Theory of Solitons

The Inverse Scattering Method

S. Novikov
S. V. Manakov
L. P. Pitaevskii
V. E. Zakharov

Consultants Bureau • New York and London

Library of Congress Cataloging in Publication Data

Teoriia solitonov. English.
Theory of solitons.

(Contemporary Soviet mathematics)
Translation of: Teoriia solitonov.
Bibliography: p.
Includes index.
1. Inverse problems (Differential equations) 2. Solitons. 3. Scattering (Mathematics) I. Novikov, Sergei Petrovich. II. Title. III. Series.
QA371.T4213 1984 515.3′55 83-21051
ISBN 0-306-10977-8

A Division of Plenum Publishing Corporation
233 Spring Street, New York, N.Y. 10013

Printed in the United States of America

CONTEMPORARY SOVIET MATHEMATICS

Series Editor: Revaz Gamkrelidze, *Steklov Institute, Moscow, USSR*

THEORY OF SOLITONS: The Inverse Scattering Method
S. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov

Preface

In the last two decades a large number of investigations in different areas of physics have been devoted to the study of nonlinear wave processes, for instance, various questions relating to the theory of plasma and nonlinear optics (except the classical problems of hydrodynamics). Simple 'model' nonlinear wave equations were constructed in the course of the development of the nonlinear wave theory; in some sense, these equations are universal, i.e., they may be encountered, just like the classical d'Alembert linear equation, in diverse physical problems. Examples of such equations are the Kortweg-de Vries equation (KdV), the nonlinear Schrödinger equation, and the sine-Gordon equation. These equations exhibit, at least in the one-dimensional case, a remarkable mathematical property. They possess hidden algebraic symmetry, as a result of which they can be 'integrated' by the so-called inverse method for an auxiliary linear operator. This book is largely concerned with this method and its generalization, the main purpose being to give an elementary, as far as possible, presentation of this method and all the necessary preliminaries from the theory of scattering, Riemann surfaces, Hamiltonian systems and others. A special chapter is devoted to the asymptotic behavior of solutions over large time intervals. In certain cases important qualitative results may be obtained by means of the methods that are not related to the scattering theory (see § 4, Chapter IV) but resemble the classical 'averaging method' of Bogolyubov and others.

To illustrate the 'universality' of KdV and of the nonlinear Schrödinger equation, we give their derivations for certain physical problems in the Introduction.

The history of the inverse scattering method dates back to the works of Gardner, Greene, Kruskal and Miura [3]. Today the literature dealing with the inverse method runs to several hundreds of papers and they hardly yield to a survey. In this text we cite only those reviews and collected works that deal with the development of this new method [1 - 26]. We felt it necessary to supplement the references with a list of fundamental works, i.e., [27 - 40] for Chapter I, [41 -51] for Chapter II, [52 - 59, 63] for Chapter III and [60 - 62] for Chapter IV. Appendix references start at [66].

We shall briefly trace the history of the soliton theory preceding the publication of [3].

A soliton solution for long waves on the surface of a liquid was first found by Boussinesq in 1872. The KdV was first derived by Kortweg and de Vries in 1895; they expressed periodic (cnoidal) waves in terms of elliptic functions. In the succeeding years these results were improved step by step, ending in a strict proof presented by Lavrent'ev and Fridrikhs to demonstrate the existence of solitons on the surface of a liquid of finite depth. The history of this epoch can be found in the excellent book by Stoker [14].

Interest in solitons was revived in connection with plasma studies. In 1958, Sagdeev [13b] postulated that solutions can propagate in plasma similar to solitons on a liquid surface. Gardner and Morikawa [4] noticed a direct analogy between the equations for shallow water. From this point onwards the KdV gains general recognitions in physics and soon attempts were made to use it in various wave topics (Whitham [15a]). At the same time, advances in nonlinear optics (Akhmanov and Khokhlov [1], Bloembergen [2]) diverted the general attention to three-wave parametric interaction and later to the nonlinear Schrödinger equation (Kadomtsev and Karpman [8]).

The discovery of the inverse scattering method was preceded by numerical simulation of the KdV. As early as 1954, Fermi, Pasta and Ulam [16], while studying the behavior of a nonlinear oscillator system with the help of a computer, detected anomolously slow stochastization of this dynamical system. In 1964 Kruskal and Zabuskii [3], using digital simulation, concluded that solitons in the KdV formalism suffer elastic collision. This result gave an impetus to new analytical studies. Soon an unending series of laws of conservation were discovered and finally, in 1967, this progress was crowned with the discovery of the inverse scattering method [3].

