DYNAMIC FRACTURE MECHANICS
VOLUME 2: PROPAGATING CRACKS
Revised Edition

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V. Z. Parton
V. G. Boriskovsky

Institute of Chemical Engineering
Moscow, USSR

Translated by
Ram S. Wadhwa

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PREFACE

The second volume of *Dynamic Fracture Mechanics* is a logical continuation of the investigations that were started in the first volume. It should be recalled that we began with a description of the basic concepts of dynamic fracture mechanics, and then considered analytical and numerical methods for determining stress intensity factors in two- and three-dimensional bodies with stationary linear, curvilinear, plane, and penny-shaped cracks subjected to harmonic and impact loading. In this volume, we shall consider the laws of crack propagation with a constant or variable velocity in elastic and elastic-plastic bodies and an elastic lattice. We shall also describe numerical and experimental methods for determining the stress intensity factors in bodies with running cracks, as well as methods for the arrest of cracks.

While preparing this volume we adhered to the notation and method of presentation used in the first volume, since it presents the formulation and solution of the problems in a simple and comprehensible manner. This spares us the task of writing a new extensive preface, but a few remarks are in order.

Over the last few decades, fracture mechanics has come to be recognized as a separate branch of mechanics of deformable solids. The main aim of these investigations is to determine the load-carrying capacity of bodies and structures by taking into account the initial distribution and possible propagation of cracks. The results obtained are used to ensure the strength, reliability, and long-life of structures, and to work out effective means of nondestructive testing in order to prevent accidents that may have serious economic and social repercussions. However, it is obvious that investigations of fracture mechanics are also important for techniques involving controlled destruction, e.g., for the extraction of mineral deposits, drilling of wells, and cutting of metals.

The solution of these problems in fracture mechanics involves the construction of fracture models and the development of analytical and numerical methods of solving problems for bodies with stationary and running cracks within the framework of the theory of elasticity, plasticity, viscoelasticity, and for nonlinear media.

The successful practical application of fracture mechanics can primarily be attributed to the mechanics of quasi-static cracks. In this case, methods have been worked out and standardized to answer questions concerning the stability of an existing arterial crack under quasi-stationary loads.

As regards dynamic fracture mechanics, which analyzes the stability of stationary loads subjected to dynamic loading and processes of crack propagation, the theoretical investigations cannot be backed up by practical recommendations, for the time being. This is due to the extremely complicated behavior of fracture mechanics, and also to the existing disproportionality between the development of theoretical and experimental methods in dynamic fracture mechanics. For many years, progress in this field was associated with the solution (by analytical and numerical methods) of simulation problems in idealized situations. This left open the question of a correspondence between the idealized situation and the real conditions of dynamic fracture, as well as the experimental confirmation of theoretical results.

However, the number of articles devoted to experimental methods in dynamic fracture mechanics has increased sharply over the last few years and has necessitated a reconsideration of many basic concepts. In this field a presentation of the results is required in which "the leitmotif, the ever recurring melody, is that two things are indispensable in any reasoning, in any description we shape of a segment of reality; to submit to experience and to face the language that is used, with unceasing logical criticism."*

Accordingly, considerable attention is paid in this book to a description of the experimental methods in dynamic fracture mechanics and to a comparison of theoretical and experimental results.

With this end in view, the authors have supplemented the results of their investigations with other important results, including those obtained by American scientists, who have made a significant contribution towards the development of this field.

The authors are privileged to express their gratitude to Profs. V. M. Finkel, G. L. Khesin, and L. I. Slepyan, and also to Dr. V. M. Markochev for the material they provided.

V. Z. Parton
V. G. Boriskovsky

*Richard von Mises, Mathematical theory of compressible fluid flow. Completed by Hilda Geiringer and G. S. S. Ludford. Academic Press, New York, 1958.

