbf{SX}_k$. If $k \geq 1$ we can then write

$$\partial T = \sum_{j=1}^{M} \mathbf{t}(\Delta_{k-1}(j), \sigma(j), \eta(j)) \qquad (\text{some } M, \Delta_{k-1}(j), \sigma(j), \eta(j))$$

where $\Delta_{k-1}(1), \dots, \Delta_{k-1}(M)$ are distinct members of $\mathbf{SX}_{k-1}$.

§6. The cone over a polyhedral chain.

If $T = \sum_i z_i[\![p_0(i), \dots, p_k(i)]\!]$ is an integral polyhedral chain in $\mathbf{IP}_k(\mathbf{R}^n)$ $(k \leq n-1)$ and q is a point in $\mathbf{R}^n$, then the *cone over T with vertex q* by definition equals

$$[\![q]\!] \times T = \sum_i z_i[\![q, p_0(i), \dots, p_k(i)]\!] \in \mathbf{IP}_{k+1}(\mathbf{R}^n)$$

and a short calculation shows

$$\partial[\![q]\!] \times T \begin{cases} = T - [\![q]\!] \times \partial T & \text{if } k \geq 1 \\ = T - (\partial T)[\![q]\!] & \text{if } k = 0. \end{cases}$$

Hence $T = \partial Q$ for some Q if and only if $\partial T = 0$.

§7. Mapping polyhedral chains by piecewise affine mappings.

Let T be an oriented k simplex $[\![p_0, \dots, p_k]\!]$ in $\mathbf{IP}_k(\mathbf{R}^n)$, and suppose $f: \mathbf{R}^n \to \mathbf{R}^N$ is affine on spt$T$. Then

$$f_\sharp T = f_\sharp[\![p_0, \dots, p_k]\!] = [\![f(p_0), \dots, f(p_k)]\!] \in \mathbf{IP}_k(\mathbf{R}^N);$$

equivalently

$$f_\sharp T = \Pi_\sharp \circ (1_{\mathbf{R}^n} \bowtie f)_\sharp T = \Pi_\sharp[\![(p_0, f(p_0)), \dots, (p_k, f(p_k))]\!].$$

Similarly, $f_\sharp(zT) = zf_\sharp T$ for each integer z. It is immediate that $f_\sharp \partial T = \partial f_\sharp T$. If f is piecewise affine and T is a polyhedral k chain then, in accordance with Proposition 3 and our remark in 4, there will exist a simplicial decomposition $\mathbf{SX}_*$ of $\mathbf{R}^n$ such that f is affine on each simplex of $\mathbf{SX}_*$ and

$$T = \sum_{i=1}^{M} \mathbf{t}(\Delta(i), \theta(i), \xi(i)) \quad (\text{some } M, \Delta(i), \theta(i), \xi(i))$$

where $\Delta(1), \dots, \Delta(M)$ are distinct members of $\mathbf{SX}_k$. We then set

$$f_\sharp T = \sum_{i=1}^{M} f_\sharp \mathbf{t}(\Delta(i), \theta(i), \xi(i)).$$

With this definition it is straightforward to check that

$$f_\sharp: \mathbf{IP}_*(\mathbf{R}^n) \to \mathbf{IP}_*(\mathbf{R}^N)$$

is a well defined chain mapping of degree 0, e.g. $f_\sharp$ is a dimension preserving homomorphism with $f_\sharp \circ \partial = \partial \circ f_\sharp$ whose definition is independent (on the current level) of the particular choice of $\mathbf{SX}_*$.

§8. Zero chains and piecewise affine multi-functions.

Whenever $T \in \mathbf{IP}_0(\mathbf{R}^n)$ with $\partial T = 0$ there will exist a nonnegative integer M and (not necessarily distinct) points $p_1, \dots, p_M, q_1, \dots, q_M \in \mathbf{R}^n$ such that $T = \sum_{i=1}^M [\![q_i]\!] - \sum_{j=1}^M [\![p_j]\!]$. For such T we define

$$\mathbf{G}(T) = \inf \left\{ \sum_{i=1}^{M} |q_i - p_{\sigma(i)}| : \sigma \text{ is a permutation of } \{1, \dots M\} \right\}.$$

This is equivalent to setting

$$\mathbf{G}(T) = \inf \{\mathbf{M}(Q) : Q \in \mathbf{IP}_1(\mathbf{R}^n) \text{ with } \partial Q = T\};$$

the optimal Q equals $\sum_{i=1}^{M} [\![p_{\sigma(i)}, q_i]\!]$ for the right σ. For each fixed integer z_0, there is a corresponding metric $\mathbf{G}$ on $\mathbf{IP}_0(\mathbf{R}^n) \cap \{T : \partial T = z_0\}$ defined by setting $\mathbf{G}(S, T) = \mathbf{G}(S - T)$.

