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HIGHER ALGEBRA

ИЗДАТЕЛЬСТВО • МИР •

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КУРС ВЫСШЕЙ АЛГЕБРЫ

ИЗДАТЕЛЬСТВО «НАУКА»
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INTRODUCTION

The education of the mathematics major begins with the study of three basic disciplines: mathematical analysis, analytic geometry and higher algebra. These disciplines have a number of points of contact, some of which overlap; together they constitute the foundation upon which rests the whole edifice of modern mathematical science.

Higher algebra—the subject of this text—is a far-reaching and natural generalization of the basic school course of elementary algebra. Central to elementary algebra is without doubt the problem of solving equations. The study of equations begins with the very simple case of one equation of the first degree in one unknown. From there on, the development proceeds in two directions: to systems of two and three equations of the first degree in two and, respectively, three unknowns, and to a single quadratic equation in one unknown and also to a few special types of higher-degree equations which readily reduce to quadratic equations (quartic equations, for example).

Both trends are further developed in the course of higher algebra, thus determining its two large areas of study. One—the foundations of linear algebra—starts with the study of arbitrary systems of equations of the first degree (linear equations). When the number of equations equals the number of unknowns, solutions of such systems are obtained by means of the theory of determinants. However, the theory proves insufficient when studying systems of linear equations in which the number of equations is not equal to the number of unknowns. This is a novel feature from the standpoint of elementary algebra, but it is very important in practical applications. This stimulated the development of the theory of matrices, which are systems of numbers arranged in square or rectangular arrays made up of rows and columns. Matrix theory proved to be very deep and has found application far beyond the limits of the theory of systems of linear equations. On the other hand, investigations into systems of linear equations gave rise to multidimensional (so-called vector or linear) spaces. To the nonmathematician, multidimensional space (four-dimensional, to begin with) is a nebulous and often confusing concept. Actually, however, the notion is a strictly mathematical one, mainly algebraic, and serves as an important tool in a variety of mathematical investigations and also in physics.

The second half of the course of higher algebra, called the algebra of polynomials, is devoted to the study of a single equation in one unknown but of arbitrary degree. Since there is a formula for solving quadratic equations, it was natural to seek similar formulas for higher-degree equations. That is precisely how this division of algebra developed historically. Formulas for solving equations of third and fourth degree were found in the sixteenth century. The search was then on for formulas capable of expressing the roots of equations of fifth and higher degree in terms of the coefficients of the equations by means of radicals, even radicals within radicals. It was futile, though it continued up to the beginning of the nineteenth century, when it was proved that no such formulas exist and that for all degrees beyond the fourth there even exist specific examples of equations with integral coefficients whose roots cannot be written down by means of radicals.

One should not be saddened by this absence of formulas for solving equations of higher degrees, for even in the case of third and fourth degree equations, where such formulas exist, computations are extremely involved and, in a practical sense, almost useless. On the other hand, the coefficients of equations one encounters in physics and engineering are usually quantities obtained in measurements. These are approximations and therefore the roots need only be known approximately, to within a specified accuracy. This led to the elaboration of a variety of methods of approximate solution of equations; only the most elementary methods are given in the course of higher algebra.

However, in the algebra of polynomials the main thing is not the problem of finding the roots of equations, but the problem of their existence. For example, we even know of quadratic equations with real coefficients that do not have real-valued roots. By extending the range of numbers to include the collection of complex numbers, we find that quadratic equations do have roots and that this holds true for equations of the third and fourth degree as well, as follows from the existence of formulas for their solution. But perhaps there are equations of the fifth and higher degree without a single root even in the class of complex numbers. Will it not be necessary, when seeking the roots of such equations, to pass from complex numbers to a still bigger class of numbers? The answer to this question is contained in an important theorem which asserts that any equation with numerical coefficients, whether real or complex, has complex-valued (real-valued, as a special case) roots; and, generally speaking, the number of roots is equal to the degree of the equation.