CONTENTS

CHAPTER
ONE

CRACK PROPAGATION WITH CONSTANT AND VARIABLE VELOCITIES

Problems concerning the propagation of cracks in plates were first formulated and solved in [10, 34]. The problem of the appearance of a crack (at the initial moment of time $t = 0$) and its propagation in both directions (starting from a zero initial length) with a constant velocity under the action of a uniform tensile stress was considered in [34], while the sudden appearance of a semi-infinite cut at the instant $t = 0$ in a tensile stress field and its propagation with a constant velocity were discussed in [10]. Naturally, the solution of both these problems can be regarded as a calibration test for determining the applicability of numerical methods in the investigation of crack propagation. A comparison of analytical and numerical results is carried out for the initial instants of time (until the arrival at the crack tip of the waves scattered from the edge or from the opposite crack tip), since the analytical results are obtained for inifinite bodies. In the case of a stationary crack, the solution obtained in [10] coincides with that for a plate with a semi-inifinite crack subjected to an instantaneous tensile stress applied to the crack faces. This solution can also be used for calibration of numerical results. It should be observed that both these results are particular cases of the general solution of the problem of crack propagation with an arbitrary velocity under the action of arbitrary loads.

The problem of crack growth at a variable speed ($0 < \upsilon < c_2$) for an elastic body under antiplane strain was solved by B.V. Kostrov [92]. For plane strain, the problem was solved by L.B. Freund [64, 65] (for time-independent loads) and B.V. Kostrov [93] (for arbitrary loading). The latter solution is quite cumbersome and hard to realize in actual practice. It is better to solve this type of problem by using methods which may be less universal but present the results in a more comprehensible form. The number of quadratures in the general solution for a semi-infinite crack ($0 < v < c_R$) was reduced from five to four by L.I. Slepyan [207], while an analysis of the general solution and functions, in terms through which the solution is presented, was carried out in [199]. The axisymmetric problem for a penny-shaped crack propagating with a variable velocity was solved in [197].

Problems of cracks propagating with supercritical velocities* were analyzed in [12] for wedging out at a constant velocity ($c_R < v < c_2$), and in [95] for crack propagation with a variable velocity in the same range.

The solution of the problem in which the velocity of crack propagation passes through the critical value (in the course of its motion), say, from the interval $0 < v < c_R$ to $c_R < v < c_2$, or vice versa, was obtained in [217, 218]. The interacoustic range of crack propagation was also investigated in these works.

It is well known that the numerical realization of dynamic problems of fracture mechanics is quite complicated. Hence, an approximation has been worked out in [192, 207], based on a simplification of the expression for the solution of Lamb's problem. A detailed analysis of the approximation technique and a comparison of the results obtained by this method with the exact solution are contained in [199].

The propagation of a semi-infinite crack whose faces are suddenly subjected to uniformly distributed stresses (this is equivalent to the sudden appearance of a semi-infinite crack in a stressed body) is investigated in [130] (antiplane problem) and [196] (plane problem). The increase in load applied to the faces of a semi-infinite crack is discussed in [195], where it is shown that it may lead to a decrease in the velocity of crack propagation. If a load applied at a certain distance from the edge of a crack moves its faces apart, a collision of the crack faces takes place in the beginning. This process has been described in [198].

A number of self-similar problems concerning the propagation of finite cracks have been investigated. A review of the works devoted to self-similar problems in the theory of elasticity is contained in [4].

In this chapter we shall consider three alternative approaches to the solution of problems of crack propagation with variable velocity. The solution of several particular problems will also be presented.

1.1 SELF-SIMILAR PROBLEMS

We introduce the polar system of coordinates with its origin at the center of a crack propagating in both directions. The components of the displacement vector are expressed in the normal form through two wave potentials φ and ψ:

$$u_r = \frac{\partial \varphi}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}; \quad u_\theta = \frac{1}{r}\frac{\partial \varphi}{\partial \theta} - \frac{\partial \psi}{\partial r}; \quad u_z = 0 \tag{1.1}$$

*The solution of corresponding mixed problems is of particular interest for describing collisions of elastic bodies. The possibility of induced propagation of cracks with supersonic velocities is discussed in [20].

