Preserver Problems on Spaces of Matrices

Xian Zhang Xiaomin Tang Chongguang Cao

（矩阵空间的保持问题）

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Preface

One of the very active research areas in matrix theory is the study of preserver problems, including linear preserver problems, additive preserver problems and multiplicative preserver problems, which concern the classification of maps on matrices or operators that preserve certain special properties; see books [256], [390], [594] and survey papers [229], [336], [347], [414], [415], [453] for some general background.

The earliest papers on the study of linear preserver problems such as G. Frobenius [219] date back to 1897. G. Frobenius showed that every linear map f on the linear space of $n \times n$ complex matrices preserving the determinant has certainly the form $f(A) = PAQ$ or $f(A) = PA^{\mathrm{T}}Q$, where P and Q are invertible matrices with $\det(PQ) = 1$, and A^{T} denotes the transpose of A. Since the linear preserver problems have been studied extensively, and many interesting results have been obtained and discovered. Interestingly, it has been found that the linear maps of matrix spaces that preserve prescribed invariants have the same general form as those above. In 1971, M. Marcus [415] explained this phenomenon by showing that most such results come down to finding the linear maps preserving rank-1 matrices, while linear rank-1 preservers are just the forms Frobenius had found, except that the condition on the determinants of P and Q is omitted. An analogous explanation was given by A. Guterman, C.K. Li and P. Šemrl [229] in 2000, they showed that some linear preserver problems on matrix or operator algebras can be solved by reducing the problems to linear idempotence preservers, while linear idempotence preservers has often the forms similar to that Frobenius had found.

Until 1991, M. Omladič and P. Šemrl [433], by "additive maps" instead of "linear maps", started to study additive preserver problems. He showed that a spectrum-preserving additive surjections ϕ from $B(X)$ to $B(Y)$ is either of the form $\phi(T) = ATA^{-1}$ for a linear isomorphism A of X onto Y or of the form $\phi(T) = BT^{*}B^{-1}$ for a linear isomorphism B of X^{*} onto Y, where $B(X)$ is the algebra of all bounded linear operators on a Banach space X. Two years later, M. Omladič and P. Šemrl [434] investigated that the general form of rank-1 preserving additive surjections on the space of all finite rank operators on a Banach space is very similar to that shown in [433]. The results of these two additive preserver problems on operator algebras can be translated to complex/real matrices, but not to matrices over a general field (even ring). The study of additive preserver problems on algebras of matrices over fields was started in 1996 by C.G. Cao and X. Zhang, the first and third authors of this book (see [122]). They found every idempotence-preserving additive map ϕ on the algebras $M_n(\mathbf{F})$ of $n \times n$ matrices over any field $\mathbf{F}$ of characteristic except 2 has one of the forms: (i) $\phi(X) = \sigma(\mathrm{tr}X)$ for an additive group homomorphism from $\mathbf{F}$ to $M_n(\mathbf{F})$ with $\sigma(1) = 0$, where $\mathrm{tr}X$ denotes the trace of X; (ii) $\phi(X) = PX^{\delta}P^{-1} + \sigma(\mathrm{tr}X)$ for an invertible matrix P and an injective field endomorphism of $\mathbf{F}$; and (iii) $\phi(X) = P(X^{\delta})^{\mathrm{T}}P^{-1} + \sigma(\mathrm{tr}X)$. Thereby, they also determined the general form of additive tripotence preservers, additive preservers of inverses of matrices, and additive preservers of group inverses of matrices. Up to now, many results about linear preserver problems have been extended to the corresponding additive preserver problems. However, it was found from the recent study such as [612], [614] (for more explanations see Chapter 1 and Chapter 2 below) that the majority of extensions from linear preserver problems to the corresponding additive preserver problems are not trivial.

