A TREATISE

ON

UNIVERSAL ALGEBRA

WITH APPLICATIONS.

BY

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PREFACE.

IT is the purpose of this work to present a thorough investigation of the various systems of Symbolic Reasoning allied to ordinary Algebra. The chief examples of such systems are Hamilton's Quaternions, Grassmann's Calculus of Extension, and Boole's Symbolic Logic. Such algebras have an intrinsic value for separate detailed study; also they are worthy of a comparative study, for the sake of the light thereby thrown on the general theory of symbolic reasoning, and on algebraic symbolism in particular.

The comparative study necessarily presupposes some previous separate study, comparison being impossible without knowledge. Accordingly after the general principles of the whole subject have been discussed in Book I. of this volume, the remaining books of the volume are devoted to the separate study of the Algebra of Symbolic Logic, and of Grassmann's Calculus of Extension, and of the ideas involved in them. The idea of a generalized conception of space has been made prominent, in the belief that the properties and operations involved in it can be made to form a uniform method of interpretation of the various algebras.

Thus it is hoped in this work to exhibit the algebras both as systems of symbolism, and also as engines for the investigation of the possibilities of thought and reasoning connected with the abstract general idea of space. A natural mode of comparison between the algebras is thus at once provided by the unity of the subject-matters of their interpretation. The detailed comparison of their symbolic structures has been adjourned to the second volume, in which it is intended to deal with Quaternions, Matrices, and the general theory of Linear Algebras. This comparative anatomy of the subject was originated by B. Peirce's paper on Linear Associative Algebra*, and has been carried forward by more recent investigations in Germany.

* First read before the National Academy of Sciences in Washington, 1871, and republished in the *American Journal of Mathematics*, vol. iv., 1881.

The general name to be given to the subject has caused me much thought: that finally adopted, Universal Algebra, has been used somewhat in this signification by Sylvester in a paper, *Lectures on the Principles of Universal Algebra*, published in the *American Journal of Mathematics*, vol. vi., 1884. This paper however, apart from the suggestiveness of its title, deals explicitly only with matrices.

Universal Algebra has been looked on with some suspicion by many mathematicians, as being without intrinsic mathematical interest and as being comparatively useless as an engine of investigation. Indeed in this respect Symbolic Logic has been peculiarly unfortunate; for it has been disowned by many logicians on the plea that its interest is mathematical, and by many mathematicians on the plea that its interest is logical. Into the quarrels of logicians I shall not be rash enough to enter. Also the nature of the interest which any individual mathematician may feel in some branch of his subject is not a matter capable of abstract argumentation. But it may be shown, I think, that Universal Algebra has the same claim to be a serious subject of mathematical study as any other branch of mathematics. In order to substantiate this claim for the importance of Universal Algebra, it is necessary to dwell shortly upon the fundamental nature of Mathematics.

Mathematics in its widest signification is the development of all types of formal, necessary, deductive reasoning.

The reasoning is formal in the sense that the meaning of propositions forms no part of the investigation. The sole concern of mathematics is the inference of proposition from proposition. The justification of the rules of inference in any branch of mathematics is not properly part of mathematics: it is the business of experience or of philosophy. The business of mathematics is simply to follow the rule. In this sense all mathematical reasoning is necessary, namely, it has followed the rule.

Mathematical reasoning is deductive in the sense that it is based upon definitions which, as far as the validity of the reasoning is concerned (apart from any existential import), need only the test of self-consistency. Thus no external verification of definitions is required in mathematics, as long as it is considered merely as mathematics. The subject-matter is not necessarily first presented to the mind by definitions: but no idea, which has not been completely defined as far as concerns its relations to other ideas involved in the subject-matter, can be admitted into the reasoning. Mathematical definitions are always to be construed as limitations as well as definitions;

namely, the properties of the thing defined are to be considered for the purposes of the argument as being merely those involved in the definitions.

Mathematical definitions either possess an existential import or are conventional. A mathematical definition with an existential import is the result of an act of pure abstraction. Such definitions are the starting points of applied mathematical sciences; and in so far as they are given this existential import, they require for verification more than the mere test of self-consistency.

Hence a branch of applied mathematics, in so far as it is applied, is not merely deductive, unless in some sense the definitions are held to be guaranteed a priori as being true in addition to being self-consistent.