Taking these relations into account, we can present the components of the stress tensor in the following form:

$$\sigma_{rr}=2\mu\frac{1-\nu}{1-2\nu}\nabla^2\varphi-2\mu\left(\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial\psi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta}\right)$$

$$\sigma_{\theta\theta}=2\mu\frac{\nu}{1-2\nu}\nabla^2\varphi+2\mu\left(\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial\varphi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta}\right) \tag{1.2}$$

$$\sigma_{zz}=2\mu\frac{\nu}{1-2\nu}\nabla^2\varphi,\quad \sigma_{\theta z}=\sigma_{rz}=0$$

$$\sigma_{r\theta}=\mu\left(\frac{2}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}-\frac{2}{r^2}\frac{\partial\varphi}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{\partial^2\psi}{\partial r^2}\right)$$

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{1.3}$$

The potentials φ and ψ satisfy the wave equations

$$\nabla^2\varphi=\frac{1}{c_1^2}\frac{\partial^2\varphi}{\partial t^2},\quad \nabla^2\psi=\frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2} \tag{1.4}$$

As usual, the spatial variable r and time t are introduced in the solution of the self-stimulating problem with the help of a single variable $\omega = r/t$. Going from variables r,θ,t to variables ω,θ, the second derivatives of the potentials $\varphi(r,\theta,t)$ and $\psi(r,\theta,t)$ with respect to time can be written in the form

$$\frac{\partial^2\varphi}{\partial t^2}=\Phi(\omega,\ \theta),\quad \frac{\partial^2\psi}{\partial t^2}=\Psi(\omega,\ \theta) \tag{1.5}$$

Functions Φ and Ψ satisfy the equations

$$\omega^2[1-(\omega/c_1)^2]\frac{\partial^2\Phi}{\partial\omega^2}+[1-2(\omega/c_1)^2]\frac{\partial\Phi}{\partial\omega}+\frac{\partial^2\Phi}{\partial\theta^2}=0$$

$$\omega^2[1-(\omega/c_2)^2]\frac{\partial^2\Psi}{\partial\omega^2}+[1-2(\omega/c_2)^2]\frac{\partial\Psi}{\partial\omega}+\frac{\partial^2\Psi}{\partial\theta^2}=0 \tag{1.6}$$

Using the substitution of variables [202]

$$\omega=c_1\,\mathrm{sech}(-\lambda_1),\quad \omega=c_2\,\mathrm{sech}(-\lambda_2)$$

we can transform the system (1.6) as follows:

$$\frac{\partial^2\Phi}{\partial\lambda_1^2}+\frac{\partial^2\Phi}{\partial\theta^2}=0;\quad \frac{\partial^2\Psi}{\partial\lambda_2^2}+\frac{\partial^2\Psi}{\partial\theta^2}=0 \tag{1.7}$$

Hence the functions Φ and Ψ can be presented as the real parts of the complex functions

$$\Phi=\mathrm{Re}\,[w_1(\lambda_1-i\theta)]=\mathrm{Re}\,(\Phi+i\chi_1)$$

$$\Psi=\mathrm{Re}\,[w_2(\lambda_2-i\theta)]=\mathrm{Re}\,(\Psi+i\chi_2) \tag{1.8}$$

The derivatives of $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ with respect to ω can be expressed in terms of Φ and Ψ:

$$\frac{\omega^2}{\mu}\frac{\partial\sigma_{\theta\theta}}{\partial\omega} = -[2-(\omega/c_2)^2]\frac{\partial\Phi}{\partial\omega} - \frac{2}{\omega}\frac{\partial\Psi}{\partial\theta}$$
$$\frac{\omega^2}{\mu}\frac{\partial\sigma_{r\theta}}{\partial\omega} = -[2-(\omega/c_2)^2]\frac{\partial\Psi}{\partial\omega} + \frac{2}{\omega}\frac{\partial\Phi}{\partial\theta} \tag{1.9}$$

With the help of Cauchy-Riemann conditions we can present these equations in the form

$$\frac{\omega^2}{\mu}\frac{\partial\sigma_{\theta\theta}}{\partial\omega} = -[2-(\omega/c_2)^2]\frac{\partial\Phi}{\partial\omega} + 2[1-(\omega/c_2)^2]^{1/2}\frac{\partial\chi_2}{\partial\omega}$$
$$\frac{\omega^2}{\mu}\frac{\partial\sigma_{r\theta}}{\partial\omega} = -[2-(\omega/c_2)^2]\frac{\partial\Psi}{\partial\omega} - 2[1-(\omega/c_1)^2]^{1/2}\frac{\partial\chi_1}{\partial\omega} \tag{1.10}$$