Multiplicative preserver problems have the closed relation with the other areas (e.g., classical group, semigroup theory, group theory, etc.), and hence it has a long history of study. To some extent, the study of multiplicative preserver problems is to determine multiplicative group or semigroup homomorphisms with some additional condition. For this reason, many multiplicative preserver problems on matrix algebras can be solved by using some results about classical group.

Though a great deal of papers about preserver problems have been published, there are only three books, S.W. Liu and D.B. Zhao [390], X. Zhang and C.G. Cao [594], and J.C. Hou and J.L. Cui [256], which were published in 1997, 2001 and 2002, respectively. S.W. Liu and D.B. Zhao [390] emphasize to introduce linear preserver problems on matrices over rings (e.g., commutative local rings, commutative principal ideal domains, etc.), Zhang and Cao [594] emphasize to introduce additive preserver problems on matrices over fields, and J.C. Hou and J.L. Cui [256] emphasize to introduce linear preserver problems on operators of Banach spaces.

The aim of this book is to provide an introduction for current advances of preserver problems on matrices and present the basic methods for studying preserver problems. In order to the content is self-contained, we took some conclusions and their proofs from other books or papers, which will be convenient to readers. It is also intended to offer a collection of references on preserver problems on matrices.

The book is addressed to final year undergraduates of mathematics. It is also intended for graduates and research level mathematicians. It is hoped that the book will be suitable for postgraduate or as a reference.

Dr. X.M. Tang and I are previous graduates of Prof. C.G. Cao. We have cooperated to study preserver problems for ten years at least, and have published one book and about 100 papers on preserver problems. Our study on preserver problems have been supposed by the Chinese Natural Science Foundations under Grant No. 10271021 and 19571019, and by the Natural Science Foundations of Heilongjiang Province under Grant No. A01-07, 9505 and 9719. For these excellent achievements, we was awarded by the Second Class Science and Technology Progress Prize of Heilongjiang Province in 2006, and the Second Class Science and Technology Progress Prizes of Heilongjiang Education Committee in 2005 and 1998, respectively.

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I would appreciate any comments and corrections from the readers.

Xian Zhang

Harbin

Octerber 2006

Notation

Unless otherwise stated, the following notation will be used in the whole book.

$\mathbb{Z}$	ring of integral numbers
$\mathbb{Z}_2$	two-element field
$\mathbb{Z}_3$	three-element field
$\mathbb{R}$	field of real numbers
$\mathbb{C}$ or $\mathbf{C}$	field of complex numbers
$\mathbb{Q}$	field of rational numbers
$\mathbf{Q}$	real quaternion division ring
$\mathbf{F}$	a field
$\mathbf{D}$	real finite dimensional division algebra
$\mathbf{R}$	a ring
$\mathbf{R}^*$	set of all units of $\mathbf{R}$
$\mathrm{ch}\mathbf{R}$	characteristic of $\mathbf{R}$
$M_{mn}(\mathbf{R})$ or $M_{m\times n}(\mathbf{R})$	set of $m \times n$ matrices over a ring $\mathbf{R}$
$M_n(\mathbf{R})$	set $M_{nn}(\mathbf{R})$
$\mathbf{R}^n$	set $M_{n1}(\mathbf{R})$
$M^k_{mn}(\mathbf{F})$ or $M^k_{m\times n}(\mathbf{F})$	subset of $M_{mn}(\mathbf{F})$ consisting of all rank-k matrices
$\Upsilon^k_n(\mathbf{F})$	subset of $M_n(\mathbf{F})$ consisting of all matrices with rank k at most, where $k \geqslant 1$
$\mathfrak{T}_n(\mathbf{F})$	subset of $M_n(\mathbf{F})$ consisting of all tripotent matrices
$\mathfrak{I}_n(\mathbf{F})$	subset of $M_n(\mathbf{F})$ consisting of all idempotent matrices
$\langle\mathfrak{I}_n(\mathbf{F})\rangle$	additive group generalized additively by elements of $\mathfrak{I}_n(\mathbf{F})$
$\Gamma_n(\mathbf{F})$	subset of $M_n(\mathbf{F})$ consisting of all involutory matrices