Such, in brief, is the basic content of the course of higher algebra. It must be stressed that higher algebra is only the starting point of the vast science of algebra which is very rich, extremely ramified and constantly expanding. Let us attempt, even more sketchily, to survey the various branches of algebra which, in the main, lie beyond the scope of the course of higher algebra.

Linear algebra, which is a broad field devoted mainly to the theory of matrices and the associated theory of linear transformations of vector spaces, includes also the theory of forms, the theory of invariants and tensor algebra, which plays an important role in differential geometry. The theory of vector spaces is further developed outside the scope of algebra, in functional analysis (infinite-dimensional spaces). Linear algebra continues, so far, to occupy first place among the numerous branches of algebra as to diversity and significance of its applications in mathematics, physics and the engineering sciences.

The algebra of polynomials, which over many decades has been growing as a science concerned with one equation of arbitrary degree in one unknown, has now in the main completed its development. It was further developed in part in certain divisions of the theory of functions of a complex variable, but basically grew into the theory of fields, which we will speak of later on. Now the very difficult problem of systems of equations of arbitrary degree (not linear) in several unknowns—it embraces both divisions of the course of higher algebra and is hardly touched on in this text—actually has to do with a special branch of mathematics called algebraic geometry.

An exhaustive treatment of the problem of the conditions under which an equation can be solved in terms of radicals was given by the French mathematician Galois (1811-1832). His investigations pointed out new vistas in the development of algebra and led, in the twentieth century, after the work of the German woman-algebraist E. Noether (1882-1935), to the establishment of a fresh viewpoint on the problems of algebraic science. There is no doubt now that the central problem of algebra is not the study of equations. The true subject of algebraic study is algebraic operations, like those of addition and multiplication of numbers, but possibly involving entities other than numbers.

In school physics one deals with the operation of composition of forces. The mathematical disciplines studied in the junior courses of universities and teachers' colleges provide numerous examples of algebraic operations: the addition and multiplication of matrices and functions, operations involving vectors, transformations of space, etc. These operations are usually similar to those involving numbers and bear the same names, but occasionally some of the properties which are customary in the case of numbers are lost. Thus, very often and in very important instances, the operations prove to be noncommutative (a product is dependent on the order of the factors), at times even nonassociative (a product of three factors depends on the placing of parentheses).

A very systematic study has been made of a few of the most important types of algebraic systems (or structures), that is, sets composed of entities of a certain nature for which certain algebraic operations have been defined. Such, for example, are fields. These are algebraic systems in which (like in the systems of real and complex numbers) are defined the operations of addition and multiplication, both commutative and associative, connected by the distributive law (the ordinary rule of removing brackets holds) and possessing the inverse operations of subtraction and division. The theory of fields was a natural area for the further development of the theory of equations, while its principal branches—the theory of fields of algebraic numbers and the theory of fields of algebraic functions—linked it up with the theory of numbers and the theory of functions of a complex variable, respectively. The present course of higher algebra includes an elementary introduction to the theory of fields, and some portions of the course—polynomials in several unknowns, the normal form of a matrix—are presented directly for the case of an arbitrary base field.

Broader than a field is the concept of a ring. Unlike the field, division is not required here and, besides, multiplication may be noncommutative and even nonassociative. The simplest instances of rings are the set of all integers (including negative numbers), the set of polynomials in one unknown and the set of real-valued functions of a real variable. The theory of rings includes such old branches of algebra as the theory of hypercomplex numbers and the theory of ideals. It is related to a number of mathematical sciences (functional analysis being one) and has already made inroads into physics. The course of higher algebra actually contains only the definition of a ring.

Still greater in its range of applications is the theory of groups. A group is an algebraic system with one basic operation, which must be associative but not necessarily commutative, and must possess an inverse operation (division if the basic operation is multiplication). Such, for example, is the set of integers with respect to the operation of addition and also the set of positive real numbers with respect to the operation of multiplication. Groups were already important in the theory of Galois, in the problem of the solvability of equations in terms of radicals; today groups are a powerful tool in the theory of fields, in many divisions of geometry, in topology, and also outside mathematics (in crystallography and theoretical physics). Generally speaking, within the sphere of algebra, group theory takes second place after linear algebra as to its range of applications. Our course of higher algebra contains a chapter on the fundamentals of the theory of groups.