Suppose that a finite crack propagates in two opposite directions with a constant velocity v. A uniform tensile stress field is applied at a distance from the crack in such a way that the following conditions are satisfied:

$$\sigma_{\theta\theta}(\omega,\ 0) = \sigma_{\theta\theta}(\omega,\ \pi) = 0, \quad |\omega| < v$$
$$u_\theta(\omega,\ 0) = u_\theta(\omega,\ \pi) = 0, \quad |\omega| \geqslant v \tag{1.11}$$
$$\sigma_{r\theta}(\omega,\ 0) = \sigma_{r\theta}(\omega,\ \pi) = 0; \ -\infty < \omega < \infty$$

$$\Phi(\omega,\ \theta) = \begin{cases} \dfrac{2}{\mu} q^{(1)} c_2^2, & \omega = c_1 \\ 0, & \omega > c_1 \end{cases} \tag{1.12}$$
$$\Psi(\omega,\ 0) = 0; \ \omega \geqslant c_2$$

Let us carry out another transformation

$$\zeta_1 = \xi_1 + i\eta_1 = \operatorname{sech}(\lambda_1 - i\theta)$$

mapping the semicircle $0 \leqslant \theta \leqslant \pi, \omega \leqslant c_1$ onto the halfplane $\eta_1 \geqslant 0$. The semicircle without diameter is mapped onto the interval $\eta_1 = 0,\ 1 < |\zeta_1| < \infty$ of the real axis, while the diameter is mapped onto the segment $\eta_1 = 0,\ |\zeta_1| < 1$. Similarly, the transformation

$$\zeta_2 = \xi_2 + i\eta_2 = \operatorname{sech}(\lambda_2 - i\theta)$$

is also applied to the function Ψ. Formula (1.12) is then transformed into

$$\operatorname{Re}[w_1'(\zeta_1)] = 0 \ \text{at} \ \eta_1 = 0, \ |\xi_1| > 1$$
$$\operatorname{Re}[w_2'(\zeta_2)] = 0 \ \text{at} \ \eta_2 = 0, \ |\xi_2| > 1 \tag{1.13}$$

while the second and third conditions in (1.11) can be written in the following form:

$$\operatorname{Im}[w_1'(\zeta_1)] = 0; \ \eta_1 = 0; \ (v/c_1) < |\xi_1| < 1$$
$$\operatorname{Re}[w_2'(\zeta_2)] = 0; \ \eta_2 = 0; \ |\xi_2| > (v/c) \tag{1.14}$$

For $0 < \omega < v$, when $\omega = \pm c_1\xi_1 = \pm c_2\xi_2$, conditions $\sigma_{\theta\theta} = 0$ and $\sigma_{r\theta} = 0$ assume the form

$$c_1 [2-(\omega/c_2)^2] \operatorname{Re} [w_2'(\zeta_2)] + 2c_2 [1-(\omega/c_1)^2]^{1/2} \operatorname{Im} [w_1'(\zeta_1)] = 0$$

$$c_2 [2-(\omega/c_2)^2] \operatorname{Re} [w_1'(\zeta_1)] - 2c_1 [1-(\omega/c_2)^2]^{1/2} \operatorname{Im} [w_2'(\zeta)] = 0 \tag{1.15}$$

Assuming that the stresses at the crack tip ($\omega = v$) are singular and have a singularity of the type $(v-\omega)^{-1/2}$ for $\omega_1'(\zeta_1)$) and $\omega_2'(\zeta_2)$ we obtain

$$w_1'(\zeta_1) = O\,(v^2 - c_1^2\zeta_1^2)^{-1/2} \quad \text{at} \quad |c_1\zeta_1| \to v$$

$$w_2'(\zeta_2) = O\,(v^2 - c_2^2\zeta_2^2)^{-1/2} \quad \text{at} \quad |c_2\zeta_2| \to v \tag{1.16}$$

