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Abstract Analytic Function Theory and Hardy Algebras

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Preface

The present work wants to be the systematic presentation of a functional-analytic theory. It is an abstract version of those parts of classical analytic function theory which can be circumscribed by boundary value theory and Hardy spaces H^p. The fascination of the field comes from the fact that famous classical theorems of typical complex-analytic flavor appear as instant outflows of an abstract theory the tools of which are standard real-analytic methods such as elementary functional analysis and measure theory. The abstract theory started about twenty years ago in papers of Arens and Singer, Gleason, Helson and Lowdenslager, Bochner, Bishop, Wermer,... and went through several steps of abstraction (Dirichlet algebras, logmodular algebras,...). It never ceased to radiate back and illuminate the concrete classical theory. We present the ultimate step of abstraction which has been under work for about ten years.

The central concept is the abstract Hardy algebra situation. It is comprehensive as well as pure and simple and permits to build up a coherent theory of remarkable and pleasant width and depth. We attempt to present a systematic account in Chapters IV-IX. The abstract Hardy algebra situation can be looked upon as a local section of the abstract function algebra situation. To achieve the localization is the main business of the abstract F.and M.Riesz theorem and of the resultant Gleason part decomposition procedure. These are central themes in Chapters II-III devoted to the abstract function algebra situation. Chapter I presents the concrete unit disk situation in such a spirit as to prepare the abstract concepts. Chapter X is devoted to standard applications of the abstract theory to polynomial and rational approximation in the complex plane and is the most conventional part of the book.

In comparison with the respective parts of the earlier treatises on uniform algebras, the most comprehensive of which is GAMELIN [1969], the present work contains numerous new results. In concepts and systematization it is shaped after the work of König. Most of the chapters contain substantial new material. A prime point is the systematic use of the associated algebra $H^{\#}$. The most important individual new result is perhaps the approximation theorem VI.4.1. Let us also quote Section VI.5 on the Marcel Riesz estimation for the abstract conjugation after fundamental results of Pichorides in the unit disk situation. For more details we refer to the Introductions and Notes to the individual chapters.

In its overall structure and in certain parts the present work resembles the lectures on function algebras which König held in 1967/68 at the California Institute of Technology in Pasadena/California, and which in part had been distributed in a provisional form. He wants to express his warmest thanks to Wim Luxemburg and Charles DePrima who were his hosts in those days, and likewise to Gunter Lumer to whom he owes the participation in the Function Algebra Seminar at the University of Washington in Seattle/Washington in 1970. Above all he sends his deepest thanks to Galen Seever and to Kôzô Yabuta for most valuable and pleasant cooperation, and he wants to include his former student Klaus Barbey who started to participate with the elaboration of 1970/71 lecture notes which formed the next step in the evolution of the present text.

In conclusion we want to express our sincere thanks to Michael Neumann for his active interest and valuable work in connection with a common seminar, to Ulla Faust and Gisela Schirmbeck who typed the final text with impressive care and thoughtfulness, to Horst Loch who read most of the proofs with distinctive care, and to Karla May and Gerd Rodé for their kind assistance.

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Contents

Chapter I

Boundary Value Theory

for Harmonic and Holomorphic Functions in the Unit Disk

The present chapter describes the concrete situation which forms the basic model for the subsequent abstract theory. It leads up to the point where the abstract theory can be put into action. The abstract theory will then illuminate the reasons for which the individual classical theorems are valid.

1. Harmonic Functions

For G an open subset of the complex plane $\mathbb{C}$ let $\mathrm{Harm}(G)$ denote the class of harmonic functions $G\to\mathbb{C}$, $\mathrm{Harm}^\infty(G)$ the class of bounded functions in $\mathrm{Harm}(G)$, and $\mathrm{CHarm}(G)$ the class of those functions in $\mathrm{Harm}(G)$ which admit continuous extensions $\overline{G}\to\mathbb{C}$. Here $\overline{G}$ means the closure of G relative to the Riemann sphere so that $\infty\in\overline{G}$ if G is unbounded. Furthermore let $\mathrm{Hol}(G)$, $\mathrm{Hol}^\infty(G)$ and $\mathrm{CHol}(G)$ denote the respective classes of holomorphic functions $G\to\mathbb{C}$.