In recent decades an entirely new branch of algebra—lattice theory—has come to the fore. A lattice is an algebraic system with two operations—addition and multiplication. These operations must be commutative and associative and must also satisfy the following requirements: both the sum and the product of an element with itself must be equal to the element; if the sum of two elements is equal to one of them, then the product is equal to the other, and conversely. An example of a lattice is the system of natural numbers relative to the operations of taking the least common multiple and the greatest common divisor. Lattice theory has interesting ties with the theory of groups and the theory of rings, and also with the theory of sets; one old branch of geometry (projective geometry) actually proved to be a part of the theory of lattice. It is also worth mentioning the expansion of lattice theory into the theory of electric circuits.

Certain similarities between parts of the theories of groups, rings and lattices led to the development of a general theory of algebraic systems (or universal algebras). The theory has only taken a few steps but its general outlines are evident and certain links with mathematical logic that have been perceived point to a rich future in this area.

The foregoing scheme does not of course embrace the whole range of algebraic science. For one thing, there are a number of divisions of algebra bordering on other areas of mathematics, such as topological algebra, which deals with algebraic systems in which the operations are continuous relative to some convergence defined for the elements of the systems. An example is the system of real numbers. Closely related to topological algebra is the theory of continuous (or Lie) groups, which has found numerous applications in a broad range of geometrical problems, in theoretical physics and hydrodynamics. Incidentally, the theory of Lie groups is characterized by such an interweaving of algebraic, topological, geometric and function-theoretic methods as to be more properly considered a special branch of mathematics altogether. Next we have the theory of ordered algebraic systems which arose out of investigations into the fundamentals of geometry and has found applications in functional analysis. Finally, there is differential algebra which has established fresh relationships between algebra and the theory of differential equations.

Quite naturally, the flowering of algebraic science so evident today is not accidental, but is an organic part of the general advance of mathematics and is due, in large measure, to the demands made upon algebra by the other mathematical sciences. On the other hand, the development of algebra itself has exerted a far-reaching influence on the elaboration of allied branches of science; this influence has been particularly enhanced by the spread of applications so characteristic of modern algebra. One is often tempted to speak of an "algebraization" of mathematics.

We conclude this rather sketchy survey of algebra with a general historical background.

Babylonian and, later, ancient Greek mathematicians studied certain problems of algebra, in particular the solution of simple equations. The peak of algebraic investigations during this period was reached in the works of the Greek mathematician Diophantos of Alexandria (third century). These studies were then extended by mathematicians of India: Aryabhata (sixth century), Brahmagupta (seventh century), and Bhaskara (twelfth century). In China, algebraic problems got an early start: Ch'ang Ts'ang (second century B.C.), Ching Chou-chan (first century A.D.). An outstanding Chinese algebraist was Ch'in Chiu-shao (thirteenth century).

A major contribution to the development of algebra was made by scholars of the Middle East whose writings were in Arabic, particularly the Uzbek scholar Muhammad al-Khowârizmî (ninth century) and the Tajik mathematician and poet Omar Khayyam (1040-1123). In particular, the very term "algebra" came from the title of al-Khowârizmî's treatise *Hisâb al-jabr w'al-muqâ-balah.*

The above-mentioned studies of Babylonian, Greek, Indian, Chinese, and Central-Asian algebraists have to do with those problems of algebra which constitute the present school course of elementary algebra and only occasionally touch on equations of the third degree. That, in the main, was the range of problems that interested medieval European algebraists and those of the Renaissance, such as the Italian mathematician Leonardo of Pisa (Fibonacci) (twelfth century) and the founder of present-day algebraic symbolism, the Frenchman Vieta (or Viète) (1540-1603). We have already mentioned that in the sixteenth century methods were found for solving equations of the third and fourth degree; here we must mention the names of the Italians Ferro (1465-1526), Tartaglia (1500-1557), Cardano (1501-1576) and Ferrari (1522-1565).