Further development of this method began with those works which revealed the algebraic mechanism underlying the technique used in [3]; a theory was developed for the KdV as a Hamiltonian system. Already in the early seventies other types of important nonlinear equations were known that could be integrated by the inverse scattering technique - the nonlinear Schrödinger equation, sine-Gordon equation, in particular. A scheme for determining the periodic solutions, which needs a profound use of the Hamiltonian formalism in combination with algebraic geometric techniques, was only found in 1974-75 even for the KdV. This epoch is reviewed in [6]. The recent advances cover the following topics: 1) new physically important systems that can be integrated by the inverse scattering method or its generalizations; 2) development of scattering techniques and geometric methods for finding the solutions; and 3) construction of quantum relativistic invariant models in which exact integrability is conserved.

We feel that the potentialities of exact solutions known today have still not been put to full use in calculating physical effects. We hope, however, that this book, while of help in mastering the inverse scattering method - undoubtedly one of the most elegent discoveries of mathematical physics in the 20th century - will also aid in advancing its applications.

V. E. Zakharov, S. V. Manakov,
S. P. Novikov, L. P. Pitaevskii

Contents

Introduction
Weak Nonlinearity and Dispersion

A wide class of wave processes occurring in various homogeneous media are usually described by a wave equation of the type

$$\frac{\partial^2\psi}{\partial t^2} = u_0^2 \frac{\partial^2\psi}{\partial x^2}, \tag{1}$$

which describes the propagation of persistent waves travelling with a constant velocity u_0. Three main assumptions are made in deriving this equation. First, there is no dissipation, namely, Eq.(1) is invariant with respect to time inversion: $t \to -t$. Second, the amplitude of oscillations are sufficiently small. Consequently, the terms nonlinear in ψ are small, and therefore they can be disregarded. Finally, in the wavelength range under consideration there should not be any dispersion, i.e., propagation velocity should not depend on the frequency and wavelength. Generally (the case considered in the beginning), this means that we can take the wavelength to be quite large.

Note that Eq.(1) is universal — its form does not depend on the specific properties of the medium; they affect only the velocity u_0. If we reject these three assumptions, viz., absence of dissipation, nonlinearity and dispersion, then, Eq.(1), of course, ceases to be versatile, and each medium has to be described by its own system of equations. Surprisingly, however, if these three effects are not rejected altogether, but are assumed to be small, once again we obtain an equation whose form is the same for a wide class of phenomena.

Korteweg - de Vries Equation. The first question that arises here is whether there is any sense in accounting for these small corrections, i.e., is there any case where these small corrections give rise to new qualitative effects? Otherwise they would have certainly been devoid of interest. It is a simple matter to understand that if these processes were to last for a long time, all these effects would essentially change the solution. Indeed, even a negligible amount of dissipation of energy, if it were to persist over a sufficiently long period, would lead to decaying of waves. Dispersion would blur the wave packet: this would, in a sufficiently large time period, so distort the solution that it can hardly be recognized. As regards nonlinear effects, we can say that they are responsible for "steepening the front" in the solution - a phenomenon which is in no way negligible.

Our aim here is to derive an approximate equation that would correctly describe these 'large effects due to small corrections'. We shall confine ourselves to conventional systems, thereby completely disregarding dissipation.

The solution of Eq.(1) in the form of a sum of two waves travelling in opposite directions is

$$\psi = \psi_1 (x - u_0 t) + \psi_2 (x + u_0 t). \tag{2}$$

It is easy to verify that if the small nonlinearity and dispersion are taken into account, we can treat waves travelling, independently of each other, in different directions. The physical reason for this is that these waves move past each other so fast that the corrections fail to get 'accumulated'. Hence we can considerably simplify the problem. Each wave in (2) satisfies a first-order equation: a wave travelling in the positive direction of x-axis, in particular, satisfies the equation

$$\frac{\partial \psi}{\partial t} + u_0 \frac{\partial \psi}{\partial x} = 0. \tag{3}$$

We thus have to search for the correction precisely for this equation.