The boundary value theory for the above function classes for unit disk $D=\{z\in\mathbb{C}:|z|<1\}$ and unit circle $S=\{s\in\mathbb{C}:|s|=1\}$ is dominated by the Poisson kernel $P:D\times S\to\mathbb{R}$. It is defined to be

$$P(z,s) = \mathrm{Re}\,\frac{s+z}{s-z} = \frac{s}{s-z} + \frac{s\overline{z}}{1-s\overline{z}} = \frac{1-|z|^2}{|s-z|^2} \qquad \forall\ z\in D \text{ and } s\in S,$$

$$P(Re^{iu},e^{iv}) = \frac{1-R^2}{1-2R\cos(u-v)+R^2} \qquad \forall\ 0\leqq R<1 \text{ and real } u,v.$$

We list some immediate properties. i) P is continuous on $D\times S$ and

$$0 < \frac{1-|z|}{1+|z|} \leqq P(z,s) \leqq \frac{1+|z|}{1-|z|} \qquad \forall\ z\in D \text{ and } s\in S.$$

ii) $P(R,e^{iv})\leqq P(R,e^{iu})$ for real u,v with $|u|\leqq|v|\leqq\pi$ and $0\leqq R<1$.

iii) $P(R,e^{it})\to 0$ for $R\uparrow 1$ pointwise in $0<|t|\leqq\pi$ and hence uniformly in $\delta\leqq|t|\leqq\pi$ for each $\delta>0$.

iv) $P(z\alpha, s\alpha)=P(z,s)$ for $z\in D$ and $s,\alpha\in S$.

v) $P(R\alpha,\beta)=P(R\beta,\alpha)$ for $\alpha,\beta\in S$ and $0\leqq R<1$.

And we recall from elementary analytic function theory the basic representation theorem. Here λ denotes one-dimensional Lebesgue measure on S with the normalization $\lambda(S)=1$.

1.1 REPRESENTATION THEOREM: For $f\in Hol(D)$ we have

$$f(Rz) = \int_S \frac{s}{s-z} f(Rs)d\lambda(s) = i\mathrm{Im}f(0) + \int_S \frac{s+z}{s-z}\mathrm{Re}f(Rs)d\lambda(s)$$

$$= \int_S P(z,s)f(Rs)d\lambda(s) \qquad \forall\ z\in D \text{ and } 0\leqq R<1.$$

Hence for $f\in Harm(D)$ we have

$$f(Rz) = \int_S P(z,s)f(Rs)d\lambda(s) \qquad \forall\ z\in D \text{ and } 0\leqq R<1.$$

In particular we have $\int_S P(z,s)d\lambda(s) = 1$ for $z\in D$.

We turn to the boundary behaviour of the functions in Harm(D). For $f:D\to\mathbb{C}$ and $0\leqq R<1$ we put $f_R: f_R(s)=f(Rs)$ $\forall s\in S$.

1.2 REMARK: Let $1\leqq p\leqq\infty$. For $f\in Harm(D)$ then

$$||f_R||_{L^p(\lambda)} := \begin{cases} \left(\int_S |f(Rs)|^p d\lambda(s)\right)^{\frac{1}{p}} & \text{for } 1\leqq p<\infty \\ \mathop{\mathrm{Max}}_{s\in S}|f(Rs)| & \text{for } p=\infty \end{cases}$$

is monotone increasing in $0\leqq R<1$.