Function $W_2'\zeta_2$ can be found from the second equations in (1.13), (1.14), and (1.16):

$$w_2'(\zeta_2) = \frac{dw_2}{d\zeta_2} = -\frac{2Ac_2^3}{c_1\,(v^2 - c_2^2\zeta_2^2)^{3/2}} \tag{1.17}$$

Similarly, we get

$$w_1'(\zeta_1) = \frac{dw_1}{d\zeta_1} = \frac{iA\,(2c_2^2 - c_1^2\zeta_1^2)}{(1-\zeta_1^2)^{1/2}\,(v^2 - c_1^2\zeta_1^2)^{3/2}} \tag{1.18}$$

Constant A is real and is determined from the condition of uniform extension at infinity:

$$A = \frac{c_1^2 v^4 q^{(1)}}{\mu c_2^2 G\,(\delta_1, \delta_2)} \tag{1.19}$$

Here

$$G\,(\delta_1, \delta_2) = \left[4 - \frac{(1+\delta_2^2)^2}{\delta_1^2}\right] E\,(\delta_1) - \left[\varepsilon_1^4 + \varepsilon_1^2\,\frac{(1+\delta_2^2)^2}{\delta_1^2}\right] K\,(\delta_1)$$
$$-\,8E\,(\delta_2) + 4\varepsilon_2^2 K\,(\delta_2) \tag{1.20}$$

while K and E are complete elliptic integrals of the first and second kind, respectively.

The solution obtained in this way can be used to determine the stressed state in the vicinity of a crack tip and the stress intensity factor, which is equal to

$$K_I = F\,(\delta_1, \delta_2)\, q^{(1)} \sqrt{\pi v t} \tag{1.21}$$

where

$$F\,(\delta_1, \delta_2) = \delta_1\,\{R_*(\delta_1, \delta_2)\,K\,(\delta_1) - 4\delta_1^2(1-\delta_2^2)\,K\,(\delta_2)$$
$$-\,[4\delta_1^2 + (1+\delta_2^2)^2]\,E\,(\delta_1) + 8\delta_1^2 E\,(\delta_2)\}^{-1} \tag{1.22}$$

The normal stresses on the line $y = 0$ are given by

$$\sigma_{yy} = \frac{q^{(1)}\sqrt{vt}}{\sqrt{2r}}\,F\,(\delta_1, \delta_2)\,R_*(\delta_1, \delta_2) \tag{1.23}$$

Figure 1.1 presents the numerical values of the normalized stress $\sqrt{2r}\sigma_{yy} / q^{(1)}\sqrt{c_1 t}$ as

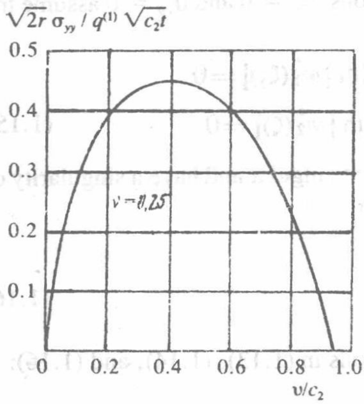

Figure 1.1 Normal stress in the crack plane as a function of its velocity ($\nu = 0.25$).

a function of v/c_2 for $\nu = 0.25$. It should be noted that the normal stress increases at first with crack speed, attains a maximum at about $v \approx 0.37c_2$, and then decreases to zero as the crack velocity becomes equal to the velocity of Rayleigh waves.

Let us study the anti plane analog of this problem. In this case, the displacement field in polar coordinates is given by

$$u_r = u_\theta = 0; \quad u_z = w(r, \theta, t) \tag{1.24}$$

and the nonzero components of stress are

$$\sigma_{rz} = \mu \frac{\partial w}{\partial r}; \quad \sigma_{\theta z} = \frac{\mu}{r} \frac{\partial w}{\partial \theta} \tag{1.25}$$

The equation of motion in polar coordinates has the form

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} = \frac{\mu}{c_2^2} \frac{\partial^2 w}{\partial t^2} \tag{1.26}$$

and the initial conditions of the problem are

$$\sigma_{yz} = q^{(3)}, \ \sigma_{xz} = 0, \frac{\partial w}{\partial t} = 0 \quad \text{at} \quad t \leqslant 0, \ r \geqslant 0 \tag{1.27}$$