It is a simple matter to find the correction due to dispersion. In the medium we are dealing with, let the exact law of dispersion for linear waves be of the form

$$\omega = ku (k). \tag{4}$$

As $k \to 0$, the velocity u should tend to u_0. In general, u(k) is an analytical function of k which can be expanded as a power series in k. We can readily show that, in the absence of dissipation, this expansion is a power series in k^2. Indeed, the dispersion law (4) is derived from a certain system of linear differential equations with real coefficients. Therefore, the solution of this system can be represented as a dependence of $i\omega$ on ik with real coefficients. In order that ω in (4) may be real (indicative of no dissipation), it is necessary that u should be expandable in even powers of ik.

By virtue of the aforementioned, for small k, the function $\omega(k)$ can be written with due regard only for the first corrections:

$$\omega = u_0 k - \beta k^3. \tag{5}$$

Immediately, it is evident that in order to obtain a correct dispersion law (5), we have to add a term containing the third derivative to (3):

$$\frac{\partial\psi}{\partial t} + u_0 \frac{\partial\psi}{\partial x} + \beta \frac{\partial^3\psi}{\partial x^3} = 0. \tag{6}$$

Now we shall take up nonlinearity. This is more convenient to do if we take into account that for the traditional systems, which we are dealing with here, exact laws of conservation of certain quantities always hold valid. (It may be the law of conservation of the number of particles.) Write one of these laws as

$$\frac{\partial\psi}{\partial t} + \frac{\partial j}{\partial x} = 0. \tag{7}$$

(Since in linear approximation all oscillating quantities satisfy the same wave equation, we can always assume that Eq. (1) is written for the perturbation of conserved quantity ψ.) Derivation lies in approximately expressing j through ψ. From a comparison of (6) and (7), it is seen that in an approximation linear in ψ we have $j = u_0\psi + \beta\partial^2\psi/\partial x^2$. In the next approximation, a term containing the second power of ψ appears:

$$j = u_0\psi + \beta\frac{\partial^2\psi}{\partial x^2} + \frac{\alpha}{2}\psi^2,$$

where α is a constant. Thus we obtain the unknown equation containing the first nonvanishing corrections:

$$\frac{\partial\psi}{\partial t} + u_0 \frac{\partial\psi}{\partial x} + \beta \frac{\partial^3\psi}{\partial x^3} + \alpha\psi \frac{\partial\psi}{\partial x} = 0. \tag{8}$$

Now on making the change of variables:

$$\xi = x - u_0 t, \quad \psi = \frac{\beta}{\alpha}\eta.$$

our equation reduces to the conventional form of Korteweg-de Vries equation:

$$\frac{\partial \eta}{\partial t} + \frac{\partial^3 \eta}{\partial \xi^3} + \eta \frac{\partial \eta}{\partial \xi} = 0. \tag{8a}$$

In order to avoid misunderstanding, let us note that these considerations cover most general cases where there is no special reason to take account of the nontypical dispersion relations and nonlinear terms. Nevertheless, we can imagine a case, where by virtue of symmetry considerations, the expansion of j may contain only odd powers of ψ. Then, nonlinearity of the type $\eta^2 \partial\eta/\partial x$ will appear in (8). Probably, more complicated nonanalytical functions $j(\psi)$ may also exist. The procedure of deriving the KdV is quite convenient in real calculations, because the coefficients α and β can be determined independently.

We shall illustrate this procedure by a few examples. First, let us determine the coefficient α in the hydrodynamics of a medium with polytropic equation of state $p = C\rho^{\gamma}$. (For adiabatic motion of an ideal gas with constant heat capacity, γ is equal to the ratio of heat capacities c_p/c_V.) We shall use the change in the density $\rho'(\rho = \rho_o + \rho'$, where ρ_o is the unperturbed density) as the quantity ψ. Then Eq. (7) turns into the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0, \tag{9}$$

which has to be supplemented with the Euler equation:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial v^2}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{10}$$

Correct to the second-order terms, we can express the relationship between p and ρ' as $p - p_o = u_o^2 \left(\rho' + \frac{\gamma-1}{2\rho_o}\rho'^2\right)$, where $u_o = (dp/d\rho)^{\frac{1}{2}}$ is the 'unperturbed' wave velocity. The system of equations (9) and (10) can therefore be represented, correct to the second-order terms, as

$$\begin{aligned} \frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v}{\partial x} &= -\frac{\partial (\rho' x)}{\partial x}, \\ \frac{\partial v}{\partial t} + \frac{u_0^2}{\rho_0}\frac{\partial \rho'}{\partial x} &= -\frac{u_0^2(\gamma-2)}{2\rho_0^2}\frac{\partial \rho'^2}{\partial x} - \frac{1}{2}\frac{\partial v^2}{\partial x}. \end{aligned} \tag{11}$$