$\Delta_n(\mathbf{F})$	subset of $M_n(\mathbf{F})$ consisting of all square-zero matrices
$\Delta_n^k(\mathbf{F})$	subset of $\Delta_n(\mathbf{F})$ consisting of all rank-k matrices
$sl_n(\mathbf{F})$	subset of $M_n(\mathbf{F})$ consisting of all trace-zero matrices
$S_n(\mathbf{F})$	set of $n \times n$ symmetric matrices over $\mathbf{F}$
$S_n^k(\mathbf{F})$	subset of $S_n(\mathbf{F})$ consisting of all rank-k matrices
$T_n(\mathbf{F})$	set of $n \times n$ triangular matrices over $\mathbf{F}$
$T_n^k(\mathbf{F})$	subset of $T_n(\mathbf{F})$ consisting of all rank-k matrices
$K_n(\mathbf{F})$	set of $n \times n$ alternate matrices over $\mathbf{F}$
$K_n^k(\mathbf{F})$	subset of $K_n(\mathbf{F})$ consisting of all rank-k matrices
$H_n(\mathbf{C})$	set of complex $n \times n$ Hermitian matrices
$H_n^k(\mathbf{C})$	subset of $H_n(\mathbf{C})$ consisting of all rank-k matrices
$GL_n(\mathbf{F})$	general linear group over $\mathbf{F}$
$SL_n(\mathbf{F})$	special linear group over $\mathbf{F}$
$O_n(\mathbf{D})$	subset of $M_n(\mathbf{D})$ consisting of all unitary matrices
$\theta_+(V)$	set of rank-additive matrix pairs in a subspace V of $M_{mn}(\mathbf{F})$
$\theta_-(V)$	set of rank-sum-minimal matrix pairs in a subspace V of $M_{mn}(\mathbf{F})$
$\theta_+^s(V)$	subset of $\theta_+(V)$ consisting of all matrix pairs (X, Y) with $\operatorname{rank}(X+Y) \leqslant s$
$\bar{a}$	conjugate of the complex number a
$\bar{A}$	conjugate of the complex matrix A
$\prod$	product sign
$\sum$	sum sign
I_n or I	$n \times n$ identity matrix
$0_{m\times n}$ or 0	$m \times n$ zero matrices
A^{T}	transpose of a matrix A
A^*	conjugate transpose of a matrix A
A^{ad}	adjoint matrix of the matrix A
$\operatorname{rank} A$	rank of a matrix A
$\det A$	determinant of a square matrix A
$\operatorname{tr} A$	trace of a square matrix A

$A\{1\}$	set of all $\{1\}$-inverses of the matrix A
$A\{2\}$	set of all $\{2\}$-inverses of the matrix A
$A\{1,2\}$	set of all $\{1,2\}$-inverses of the matrix A
A^+	Moore-Penrose inverse of the matrix A
$A^\#$	group inverse of the matrix A
A^{-1}	inverse of the nonsingular matrix A
A^ρ	matrix $[\rho(a_{ij})]$ for a map $\rho : \mathbf{R} \to \mathbf{R}$ and a matrix $A \in M_{mn}(\mathbf{R})$
E_{ij}	matrix with 1 in the (i,j)th entry and 0 elsewhere
X_{ij}	matrix $I - E_{ii} - E_{jj}$
X_{ijk}	matrix $I - E_{ii} - E_{jj} - E_{kk}$
D_{ij}	matrix $E_{ij} + E_{ji}$ when $i \neq j$
W_{ij}	matrix $E_{ij} - E_{ji}$ when $i \neq j$
$D_i(a)$	matrix $I + (a-1)E_{ii}$ with $a \neq 0$
$T_{ij}(b)$	matrix $I + bE_{ij}$
J	matrix $\sum_{i=1}^{n} E_{i\,n+1-i}$
e_i	the ith column of identity matrix
J_n	matrix whose all entries are 1
$\Upsilon_n^0(\mathbf{F})$	set $\{aE_{ij} \,\vert\, a \in \mathbf{F},\ i,j \in \langle n\rangle\}$
$\ker \phi$	kernel of a group/semigroup homomorphism ϕ
$N(A)$	kernel space of the matrix A
$N^{\mathrm{T}}(A)$	set $\{A^{\mathrm{T}} \,\vert\, A \in N(A)\}$
(a,b)	commutator of two elements a and b in a group
(G,G)	commutator subgroup of the group G
$C(G)$	central subgroup of the group G
$\langle m\rangle$	set $\{1,2,\cdots,m\}$
$[m]$	maximal integer less than or equal to m
C_n^k	positive integer $\frac{n!}{k!(n-k)!}$
$\oplus$	direct sum
$\otimes$	Kronecker (tensor) produce
i	imaginary unit
$\mathrm{diag}(x_1,\cdots,x_n)$	diagonal matrix $\begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix}$