Along the axis of crack propagation, the boundary conditions have the form

$$\begin{aligned} &\sigma_{yz} = 0, \ y = 0, \quad |x| < l \ (l = vt) \\ &w = 0, \ y = 0, \quad |x| \geqslant l \end{aligned} \tag{1.28}$$

It can be shown with the help of the superposition principle that this problem is equivalent to the one in which the shear stresses $\sigma_{yz} = -q^{(3)}$ on the crack are specified for $t > 0$, i.e.

$$\begin{aligned} &\sigma_{yz} = -q^{(3)}, \ y = 0, \quad |x| < l \\ &w = 0, \qquad y = 0, \quad |x| \geqslant l \end{aligned} \tag{1.29}$$

while the initial conditions have the form

$$w = \frac{\partial w}{\partial t} = 0; \quad t \leqslant 0 \tag{1.30}$$

As in the case of the plane problem, we introduce the variable $\omega = r/t$ and carry out the transformation $\omega = c_2 \operatorname{sech}(-\lambda_2)$. Equation (1.16) can then be reduced to a Laplace equation in variables λ_2 and θ:

$$\frac{\partial^2 w}{\partial \lambda_2^2} + \frac{\partial^2 w}{\partial \theta^2} = 0 \tag{1.31}$$

Using the supplementary transformation

$$\zeta_3 = \zeta_3 + i\eta_3 = \operatorname{sech}(\lambda_2 + i\theta) \tag{1.32}$$

we can present the solution of (1.31) in the form

$$w = \operatorname{Re}\,[w_3(\zeta_3)] \tag{1.33}$$

We define the function $W(\zeta_3)$ as

$$W(\zeta_3) = \begin{cases} w_3'(\zeta_3), & \eta > 0 \\ -w_3'(\zeta_3), & \eta < 0 \end{cases} \tag{1.34}$$

Considering that the stresses must have a singularity of the type $1/\sqrt{r}$, we find from (1.28) that for small values of $|\zeta_3 - v/c_2|$

$$W(\zeta_3) = \frac{2q^{(3)}c_2 t}{i\pi\mu}\sqrt{\frac{v}{2c_2}}\,\frac{K(v/c_2)}{(\zeta_3 - v/c_2)} \tag{1.35}$$

Then the longitudinal shear stress intensity factor is given by

$$K_{III} = \frac{2}{\sqrt{\pi}}\, q^{(3)} \sqrt{vt\delta_2}\, K(v/c_2) \tag{1.36}$$

The dependence of K_{III} on v/c_2 is shown in Fig. 1.2.

Let us consider the self-similar problem on the growth of a penny-shaped (plane circular) crack with a constant velocity under a uniform tension. For this purpose, we make use of Eqs. (1.7)–(1.10), Vol. 1. Applying Laplace and Hankel transformations

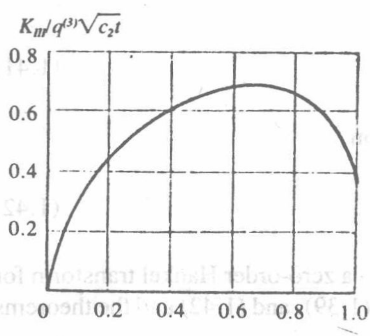

Figure 1.2 Dependence of the stress intensity factor of anti-plane shear on the crack velocity.

and an inverse Hankel transformation to Eqs. (1.10), Vol. 1, we obtain the integral representations for the Laplace transforms of potentials φ and ψ:

$$\varphi(r, z, p)=\int_0^\infty A_1(p, s) J_0(rs) e^{-\gamma_1 z}\, ds$$
$$\psi(r, z, p)=\int_0^\infty A_2(p, s) J_1(rp) e^{-\gamma_2 z}\, ds \qquad (1.37)$$