To eliminate v, differentiate the first equation with respect to t and substitute $\frac{\partial v}{\partial t}$ from the second equation. Thus, we obtain

$$\left(\frac{\partial}{\partial t}-u_0\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)\rho'=$$
$$=-\frac{\partial^2(\rho'v)}{\partial x\,\partial t}+\frac{1}{2}\rho_0\frac{\partial^2 v^2}{\partial x^2}+\frac{1}{2}\frac{u_0\gamma}{\rho_0}\frac{\partial^2\rho'^2}{\partial x^2}. \quad (12)$$

Now, in the left-hand side of (12) we can, with sufficient accuracy, make the substitution $\frac{\partial}{\partial t} - u_o\frac{\partial}{\partial x} \simeq -2u_o\frac{\partial}{\partial x}$, and in the right-hand side, take $\frac{\partial}{\partial t} \simeq u_o\frac{\partial}{\partial x}$ and $v \simeq \frac{u_o}{\rho_o}u'$. (The last approximation follows from the preceding equality or from any equation in (11).) Eliminating $\frac{\partial}{\partial x}$ in the left-hand and right-hand sides, we obtain

$$\alpha=\frac{\gamma+1}{2\rho_0}u_0. \quad (13)$$

We shall now apply these results to an important case of ion acoustic waves in collisionless plasma, for $T_e >> T_i$. Here for ions, we can apply the hydrodynamic equation, neglecting the thermal scatter of velocities:

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{Ze}{M}\frac{\partial\varphi}{\partial x} \quad (14)$$

where M is the mass of ion.

Electrons can be taken to be distributed according to the Boltzmann law:

$$n_e=Zn_0e^{e\varphi/T_e}$$

(n_o is the equilibrium density of ions), so that the Poisson equation for the potential φ takes the form

$$\Delta\varphi=-4\pi e\,(Zn_i-Zn_0e^{e\varphi/T_e}). \quad (15)$$

To determine the coefficient α, we shall disregard dispersion, in other words discard the second derivative in Eq. (15). Thus, in place of (15), we obtain quasi-neutrality condition

$$\frac{e\varphi}{T_e}=\ln\frac{n}{n_0}.$$

Substituting $e\varphi$ found from this relationship into (14), we obtain

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{ZT_e}{M}\frac{\nabla n_i}{n_i}. \quad (16)$$

This is a simple hydrodynamics equation, in which the velocity of sound is $u_o = \sqrt{ZT_e/M}$, and the polytropic exponent is $\gamma = 1$.

Therefore for this case we have

$$\alpha = \frac{u_0}{Mn_0} = \sqrt{\frac{ZT_e}{M}}\frac{1}{Mn_0}. \tag{17}$$

In order to determine β, we shall linearize Eq. (14), assuming that all quantities are $\sim e^{i(kx-\omega t)}$:

$$-i\omega v = -ikZe\varphi/M,$$

and using the Poisson equation

$$\left(\frac{4\pi Zn_0e^2}{T_e} + k^2\right)\varphi = 4\pi Zen_i'$$

and the equation of continuity

$$-i\omega n_i' + ikn_0v = 0.$$

On equating the determinant of this system to zero, we obtain the law of dispersion

$$\omega = \sqrt{\frac{ZT_e}{M}}k(1+k^2D^2)^{-1/2} = \sqrt{\frac{ZT_e}{M}}k - \frac{1}{2}\sqrt{\frac{ZT_e}{M}}k^3D^2,$$

where $D = \sqrt{T_e/(4\pi Zn_0e^2)}$ is the Debye radius of electron screening in plasma. Thus, the coefficient β in (8) is

$$\beta = {}^1/_2 D^2\sqrt{ZT_e/M}. \tag{18}$$

A classical example of the application of the KdV is the problem of wave propagation in shallow water (in 1895, it is precisely for this case that Korteweg and de Vries derived this equation). To find α, we shall write the equation for infinitely long waves ($k \to 0$). Let h be the varying thickness of the liquid layer. Assuming that the liquid is incompressible and that the liquid in a thin layer flows almost over its surface, write the equation of continuity as follows:

$$\frac{\partial h}{\partial t} + \frac{\partial (hv)}{\partial x} = 0. \tag{19}$$

The equation for the velocity takes the form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -g\frac{\partial h}{\partial x}, \tag{20}$$

since the change in the potential energy of a liquid layer with the thickness h is $dU = g\rho dh$.