span$S$	linear span of elements of the set S
$\dim V$	dimension of the linear space V
$\|S\|$	numbers of elements of a finite set S
$S_1 \setminus S_2$	different set of two sets S_1 and S_2
$f\|_W$	restriction of the linear map $f : V \to V$ on the subspace W of V
$\bigstar$	linear map from $K_n(\mathbf{F})$ to itself exchanging the (1, 4)th entry and the $(2, 3)$th entry and the $(4, 1)$ th entry with the $(3, 2)$th entry

Contents

Chapter 1

Introduction to preserver problems

In this chapter we will give a brief introduction about preserver problems. Our main references are [229], [336], [347].

1.1 Definitions

We first define Additive Preserver Problems (APPs) and Multiplicative Preserver Problems (MPPs) as follows:

Definition 1.1.1 *Let* $(G_1, \circledast)$ *and* $(G_2, \circledast)$ *be two semigroups. A map* $\phi : G_1 \to G_2$ *is called a* $\circledast$*-preserver if* $\phi(A \circledast B) = \phi(A) \circledast \phi(B)$ *for any* $A, B \in G_1$.

A $\circledast$-preserver $\phi : G_1 \to G_2$ (respectively, $\phi : G_1 \to G_1$) is also called a semigroup homomorphism (respectively, endomorphism) from G_1 to G_2 (respectively, itself). In particular, ϕ is called a semigroup isomorphism (respectively, automorphism) if it is bijective.

Definition 1.1.2 *Let* $(G_1, \circledast)$ *and* $(G_2, \circledast)$ *be two semigroups. Characterizing* $\circledast$*-preservers from* G_1 *to* G_2*, that preserve certain properties, functions, subsets or relations invariant, is called* $\circledast$*-preserver problems. In particular,* $\circledast$*-preserver problems is called APPs (respectively, MPPs) when the operator "*$\circledast$*" is said to be additive (respectively, multiplicative).*

For APPs or MPPs, elements of G_1 and G_2 are often matrices (over a ring or semiring) or operators (in a Banach space). Now let us define Linear Preserver Problems (LPPs). This requires the concept — linear closed set.

Definition 1.1.3 *Let* $\mathcal{A}$ *be a commutative ring or semiring. A linear closed set* V *over* $\mathcal{A}$ *is a set of elements, called vectors, such that:*

(i) *Any two vectors α and β of V determine a unique vector $\alpha+\beta$ as sum;*

(ii) *Any vector $\alpha \in V$ and any scalar $c \in \mathcal{A}$ determine a scalar product $c \cdot \alpha$ in V;*

(iii) *V is an additive semigroup under the operator "+".*

Clearly, a linear space over a field $\mathbf{F}$ is a linear closed set.