Here, p is the Laplace transformation variable and $\gamma_j^2 = s_2 + (p/c_j)^2$ $(j = 1, 2)$. J_0 and J_1 are the zeroth- and first-order Bessel functions of the first kind, respectively. These relations are derived by using the zero initial conditions

$$u_r = u_z = \frac{\partial u_r}{\partial t} = \frac{\partial u_z}{\partial t} = 0; \quad t \leqslant 0 \qquad (1.38)$$

In view of symmetry about the plane $z = 0$, the equality $\sigma_{rz}(r, 0, t) = 0$ holds for all r and t. It follows, hence, that A_1 and A_2 are not independent and can be presented with the help of a single function $A(p, s)$:

$$A_1(p, s) = -\frac{2s}{\gamma_1}\left(\frac{c_2}{p}\right)^2\left[s^2+\frac{1}{2}\left(\frac{p}{c_2}\right)^2\right] A(p, s)$$
$$A_2(p, s) = -2\left(\frac{sc_2}{p}\right)^2 A(p, s) \qquad (1.39)$$

Function $A(p, s)$ is determined from the boundary conditions

$$\sigma_{zz}(r, 0, t) = -q^{(1)}(r, t), \quad 0 \leqslant r < l(t)$$
$$u_z(r, 0, t) = 0, \qquad r \geqslant l(t) \qquad (1.40)$$

where q^1 is the pressure applied to the crack surface. In accordance with the superposition principle, the solution of problems in which the load is applied at a certain distance from the crack can be obtained by superimposing the stressed state onto the stress field, due to the load (1.40) (due to a given load in the absence of a crack), but with the opposite sign.

We make use of the notation $u(r, t) = u_z(r, 0, t)$, $0 \leqslant r \leqslant l(t)$. The Laplace transformation

$$u(r, p) = \int_0^\infty u(r, t) e^{-pt}\, dt \qquad (1.41)$$

is related to the function $A(p, s)$ through the relation

$$A(p, s) = \int_0^l r u(r, p) J_0(sr) = u(p, s) \qquad (1.42)$$

In this relation, the function $u(p, s)$ can be treated as a zero-order Hankel transform for the function $u(r, p)$. Using the dependences (1.37), (1.39), and (1.42) and the theorems for inverse Laplace and Hankel transformation, we obtain an expression for the stress components σ_{zz} in the plane $z = 0$ [232]:

$$\sigma_{zz}(r, 0, t) = [\sigma_{zz}]_{v=0} - \mu \frac{c_1}{c_2^2} Q_1 - \mu D Q_2 \tag{1.43}$$

where the static solution is

$$[\sigma_{zz}]_{v=0} = -\frac{\mu}{1-\nu}\int_0^\infty s^2 u(s, t) J_0(rs)\,ds \tag{1.44}$$

$$Q_1 = \int_0^\infty s J_0(rs)\,ds \frac{\partial}{\partial t}\left\{\int_0^t J_0[sc_1(t-\tau)]\frac{\partial}{\partial \tau}[u(s, \tau)]\,d\tau\right\}$$

$$Q_2 = \int_0^\infty s^2 J_0(rs)\,ds \left\{\int_0^t \cos[sc_2\eta(t-\tau)]\frac{\partial}{\partial \tau}[u(s, \tau)]\,d\tau\right\} \tag{1.45}$$

and symbol D defines the operator

$$D[\cos(\eta x)] = \frac{8}{\pi}\left\{\int_1^{n_*}\frac{(\eta^2-1)^{1/2}}{\eta^3}\cos(\eta x)\,d\eta\right.$$

$$\left. + \int_{n_*}^\infty \frac{1-\eta^2+\left[(\eta^2-1)\left(n_*^2\eta^2-1\right)\right]^{1/2}}{\eta^3\left(n_*^2\eta^2-1\right)^{1/2}}\cos(\eta x)\,d\eta\right\} \tag{1.46}$$

Thus, the problem is reduced to determination of the displacement $u(r, t)$.