Proof: For $0\leqq r<R<1$ we obtain from 1.1

$$f(rz) = f(R\tfrac{r}{R}z) = \int_S P(\tfrac{r}{R}z,s)f(Rs)d\lambda(s) \qquad \forall\ z\in S.$$

From this the cases $p=1$, $1<p<\infty$ and $p=\infty$ require separate treatment. The cases $p=1$ and $p=\infty$ are almost obvious, so we restrict ourselves to $1<p<\infty$. For $1<q<\infty$ the conjugate exponent we obtain

$$\int_S |f(rz)|^p d\lambda(z) \leqq \int_S\left(\int_S P(\tfrac{r}{R}z,s)|f(Rs)|d\lambda(s)\right)^p d\lambda(z)$$

$$= \int_S\left(\int_S P(\tfrac{r}{R}z,s)^{1/q}\left(P(\tfrac{r}{R}z,s)^{1/p}|f(Rs)|\right)d\lambda(s)\right)^p d\lambda(z)$$

$$\leq \int_S \Big(\int_S P(\tfrac{r}{R}z,s)\,d\lambda(s)\Big)^{p/q}\Big(\int_S P(\tfrac{r}{R}z,s)\,|f(Rs)|^p d\lambda(s)\Big)d\lambda(z)$$

$$= \int_S \Big(\int_S P(\tfrac{r}{R}z,s)\,|f(Rs)|^p d\lambda(s)\Big)d\lambda(z) = \int_S |f(Rs)|^p d\lambda(s),$$

where v) above has been applied. QED.

For $1 \leqq p \leqq \infty$ we define $\mathrm{Harm}^p(D)$ to consist of the functions $f \in \mathrm{Harm}(D)$ with

$$N_p f := \lim_{R\uparrow 1} \|f_R\|_{L^p(\lambda)} = \sup_{0\leqq R<1} \|f_R\|_{L^p(\lambda)} < \infty.$$

For $p=\infty$ this coincides with the earlier definition. From

$$N_1 f \leqq N_p f \leqq N_\infty f \qquad \text{for } f \in \mathrm{Harm}(D)$$

we see that $\mathrm{Harm}^\infty(D) \subset \mathrm{Harm}^p(D) \subset \mathrm{Harm}^1(D)$. We formulate the boundary behaviour of the functions in $\mathrm{Harm}^p(D)$ in the subsequent propositions. Here $ca(S)$ denotes the class of complex-valued Baire measures on S.

<u>1.3</u> <u>PROPOSITION</u>: i) For $\theta \in ca(S)$ define the function

$$\langle\theta\rangle : \langle\theta\rangle(z) = \int_S P(z,s)\,d\theta(s) \qquad \forall\ z \in D.$$

Then $\langle\theta\rangle \in \mathrm{Harm}^1(D)$ and $N_1\langle\theta\rangle = \|\theta\|$:= total variation of θ. Furthermore for $R\uparrow 1$ we have convergence $\langle\theta\rangle_R \lambda \to \theta$ in the weak* topology $\sigma(ca(S),C(S)) = \sigma(C(S)',C(S))$.

ii) Let $1 \leqq p \leqq \infty$. For $F \in L^p(\lambda)$ consider the function

$$f = \langle F\lambda\rangle : f(z) = \int_S P(z,s)F(s)\,d\lambda(s) \qquad \forall\ z \in D.$$

Then $f \in \mathrm{Harm}^p(D)$ and $N_p f = \|F\|_{L^p(\lambda)}$. Furthermore for $R\uparrow 1$ we have convergence $f_R \to F$ in $L^p(\lambda)$-norm if $1 \leqq p < \infty$ and in the weak* topology $\sigma(L^\infty(\lambda),L^1(\lambda)) = \sigma(L^1(\lambda)',L^1(\lambda))$ if $p=\infty$.

iii) For $F \in C(S)$ we have $f = \langle F\lambda\rangle \in \mathrm{CHarm}(D)$. Furthermore for $R\uparrow 1$ we have convergence $f_R \to F$ uniformly on S.