Definition 1.1.4 *Let $\mathcal{A}$ be a commutative ring or semiring, and let V_1 and V_2 be two linear closed sets over $\mathcal{A}$. A map $\phi : V_1 \to V_2$ is called a linear preserver if ϕ satisfies*

$$\phi(aX+bY) = a\phi(X) + b\phi(Y),\ \forall a, b \in \mathcal{A},\ X, Y \in V_1.$$

Based on the definition, we can define Linear Preserver Problems (LPPs) as follows:

Definition 1.1.5 *Characterizing linear preservers between linear closed sets, that preserve certain properties, functions, subsets or relations invariant, is called LPPs .*

For LPPs, elements of V_1 and V_2 are often matrices (over a commutative ring or semiring) or operators (in a Banach space).

From Definitions 1.1.1~1.1.5 it is clear that LPPs must be APPs. However, some APPs (e.g., additive preservers of involution matrices, additive preservers of rank-k matrices for $k \geqslant 2$) are so difficult that we can only solve the corresponding LPPs at present. For the same invariant, the essential distinction between LPPs and APPs can also be found from those used techniques and conclusion of the next chapters. Here, we explain it only from an idea of producing by a basis. Let $A_1, \cdots, A_n$ be a basis of a linear space V over an infinite field $\mathbf{F}$. Then every element A in V can be written as

$$A = \sum_{i \in \langle n \rangle} a_i A_i, \tag{1.1.1}$$

where $a_1, a_2, \cdots, a_n$ are scalars determined uniquely by A.

- If $\phi : V \to V$ is an additive preserver of some invariant, then it follows from (1.1.1) that $\phi(A) = \sum_{i \in \langle n \rangle} \phi(a_i A_i)$ for every $A \in V$. Thus, in order to characterize the form of ϕ, we need determine the set $\mathcal{B}_1 = \{\phi(aA_i) |\, a \in \mathbf{F}, i \in \langle n \rangle\}$.

- If $\phi : V \to V$ is a linear preserver of some invariant, then it follows from (1.1.1) that $\phi(A) = \sum_{i\in\langle n\rangle} a_i\phi(A_i)$ for every $A \in V$. Thus, in order to characterize the form of ϕ, we need determine the set $\mathcal{B}_2 = \{\phi(A_i) \mid i \in \langle n\rangle\}$.

Therefore, LPPs and APPs have the essential distinction because $\mathcal{B}_1$ is infinite and $\mathcal{B}_2$ is finite.

Usually, all APPs, LPPs and MPPs are called preserver problems. By the objects acted on by maps, preserver problems can be mainly classified into two categories, one is preserver problems on operators, the other is preserver problems on matrices. In general, when these two categories of preserver problems have the same invariant, the conclusions of the first category preserver problems can be directly used to solve the second category preserver problems over the complex/real number field.[1] However, unless the case of the complex/real number field, the second category preserver problems cannot be solved by using the conclusions of the first category preserver problems with the same invariant (because Banach spaces are defined only over the complex/real number field).

An extensive research on preserver problems is to characterize maps between sets preserving certain properties, functions, subsets or relations invariant (for details see Section 6.8 and Chapter 11). This intimates the closed relation between preserver problems and the other areas (e.g., classical group [269], matrix geometry [270, 274, 537], Jordan homomorphism [441], ring homomorphism [297, 298], etc.).

The research of preserver problems has attracted the attention of many mathematicians (see books [256], [390], [594] and survey papers [95], [229], [336], [347], [414], [415], [453] for details). It should be mentioned that Zhang and Cao [594] introduced mainly APPs on matrices over fields, Liu and Zhao [390] introduced mainly LPPs on matrices over rings (e.g., commutative local rings, commutative principal ideal domains, etc.), and Hou and Cui [256] introduced mainly LPPs on operators of Banach spaces.

1 Sometimes, by using model theoretic algebra technique, results of some preserver problems on complex matrices can be generalized to matrices over any field of characteristic 0 (for details see Section 1.3).