The seventeenth and eighteenth centuries saw an intensive elaboration of the general theory of equations (or the algebra of polynomials) in which outstanding scholars of the time participated: Descartes (1596-1650), Sir Isaac Newton (1643-1727), d'Alembert (1717-1783) and Lagrange (1736-1813). In the eighteenth century, the Swiss mathematician Cramer (1704-1752) and Laplace (1749-1827) of France, laid the foundation of the theory of determinants. At the turn of the century, the great German mathematician Gauss (1777-1855) proved the earlier mentioned fundamental theorem on the existence of roots of equations with numerical coefficients.

The first third of the nineteenth century stands out in the history of algebra as the time when the problem of the solvability of equations by radicals was resolved. Proof of the impossibility of obtaining formulas for the solution of equations of degree five or higher was obtained by the Italian mathematician Ruffini (1765-1822) and in more rigorous form by the Norwegian Abel (1802-1829). As already mentioned, an exhaustive treatment of the problem of the conditions under which an equation admits of solution in terms of radicals was given by Galois.

Galois' theory spurred the advance of algebra in the latter half of the nineteenth century. There appeared the theory of fields of algebraic numbers and of fields of algebraic functions and the associated theory of ideals. Here, mention should be made of the German mathematicians Kummer (1810-1893), Kronecker (1823-1891), and Dedekind (1831-1916), and the Russian mathematicians E. I. Zolotarev (1847-1878) and G. F. Voronoi (1868-1908). Particular advances were made in the theory of finite groups which grew out of the research of Lagrange and Galois; this work was carried out by the French mathematicians Cauchy (1789-1857) and Jordan (1838-1922), the Norwegian Sylow (1832-1918), the German algebraists Frobenius (1849-1918) and Hölder (1859-1937). The investigations of the Norwegian S. Lie (1842-1899) initiated the theory of continuous groups.

The works of Hamilton (1805-1865) and the German mathematician Grassmann (1809-1877) laid the foundations for the theory of hypercomplex systems or, as we now say, the theory of algebras. A prominent role in the development of this branch of algebra was played (at the end of the century) by the Russian mathematician F. E. Molin (1861-1941).

Linear algebra attained great heights in the nineteenth century primarily due to the work of the English mathematicians Sylvester (1814-1897) and Cayley (1821-1895). Work continued on the algebra of polynomials; we note only the method of approximate solution of equations found by the Russian geometer N. I. Lobachevsky (1792-1856) and the work of the German Hurwitz (1859-1919). Algebraic geometry was begun in the latter part of the nineteenth century, particularly in the works of the German mathematician M. Noether (1844-1922).

In the twentieth century, algebraic studies expanded considerably and algebra, as we already know, occupies a very special place of honour in mathematics. New divisions of algebra have sprung up, including the general theory of fields (in the 1910's), the theory of rings and the general theory of groups (1920's), topological algebra and lattice theory (1930's), the theory of semigroups and the theory of quasigroups, the theory of universal algebras, homological algebra, the theory of categories (all in the 1940's and 1950's). Prominent mathematicians are presently engaged in all spheres of algebra, and in a number of countries (in the Soviet Union, for example) whole schools of algebra are in evidence.

Among the prerevolutionary Russian algebraists, noteworthy contributions to algebra were also made by S.O. Shatunovsky (1859-1929) and D. A. Grave (1863-1939). However, it was only after the Great October Revolution of 1917 that algebraic investigations in the Soviet Union reached high peaks. These studies now embrace practically all divisions of modern algebraic science and in some the work of Soviet algebraists is of a leading nature. Suffice it to name only two algebraists: N. G. Chebotarev (1894-1947), who worked in the theory of fields and Lie groups, and O. Yu. Schmidt (1891-1956), the famous polar explorer who was also a noted algebraist and founded the Soviet school of group theory.