Let us first find the static solution $[u]_{v=0}$. Omitting terms containing Q_1 and Q_2 in (1.43), we obtain

$$\sigma_{zz}(r, 0, t) = -\frac{\mu}{1-\nu}\int_0^\infty s^2 J_0(xs)\,ds \int_0^l \xi[u(\xi)]_{v=0} J_0(\xi s)\,d\xi$$

$$= \frac{\mu}{1-\nu}\int_0^\infty s J_0(xs)\,ds \int_0^l \xi \frac{\partial}{\partial \xi}[u(\xi)]_{v=0} J_1(\xi s)\,d\xi \tag{1.47}$$

where we have used the condition $u_z = 0$ for $z = 0$ and $r = l$. We multiply both sides of this equation by $1/\sqrt{r^2 - x^2}$ and integrate with respect to x

$$\int_0^l \frac{\partial}{\partial \xi}[u(\xi)]_{v=0}\frac{d\xi}{\sqrt{\xi^2 - r^2}} = \frac{1-\nu}{\mu}\frac{1}{r}\int_0^r \frac{xq^{(1)}(x)\,dx}{\sqrt{r^2-x^2}} \tag{1.48}$$

This gives Abel's equation, whose solution is known:

$$[u(r)]_{v=0} = -\frac{2(1-\nu)}{\pi\mu}\int_r^l \frac{\eta}{\sqrt{\eta^2 - r^2}}\int_0^\eta \frac{xq^{(1)}(x)\,dx}{\sqrt{\eta^2 - x^2}} \tag{1.49}$$

Let us now find the general expression for the shape of a penny-shaped crack subjected to a load $q^{(1)}(x)$. Taking into account the dynamic terms, the displacement $u(r, t)$ can be written in the form

$$u(r,\ t)=[u]_{v=0}-\frac{2(1-\nu)}{\pi}\int_r^b \frac{1}{(\eta^2-r^2)^{1/2}}\left[\frac{2(1-\nu)}{(1-2\nu)c_1}P_1-DP_2\right] \tag{1.50}$$

where

$$\begin{aligned} P_1&=\int_0^\infty \sin(s\xi)\frac{\partial}{\partial t}\int_0^t J_0[sc_1(t-\tau)]\frac{\partial}{\partial\tau}[u]_{v=0}\,d\tau ds \\ P_2&=\frac{\partial}{\partial\xi}\int_0^\infty \cos(s\xi)\int_0^t \cos[sc_2\eta(t-\tau)]\frac{\partial}{\partial\tau}[u]_{v=0}\,d\tau ds \end{aligned} \tag{1.51}$$

Equation (1.50) is solved iteratively. For the sake of simplicity, let us consider the case of constant velocity propagation of a crack subjected to a uniform tension:

$$\begin{aligned} &\sigma_{zz}(r,\ 0,\ t)=-q^{(1)}; \quad 0\leqslant r<vt \\ &u_z(r,\ 0,\ t)=0; \qquad\quad r\geqslant vt \end{aligned} \tag{1.52}$$

For the first approximation, $u^{(1)}(r, t)$, we use the static solution $[u]_{v=0}$. In this case we can integrate Eq. (1.49):

$$u^{(1)}(r,\ t)=\frac{2(1-\nu)q^{(1)}}{\pi\mu}[l^2(t)-r^2]^{1/2}=[u]_{v=0} \tag{1.53}$$

whence

$$\frac{\partial}{\partial t}[u^{(1)}(r,\ t)]=\frac{2(1-\nu)q^{(1)}vl(t)\sin(ls)}{\pi\mu s}=\frac{\partial}{\partial t}[u]_{v=0} \tag{1.54}$$

Substituting (1.54) into (1.50) and taking (1.51) into account, we obtain, after inverse transformation,

$$u^{(2)}(r, t)=(1-S)[u]_{v=0} \tag{1.55}$$

Here

$$\begin{aligned} S=&\frac{4(1-\nu)^2\varepsilon_1^2}{\pi(1-2\nu)\delta_1^2}\left[\frac{2}{\delta_1}\operatorname{arctg}\left(\frac{1-\varepsilon_1}{1+\varepsilon_1}\right)^{1/2}-\varepsilon_1\right] \\ &+(1-\nu)D\left[\frac{\varepsilon_2^2}{(\eta+\varepsilon_2)^2}\right] \end{aligned} \tag{1.56}$$

Continuing the iterative process, we obtain