Proof: 1) $\langle\theta\rangle \in \mathrm{Harm}(D)$ for $\theta \in ca(S)$ is obvious since for real-valued θ the definition represents $\langle\theta\rangle$ as the real part of a function in Hol(D).
2) In order to prove iii) it suffices to show the uniform convergence $f_R \to F$ for $R\uparrow 1$. For $0 \leqq R < 1$ and $z \in S$ we have

$$f_R(z)-F(z) = \int_S P(Rz,s)\big(F(s)-F(z)\big)d\lambda(s) = \int_S P(R,\tfrac{s}{z})\big(F(s)-F(z)\big)d\lambda(s)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P(R,e^{it})\big(F(ze^{it})-F(z)\big)dt,$$

and hence for $0<\delta<\pi$ after subdivision into $|t|\leqq\delta$ and $\delta\leqq|t|\leqq\pi$

$$|f_R(z)-F(z)| \leqq \frac{1}{2\pi}\int_{-\delta}^{\delta} P(R,e^{it})\omega(\delta)dt + 2\,||F||\,P(R,e^{i\delta}) \leqq \omega(\delta) + 2\,||F||\,P(R,e^{i\delta}),$$

where ω is the modulus of continuity of the function $F\in C(S)$. Therefore

$$\limsup_{R\uparrow 1} ||f_R - F|| \leqq \omega(\delta) \qquad \text{for each } 0<\delta<\pi,$$

so that $||f_R-F||\to 0$ for $R\uparrow 1$.

3) We next prove i). For $f=\langle\theta\rangle\in\mathrm{Harm}(D)$ and $0\leqq R<1$ we have

$$\int_S |f(Rz)|d\lambda(z) \leqq \int_S\Big(\int_S P(Rz,s)d|\theta|(s)\Big)d\lambda(z)$$

$$=\int_S\Big(\int_S P(Rs,z)d\lambda(z)\Big)d|\theta|(s) = ||\theta||,$$

therefore $f\in\mathrm{Harm}^1(D)$ and $N_1 f\leqq ||\theta||$. The weak* convergence to be shown means that $\int_S Hf_R d\lambda \to \int_S H d\theta$ for $R\uparrow 1$ for each $H\in C(S)$. But this is true since for $h=\langle H\lambda\rangle$ we know from iii) that

$$\int_S Hf_R d\lambda = \int_S\Big(\int_S H(z)P(Rz,s)d\theta(s)\Big)d\lambda(z)$$

$$= \int_S\Big(\int_S H(z)P(Rs,z)d\lambda(z)\Big)d\theta(s) = \int_S h_R(s)d\theta(s) \to \int_S Hd\theta \text{ for } R\uparrow 1.$$

And then $|\int_S Hf_R d\lambda| \leqq ||H|| \int_S |f_R|d\lambda \leqq ||H||N_1 f$ for $0\leqq R<1$ implies that $|\int_S Hd\theta| \leqq ||H||N_1 f$ for each $H\in C(S)$. This means $||\theta|| \leqq N_1 f$, so that we obtain $N_1 f= ||\theta||$.

4) In order to prove ii) we obtain

$$||f_R||_{L^p(\lambda)} \leqq ||F||_{L^p(\lambda)} \qquad \text{for } 0\leqq R<1$$

as in the proof of 1.2. Thus $f\in\mathrm{Harm}^p(D)$ and $N_p f\leqq ||F||_{L^p(\lambda)}$. Then in the case $1\leqq p<\infty$ we use the fact that $C(S)$ is dense in $L^p(\lambda)$. Thus for $H\in C(S)$

and $h=\langle H\lambda\rangle$ we have

$$\|f_R-F\|_{L^p(\lambda)} \leqq \|(f-h)_R\|_{L^p(\lambda)} + \|F-H\|_{L^p(\lambda)} + \|h_R-H\|_{L^p(\lambda)}$$

$$\leqq 2\,\|F-H\|_{L^p(\lambda)} + \|h_R-H\| \qquad \text{for } 0\leqq R<1,$$

$$\limsup_{R\uparrow 1} \|f_R-F\|_{L^p(\lambda)} \leqq 2\,\|F-H\|_{L^p(\lambda)} \qquad \text{in view of iii)},$$

and hence $f_R\to F$ in $L^p(\lambda)$-norm. From this it follows that

$$\|F\|_{L^p(\lambda)} = \lim_{R\uparrow 1} \|f_R\|_{L^p(\lambda)} = N_p f.$$

In the case $p=\infty$ the weak* convergence to be shown means that $\int_S Hf_R d\lambda \to \int_S HFd\lambda$ for $R\uparrow 1$ for each $H\in L^1(\lambda)$. But this is true since for $h=\langle H\lambda\rangle\in$ $\in\mathrm{Harm}^1(D)$ we know that $h_R\to H$ in $L^1(\lambda)$-norm and hence

$$\int_S Hf_R d\lambda = \int_S h_R F d\lambda \to \int_S HFd\lambda \qquad \text{for } R\uparrow 1.$$