A conventional mathematical definition has no existential import. It sets before the mind by an act of imagination a set of things with fully defined self-consistent types of relation. In order that a mathematical science of any importance may be founded upon conventional definitions, the entities created by them must have properties which bear some affinity to the properties of existing things. Thus the distinction between a mathematical definition with an existential import and a conventional definition is not always very obvious from the form in which they are stated. Though it is possible to make a definition in form unmistakably either conventional or existential, there is often no gain in so doing. In such a case the definitions and resulting propositions can be construed either as referring to a world of ideas created by convention, or as referring exactly or approximately to the world of existing things. The existential import of a mathematical definition attaches to it, if at all, quâ mixed mathematics; quâ pure mathematics, mathematical definitions must be conventional*.

Historically, mathematics has, till recently, been confined to the theories of Number, of Quantity (strictly so-called), and of the Space of common experience. The limitation was practically justified: for no other large systems of deductive reasoning were in existence, which satisfied our definition of mathematics. The introduction of the complex quantity of ordinary algebra, an entity which is evidently based upon conventional definitions, gave rise to the wider mathematical science of to-day. The realization of wider conceptions has been retarded by the habit of mathematicians, eminently useful and indeed necessary for its own purposes, of extending all names to apply to new ideas as they arise. Thus the name

* Cf. Grassmann, *Ausdehnungslehre von* 1844, Einleitung.

of quantity was transferred from the quantity, strictly so called, to the generalized entity of ordinary algebra, created by conventional definition, which only includes quantity (in the strict sense) as a special case.

Ordinary algebra in its modern developments is studied as being a large body of propositions, inter-related by deductive reasoning, and based upon conventional definitions which are generalizations of fundamental conceptions. Thus a science is gradually being created, which by reason of its fundamental character has relation to almost every event, phenomenal or intellectual, which can occur. But these reasons for the study of ordinary Algebra apply to the study of Universal Algebra; provided that the newly invented algebras can be shown either to exemplify in their symbolism, or to represent in their interpretation interesting generalizations of important systems of ideas, and to be useful engines of investigation. Such algebras are mathematical sciences, which are not essentially concerned with number or quantity; and this bold extension beyond the traditional domain of pure quantity forms their peculiar interest. The ideal of mathematics should be to erect a calculus to facilitate reasoning in connection with every province of thought, or of external experience, in which the succession of thoughts, or of events can be definitely ascertained and precisely stated. So that all serious thought which is not philosophy, or inductive reasoning, or imaginative literature, shall be mathematics developed by means of a calculus.

It is the object of the present work to exhibit the new algebras, in their detail, as being useful engines for the deduction of propositions; and in their several subordination to dominant ideas, as being representative symbolisms of fundamental conceptions. In conformity with this latter object I have not hesitated to compress, or even to omit, developments and applications which are not allied to the dominant interpretation of any algebra. Thus unity of idea, rather than completeness, is the ideal of this book. I am convinced that the comparative neglect of this subject during the last forty years is partially due to the lack of unity of idea in its presentation.

The neglect of the subject is also, I think, partially due to another defect in its presentation, which (for the want of a better word) I will call the lack of independence with which it has been conceived. I will proceed to explain my meaning.

Every method of research creates its own applications: thus Analytical Geometry is a different science from Synthetic Geometry, and both these sciences are different from modern Projective Geometry. Many propositions are identical in all three sciences, and the general subject-matter, Space, is the same throughout. But it would be a serious mistake in the development of one of the three merely to take a list of the propositions as they occur in the others, and to endeavour to prove them by the methods of the one in hand. Some propositions could only be proved with great difficulty, some could hardly even be stated in the technical language, or symbolism, of the special branch. The same applies to the applications of the algebras in this book. Thus Grassmann's Algebra, the Calculus of Extension, is applied to Descriptive Geometry, Line Geometry, and Metrical Geometry, both non-Euclidean and Euclidean. But these sciences, as here developed, are not the same sciences as developed by other methods, though they apply to the same general subject-matter. Their combination here forms one new and distinct science, as distinct from the other sciences, whose general subject-matters they deal with, as is Analytical Geometry from Pure Geometry. This distinction, or independence, of the application of any new algebra appears to me to have been insufficiently realized, with the result that the developments of the new Algebras have been cramped.

In the use of symbolism I have endeavoured to be very conservative. Strange symbols are apt to be rather an encumbrance than an aid to thought: accordingly I have not ventured to disturb any well-established notation. On the other hand I have not hesitated to introduce fresh symbols when they were required in order to express new ideas.