When we say that f is a *piecewise affine multi-function* we mean that, for some positive integers n and N and some simplicial decomposition $\mathbf{SX}_*$ of $\mathbf{R}^n$, f maps $\mathbf{R}^n$ to $\mathbf{IP}_0(\mathbf{R}^N)$ and the following is true. Associated with each n simplex Δ_n in $\mathbf{SX}_n$ there is a nonnegative integer M together with integers $z_1, \ldots, z_M$ and affine mappings $g_1, \ldots, g_M : \mathbf{R}^n \to \mathbf{R}^N$ such that

$$f(x) = \sum_{i=1}^{M} z_i [\![g_i(x)]\!] \qquad \text{for each } x \in \Delta_n.$$

If f is such a piecewise affine multi-function, then one defines the function $[\![1_{\mathbf{R}^n}]\!] \bowtie f : \mathbf{R}^n \to \mathbf{IP}_0(\mathbf{R}^n \times \mathbf{R}^N)$ by setting

$$([\![1_{\mathbf{R}^n}]\!] \bowtie f)(x) = \sum_{i=1}^{M} z_i [\![(x, g_i(x))]\!] = [\![x]\!] \times f(x)$$

for each x in Δ_n, etc. If f is a piecewise affine multi-function then it is straightforward to check the existence of some integer z_0 such that $\partial \circ f(x) = z_0$ for each $x \in \mathbf{R}^n$; furthermore, f is $\mathbf{G}$ continuous.

As an example, the function $f : \mathbf{R} \to \mathbf{IP}_0(\mathbf{R})$,

$$f(x) = \begin{cases} [\![x]\!] - [\![-x]\!] & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is a piecewise affine multi-function.

In view of Proposition 3, the piecewise affine multi-functions from $\mathbf{R}^n$ to $\mathbf{IP}_0(\mathbf{R}^N)$ themselves form an abelian group based on the addition within $\mathbf{IP}_0(\mathbf{R}^N)$.

§9. Local representation of multi-functions.

The analysis of piecewise affine multi-functions and multi-functions mod ν is facilitated by the following proposition.

PROPOSITION. *Suppose* Δ *is a nondegenerate* k *simplex in* $\mathbf{R}^n$ $(k \geq 1)$ *and* $z_1, \ldots, z_M$ *are integers and* $g_1, \ldots, g_M \colon \mathbf{R}^n \to \mathbf{R}^N$ *are affine functions such that*

$$\sum_{i=1}^{M} z_i [\![g_i(x)]\!] = 0$$

$$\left[\text{resp. } \sum_{i=1}^{M} z_i [\![g_i(x)]\!] \in \nu \mathbf{IP}_0(\mathbf{R}^N) \qquad \text{for some } \nu \in \{2, 3, 4, \ldots\} \right]$$

for each $x \in \Delta$. *Then there is a partitioning of* $\{1, \ldots, M\}$ *into nonempty subsets* $W_1, \ldots, W_K$ *such that, for each* $\alpha = 1, \ldots, K$,

(1) $(g_i|\Delta) = (g_j|\Delta)$ *whenever* $i, j \in W_\alpha$;

(2) $\sum_{i \in W_\alpha} z_i = 0$ $\left[\text{resp. } \sum_{i \in W_\alpha} z_i \in \nu \mathbf{Z}\right]$.

§10. Mapping polyhedral chains by piecewise affine multi-functions.

Suppose $f \colon \mathbf{R}^n \to \mathbf{IP}_0(\mathbf{R}^N)$ is a piecewise affine multi-function and T is an integral polyhedral k chain in $\mathbf{R}^n$. In accordance with Proposition 3 we can find some simplicial decomposition $\mathbf{SX}_*$ of $\mathbf{R}^n$ with respect to which the following is true.

(1) $T = \sum_{i=1}^{M} \mathbf{t}(\Delta(i), \theta(i), \xi(i))$ for some M, $\Delta(i)$, $\theta(i)$, and $\xi(i)$ where $\Delta(1), \ldots, \Delta(M)$ are distinct members of $\mathbf{SX}_k$.

(2) For each $i = 1, \ldots, M$ and each $x \in \Delta(i)$,

$$f(x) = \sum_{j=1}^{J(i)} z(i, j) [\![f(i, j)(x)]\!]$$

for some nonnegative integers $J(1), \ldots, J(M)$, some integers $z(i, j)$, and some affine functions $f(i, j) \colon \mathbf{R}^n \to \mathbf{R}^N$.