The system (19) and (20) once again formally coincides with the system of equations of hydrodynamics if h is taken to be the density and pressure is assumed to be equal to $gh^2/2$. Then

$$u_0 = \sqrt{gh_0}, \quad \gamma = 2, \tag{21}$$

so that in the equation for h'

$$\alpha = {}^3/_2\sqrt{g/h_0}.$$

To determine β we have to solve a linearized problem of vibrations with finite wavelength in a layer of thickness h_o. The dispersion law for this case[14] takes the form

$$\omega \simeq \sqrt{gh_0}\,[k - {}^1/_2\,(kh_0)^2].$$

Thus, in this case

$$\beta = {}^1/_2\sqrt{gh_0}\,h_0^2. \tag{22}$$

Nonlinear Schrödinger Equation. The nonlinear terms in the wave propagation equation considered in the previous section have, in general, the same type of nonlinearity as in the hydrodynamics equations. They are expressed in terms of the values of a varying parameter at the same time instant and its derivative. In other words, they are of 'local' nature. If, at the initial instant, perturbation was harmonic in time, this type of nonlinearity would rapidly give rise to higher harmonics and distort the initial profile.

Various other types of nonlinearity also exist. For instance, consider the electron plasma oscillations. (This example will be considered quantitatively later. Here we are interested in the general nature of the phenomena.) These oscillations take place with the electron plasma frequency $\omega_o = \sqrt{4\pi n_e e^2/m}$. Ions, because of their higher mass, will virtually not take part in these oscillations. But, by virtue of quasi-neutrality, ion density should be equal to the electron density, which cannot have a variable component with frequency of the order of ω_o. This situation does not, nevertheless, hinder plasma density from varying due to the time-averaged force acting on plasma in the field. Such a perturbation, time-invariant in the first-order approximation, will depend on time-average of field-dependent quantities. In weak fields hardly differing from harmonic ones, this implies that the dispersion law is dependent on $|\psi_0|^2$, where ψ_0 is the complex-valued field amplitude $\psi = \psi_0 \exp[i(kx-\omega t)]$. Such a type of nonlinearity readily arises on heating a medium, because if $\omega >> \nu$ (ν is the effective number of collisions), temperature will not be able to keep pace with the field oscillations, and is determined by its mean value.

It is a simple matter to obtain an equation that takes account of this type of weak nonlinear corrections for fields hardly different from harmonic ones. Let the field be of the type

$$\psi = \psi_0 e^{i(k_0 x - \omega_0 t)}, \tag{23}$$

where ψ_0 varies slowly both in space and in time. Then, the spectral expansion of the field would contain only wave vectors close to k_0. Therefore, the right-hand side of the dispersion equation $\omega = \omega(k)$ can be expanded in powers of $k - k_0$:

$$\omega = \omega(k_0) + u_0 (k - k_0) + \beta (k - k_0)^2. \tag{24}$$

For this dispersion law, we obtain the linear equation

$$i \frac{\partial \psi}{\partial t} = \omega_0 \psi + u_0 \left(\frac{1}{i} \frac{\partial}{\partial x} - k_0 \right) \psi + \beta \left(\frac{1}{i} \frac{\partial}{\partial x} - k_0 \right)^2 \psi. \tag{25}$$

If the field in the form of (23) is substituted into (25), then ψ_0 is given by the following equation:

$$i \left(\frac{\partial \psi_0}{\partial t} + u_0 \frac{\partial \psi_0}{\partial x} \right) = -\beta \frac{\partial^2 \psi_0}{\partial x^2}.$$

What now remains is to account for the nonlinearity. From the aforementioned, it follows that this nonlinearity is exhibited in the form of a dependence of the dispersion law on $|\psi_0|$. In the first approximation we can only account for the function $\omega(k_0)$, and take only a correction of the order of $|\psi_0|^2$:

$$\omega(k_0) = \omega_0 + \alpha |\psi|^2.$$

Combining these equations, we obtain

$$i \left(\frac{\partial \psi_0}{\partial t} + u_0 \frac{\partial \psi_0}{\partial x} \right) = -\beta \frac{\partial^2 \psi_0}{\partial x^2} + \alpha |\psi_0|^2 \psi_0. \tag{26}$$

If the right-hand side is equated to zero, this equation describes the propagation of a wave packet with a group velocity of $u_0 = \left(\frac{\partial \omega}{\partial k}\right)_{k=k_0}$. The right-hand side accounts for dispersion and nonlinear corrections.