And then

$$\Big|\int_S Hf_R d\lambda\Big| \leqq \|H\|_{L^1(\lambda)} \|f_R\|_{L^\infty(\lambda)} \leqq \|H\|_{L^1(\lambda)} N_\infty f \qquad \text{for } 0\leqq R<1$$

implies that $\left|\int_S HFd\lambda\right| \leqq \|H\|_{L^1(\lambda)} N_\infty f$ for each $H\in L^1(\lambda)$. This means $\|F\|_{L^\infty(\lambda)} \leqq N_\infty f$, so that we obtain $N_\infty f = \|F\|_{L^\infty(\lambda)}$. QED.

<u>1.4</u> <u>PROPOSITION</u>: i) For each $f\in\mathrm{Harm}^1(D)$ there exists a unique $\theta\in ca(S)$ with $f=\langle\theta\rangle$.

ii) Let $1<p\leqq\infty$. For each $f\in\mathrm{Harm}^p(D)$ there exists a unique $F\in L^p(\lambda)$ with $f=\langle F\lambda\rangle$.

iii) For each $f\in C\mathrm{Harm}(D)$ there exists a unique $F\in C(S)$ with $f=\langle F\lambda\rangle$.

Proof: i) The measures $f_R\lambda\in ca(S)$ fulfill

$$\|f_R\lambda\| = \|f_R\|_{L^1(\lambda)} \leqq N_1 f<\infty \quad \forall\ 0\leqq R<1.$$

Let $\theta\in ca(S)$ be a weak* limit point of these measures for $R\uparrow 1$. Thus for each $H\in C(S)$ there is a sequence $R(n)\uparrow 1$ with $\int_S Hf_{R(n)}d\lambda\to\int_S Hd\theta$. Now from 1.1 we have

$$f(Rz) = \int_S P(z,s) f_R(s)\, d\lambda(s) \qquad \forall\ z\in D \text{ and } 0\leqq R<1.$$

For fixed $z\in D$ we take $H=P(z,\cdot)$ and a suitable sequence $R=R(n)\uparrow 1$ to obtain

$$f(z) = \int_S P(z,s)d\theta(s) = \langle\theta\rangle(z) \quad \forall\ z\in D.$$

ii) For $1\leqq q<\infty$ the conjugate exponent we have $L^p(\lambda)=L^q(\lambda)'$. In view of $\|f_R\|_{L^p(\lambda)}\leqq N_p f<\infty$ for $0\leqq R<1$ there exists a weak* limit point $F\in L^p(\lambda)$ of the f_R for $R\uparrow 1$. Then we obtain $f=\langle F\lambda\rangle$ as in the proof of i). The proof of iii) is similar but simpler since the compactness argument is not needed. The uniqueness assertions are immediate from 1.3. QED.

1.5 COROLLARY (HERGLOTZ): The formula $f=\langle\theta\rangle$ defines a bijection between the nonnegative functions $f\in Harm(D)$ and the nonnegative measures $\theta\in Pos(S)$.

Proof: For $f\in Harm(D)$ nonnegative we have

$$f(0) = \int_S f(Rs)d\lambda(s) = \|f_R\|_{L^1(\lambda)} \qquad \text{for } 0\leqq R<1,$$

so that $f\in Harm^1(D)$ and $N_1 f=f(0)$. Then the assertions follow from 1.3 and 1.4. QED.