We then set

$$f_\sharp T = \sum_{i=1}^{M} \sum_{j=1}^{J(i)} z(i, j) f(i, j)_\sharp \mathbf{t}(\Delta(i), \theta(i), \xi(i)) \in \mathbf{IP}_k(\mathbf{R}^N).$$

The obvious extension of this definition defines

$$([\![1_{\mathbf{R}^n}]\!] \bowtie f)_\sharp \mathbf{E}^n \in \mathbf{IP}_{n,loc}(\mathbf{R}^n \times \mathbf{R}^N)$$

and also

$$f_\sharp \mathbf{E}^n = \Pi_\sharp \circ ([\![1_{\mathbf{R}^n}]\!] \bowtie f)_\sharp \mathbf{E}^n \in \mathbf{IP}_n(\mathbf{R}^N)$$

in case $\{x \colon f(x) \neq 0\}$ is bounded (such boundedness implies, in particular, that $\partial \circ f = 0$).

§11. Multi-functions induce chain mappings.

One of the basic properties of multi-functions is the following.

THEOREM. *Suppose* $f: \mathbf{R}^n \to \mathbf{IP}_0(\mathbf{R}^N)$ *is a piecewise affine multi-function. Then the induced mapping of polyhedral chains*

$$f_\sharp: \mathbf{IP}_*(\mathbf{R}^n) \to \mathbf{IP}_*(\mathbf{R}^N)$$

is a chain mapping of degree zero, e.g. $f_\sharp$ *is a dimension preserving homomorphism with* $\partial \circ f_\sharp = f_\sharp \circ \partial$.

PROOF: For example, use Proposition 3 and the factorization $f_\sharp = \Pi_\sharp \circ ([\![1_{\mathbf{R}^n}]\!] \bowtie f)_\sharp$.

§12. Slicing an integral polyhedral chain by an othogonal projection.

Suppose $(x_0, y_0), \ldots, (x_n, y_n) \in \mathbf{R}^n \times \mathbf{R}^N$ are the vertices of an n simplex Δ^* in $\mathbf{R}^n \times \mathbf{R}^N$ and $x_0, \ldots, x_n$ are vertices of an n simplex Δ in $\mathbf{R}^n$. Under these conditions there is a unique affine function $f: \mathbf{R}^n \to \mathbf{R}^N$ such that $f(x_i) = y_i$ for each i. By the *slice of* $T = [\![(x_0, y_0), \ldots, (x_n, y_n)]\!] = \mathbf{t}(\Delta^*, 1, \xi)$ *by* Σ *at* $x \in \mathbf{R}^n \sim \partial\Delta$ we mean the integral zero chain $\langle T, \Sigma, x\rangle$ whose value at $x \in \Delta \sim \partial\Delta$ equals

$$\operatorname{sign}(\xi(x, f(x)) \bullet \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n) \cdot [\![(x, f(x))]\!]$$

and whose value at $x \in \mathbf{R}^n \sim \Delta$ equals 0. We do not attempt to define $\langle T, \Sigma, x\rangle$ for $x \in \partial\Delta$ (Federer's treatment of slicing [**10**, 4.3] does define $\langle T, \Sigma, x\rangle$ as an appropriate real zero chain). Similarly we define

$$\langle zT, \Sigma, x\rangle = z\,\langle T, \Sigma, x\rangle \qquad \text{for each integer } z.$$

Now suppose $\Delta^*(1), \ldots, \Delta^*(M)$ are n simplexes in $\mathbf{R}^n \times \mathbf{R}^N$ whose orthogonal projections $\Delta(i) = \Sigma[\Delta^*(i)]$ $(i = 1, \ldots, M)$ are n simplexes in $\mathbf{R}^n$ and that

$$S = \sum_{i=1}^{M} \mathbf{t}(\Delta^*(i), \theta(i), \xi(i)) \in \mathbf{IP}_n(\mathbf{R}^n \times \mathbf{R}^N) \qquad (\text{some } M, \theta(i), \xi(i)).$$

By the slice of S by Σ at $x \in \mathbf{R}^n \sim \bigcup_{i=1}^{M} \partial\Delta(i)$ we mean the integral zero chain

$$\langle S, \Sigma, x\rangle = \sum_{i=1}^{M} \Big\langle \mathbf{t}(\Delta^*(i), \theta(i), \xi(i)), \Sigma, x \Big\rangle.$$