This volume is divided into seven books. In Book I. the general principles of the whole subject are considered. Book II. is devoted to the Algebra of Symbolic Logic; the results of this book are not required in any of the succeeding books of this volume. Book III. is devoted to the general principles of addition and to the theory of a Positional manifold, which is a generalized conception of Space of any number of dimensions without the introduction of the idea of distance. The comprehension of this book is essential in reading the succeeding books. Book IV. is devoted to the principles of the Calculus of Extension. Book V. applies the Calculus of Extension to the theory of forces in a Positional manifold of three dimensions. Book VI. applies the Calculus of Extension to Non-Euclidean Geometry, considered, after Cayley, as being the most general theory of distance in a Positional manifold; the comprehension of this book is not necessary in reading the succeeding book. Book VII. applies the Calculus of Extension to ordinary Euclidean Space of three dimensions.

It would have been impossible within reasonable limits of time to have made an exhaustive study of the many subjects, logical and mathematical, on which this volume touches; and, though the writing of this volume has been continued amidst other avocations since the year 1890, I cannot pretend to have done so. In the subject of pure Logic I am chiefly indebted to Mill, Jevons, Lotze, and Bradley; and in regard to Symbolic Logic to Boole, Schröder and Venn. Also I have not been able in the footnotes to this volume adequately to recognize my obligations to De Morgan's writings, both logical and mathematical. The subject-matter of this volume is not concerned with Quaternions; accordingly it is the more necessary to mention in this preface that Hamilton must be regarded as a founder of the science of Universal Algebra. He and De Morgan (cf. note, p. 131) were the first to express quite clearly the general possibilities of algebraic symbolism.

The greatness of my obligations in this volume to Grassmann will be understood by those who have mastered his two *Ausdehnungslehres.* The technical development of the subject is inspired chiefly by his work of 1862, but the underlying ideas follow the work of 1844. At the same time I have tried to extend his Calculus of Extension both in its technique and in its ideas. But this work does not profess to be a complete interpretation of Grassmann's investigations, and there is much valuable matter in his *Ausdehnungslehres* which it has not fallen within my province to touch upon. Other obligations, as far as I am aware of them, are mentioned as they occur. But the book is the product of a long preparatory period of thought and miscellaneous reading, and it was only gradually that the subject in its full extent shaped itself in my mind; since then the various parts of this volume have been systematically deduced, according to the methods appropriate to them here, with hardly any aid from other works. This procedure was necessary, if any unity of idea was to be preserved, owing to the bewildering variety of methods and points of view adopted by writers on the various subjects of this volume. Accordingly there is a possibility of some oversights, which I should very much regret, in the attribution of ideas and methods to their sources. I should like in this connection to mention the names of Arthur Buchheim and of Homersham Cox as the mathematicians whose writings have chiefly aided me in the development of the Calculus of Extension (cf. notes, pp. 248, 346, 370, and 575). In the development of Non-Euclidean Geometry the ideas of Cayley, Klein, and Clifford have been chiefly followed; and in the development of the theory of Systems of Forces I am indebted to Sir R. S. Ball, and to Lindemann.

I have added unsystematically notes to a few theorems or methods, stating that they are, as far as I know, now enunciated for the first time. These notes are unsystematic in the double sense that I have not made a systematic search in the large literatures of the many branches of mathematics with which this book has to do, and that I have not added notes to every theorem or method which happens to be new to me.

My warmest thanks for their aid in the final revision of this volume are due to Mr Arthur Berry, Fellow of King's College, to Mr W. E. Johnson, of King's College, and Lecturer to the University in Moral Science, to Prof. Forsyth, Sadlerian Professor to the University, who read the first three books in manuscript, and to the Hon. B. Russell, Fellow of Trinity College, who has read many of the proofs, especially in the parts connected with Non-Euclidean Geometry.

Mr Johnson not only read the proofs of the first three books, and made many important suggestions and corrections, but also generously placed at my disposal some work of his own on Symbolic Logic, which will be found duly incorporated with acknowledgements.

Mr Berry throughout the printing of this volume has spared himself no trouble in aiding me with criticisms and suggestions. He undertook the extremely laborious task of correcting all the proofs in detail. Every page has been improved either substantially or in expression owing to his suggestions. I cannot express too strongly my obligations to him both for his general and detailed criticism.

The high efficiency of the University Press in all that concerns mathematical printing, and the courtesy which I have received from its Officials, also deserve grateful acknowledgements.

CAMBRIDGE,

December, 1897.

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