2. Pointwise Convergence: The Fatou Theorem and its Converse

Let $f\in Harm^1(D)$. We ask for the pointwise convergence behaviour of the functions f_R for $R\uparrow 1$. The answer is the famous Fatou theorem. Besides of this abelian-type theorem we prove its tauberian converse due to Loomis. As usual the converse is not true unless an extra tauberian condition is satisfied: here we have to assume that $f\geqq 0$. The Loomis theorem will in Section V.5 be valuable for the abstract theory.

It is convenient for us to work with functions of bounded variation. The Baire measures $\theta\in ca(S)$ are in bijective correspondence with the functions of bounded variation $\Theta:[-\pi,\pi]\to\mathbb{C}$ with the normalization

$$\Theta(t) = \frac{1}{2}(\Theta(t+)+\Theta(t-)) \text{ for } |t|<\pi,\ \Theta(\pi) - \Theta(\pi-) = \Theta(-\pi+) - \Theta(-\pi),$$

and $\Theta(0)=0$: the correspondence is

$$\int_S F d\theta = \int_{-\pi}^{\pi} F(e^{it})d\Theta(t) \quad \forall\ F\in C(S).$$

We can extend $\Theta:[-\pi,\pi]\to\mathbb{C}$ to a unique function $\Theta:\mathbb{R}\to\mathbb{C}$ with the periodicity property $\Theta(t+2\pi)-\Theta(t)=\text{const}\ \forall$ real t, which then is a function of local bounded variation with the normalization $\Theta(t)=\frac{1}{2}(\Theta(t+)+\Theta(t-))\ \forall$ real t and $\Theta(0)=0$. The above correspondence reads

$$\int_S F d\theta = \int_{\tau-\pi}^{\tau+\pi} F(e^{it})d\Theta(t) \quad \forall\ F\in C(S) \text{ and each real } \tau.$$

Equivalent is

$$\theta(\{e^{it}:u<t<v\}) = \Theta(v-)-\Theta(u+) \quad \forall \text{ real } u<v\leqq u+2\pi.$$

In particular θ is nonnegative $\in Pos(S)$ iff Θ is real-valued and monotone increasing.

2.1 FATOU THEOREM: Let $f\in Harm^1(D)$ with corresponding $\theta\in ca(S)$ and $\Theta:\mathbb{R}\to\mathbb{C}$. Let $\alpha\in\mathbb{R}$ with

$$\frac{\Theta(\alpha+t)-\Theta(\alpha-t)}{2t} \to \frac{A}{2\pi} \quad \text{for } t\downarrow 0.$$

Then $f(Re^{i\alpha})\to A$ for $R\uparrow 1$.

From the above we see that

$$\frac{\Theta(\alpha+t-)-\Theta(\alpha-t+)}{2t} = \frac{1}{2t}\,\theta(\{e^{iu}:\alpha-t<u<\alpha+t\})$$

$$= \frac{1}{2\pi}\,\frac{\theta(\{e^{iu}:\alpha-t<u<\alpha+t\})}{\lambda(\{e^{iu}:\alpha-t<u<\alpha+t\})} \quad \forall\ 0<t\leqq\pi,$$

and this tends $\to \frac{1}{2\pi}\frac{d\theta}{d\lambda}(e^{i\alpha})$ for $t\downarrow 0$ for λ-almost all $e^{i\alpha}\in S$. Thus we have the subsequent corollary.

2.2 COROLLARY: Let $f\in Harm^1(D)$ with corresponding $\theta\in ca(S)$. Then the radial limit $\lim_{R\uparrow 1} f(Rs)$ exists for λ-almost all $s\in S$, and the limit function is $=\frac{d\theta}{d\lambda}\in L^1(\lambda)$.

The result can be extended from radial limits to non-tangential limits. We shall come back to this point in Section 4.

2.3 LOOMIS THEOREM: Let $f\in Harm^1(D)$ be nonnegative with corresponding $\theta\in ca(S)$ and $\Theta:\mathbb{R}\to\mathbb{C}$. Let $\alpha\in\mathbb{R}$ with $f(Re^{i\alpha})\to A$ for $R\uparrow 1$. Then