We conclude this brief survey of the historical background and modern state of algebra with the remark that most of the fields of research mentioned here lie beyond the scope of the present course of higher algebra. The aim of the survey was to help the reader to find the proper place for this text in algebraic science as a whole within the edifice of mathematics.

CHAPTER 1

SYSTEMS OF LINEAR EQUATIONS. DETERMINANTS

1. The Method of Successive Elimination of Unknowns

We begin the course of higher algebra with a study of systems of first-degree equations in several unknowns or, to use the more common term, *systems of linear equations.**

The theory of systems of linear equations serves as the foundation for a vast and important division of algebra—linear algebra—to which a good portion of this book is devoted (the first three chapters in particular). The coefficients of the equations considered in these three chapters, the values of the unknowns and, generally, all numbers that will be encountered are to be considered real. Incidentally, all the material of these three chapters is readily extendable to the case of arbitrary complex numbers which are familiar from elementary mathematics.

In contrast to elementary algebra, we will study systems with an arbitrary number of equations and unknowns; at times, the number of equations of a system will not even be assumed to coincide with the number of unknowns. Suppose we have a system of s linear equations in n unknowns. Let us agree to use the following symbolism: the unknowns will be denoted by x and subscripts: $x_1, x_2, \ldots, x_n$; we will consider the equations to be enumerated thus: first, second, . . ., sth; the coefficient of x_j in the ith equation will be given as a_{ij}**. Finally, the constant term of the ith equation will be indicated as b_i.

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* The term "linear" stems from analytic geometry, where a first-degree equation in two unknowns defines a straight line in a plane.

** We thus use two subscripts, the first indicates the position number of the equation, the second the position number of the unknown. They are to be read: a_{11} "a sub one one" and not "a eleven"; a_{34} "a sub three four" and not "a thirty-four", and are not separated by a comma.

Our system of equations will now be written as follows:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{s1}x_1 + a_{s2}x_2 + \ldots + a_{sn}x_n &= b_s \end{aligned}\right\} \qquad (1)$$

The coefficients of the unknowns form a rectangular array:

$$\begin{pmatrix} a_{11}a_{12} & \ldots & a_{1n} \\ a_{21}a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots \\ a_{s1}a_{s2} & \ldots & a_{sn} \end{pmatrix} \qquad (2)$$

called a *matrix* of *s rows* and *n columns*; the numbers a_{ij} are termed *elements* of the matrix.* If $s=n$ (which means the number of rows is equal to the number of columns), then the matrix is called *a square matrix of order n*. The diagonal of the matrix from upper left corner to lower right corner (i.e., composed of the elements $a_{11}, a_{22}, \ldots, a_{sn}$) is called the *principal diagonal*. We call a square matrix of order n a *unit matrix of order n* if all the elements of its principal diagonal are equal to unity and all other elements are zero.

The *solution* of the system of linear equations (1) is a set of n numbers $k_1, k_2, \ldots, k_n$ such that each of the equations (1) becomes an identity upon substitution of the corresponding numbers k_i, $i = 1, 2, \ldots, n$ for the unknowns x_i.**

A system of linear equations may not have any solutions; it is then called *inconsistent*. Such, for example, is the system

$$x_1 + 5x_2 = 1,$$
$$x_1 + 5x_2 = 7$$

The left members of these equations coincide, but the right members are different and so no set of values of the unknowns can satisfy both equations simultaneously.

If a system of linear equations has solutions, it is termed *consistent*. A consistent system is called *determinate* if it has a unique solution—only such are considered in elementary algebra—and *indeterminate* if there are more solutions than one. As we shall learn later on, there may even be an infinity of solutions. For instance,

* Thus, if the matrix (2) is regarded by itself (not connected with the system (1)), then the first subscript of element a_{ij} indicates the number of the row, the second the number of the column at the intersection of which the element is positioned.

** We stress the fact that the numbers $k_1, k_2, \ldots, k_n$ constitute a *single* solution of the system and not n solutions.