Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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H. P. Schlickewei E. Wirsing (Eds.)

Number Theory, Ulm 1987

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Number Theory

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Résumé
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The 15th Journées Arithmétiques were held at the University of Ulm in 1987. 148 number theorists had joined here to exchange the results of their work. The mornings were devoted to the 10 main lectures delivered to the plenum, while the afternoon lectures had to run in three parallel sections; there were 68 of them. As an organizer I tried to secure as much space and time as possible for any type of personal contact, and I like to think that this part of the conference may not have been the least fruitful one.

A proceedings volume that is refereed like any scientific journal may not duplicate work that is being published elsewhere. Thus many interesting lectures given at the conference are not found in here. But the volume gives a good cross-section of the Journées Arithmétiques 1987 and thereby of present activity in number theory.

The conference could happen only because of various contributions from many sides:

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Personally I wish to thank Professor G. Henniart who successfully battled the authorities and generally organized everything on the French side and Dr. J.H. Goguel and Frau W. Boremski, from the Mathematics Department II in Ulm. Without their constant effort I could not possibly have managed the conference.

E. Wirsing

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CONTENTS

APPLICATIONS OF CAYLEY-CHOW FORMS

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I. Introduction.

During the past ten years, Cayley-Chow forms have become a powerful tool for effective elimination. The purpose of this article is to make these recent developments available to non-specialists and to survey the applications which have already been attained. The Cayley-Chow form possesses a venerable geometric history and has appealingly intuitive properties. Indeed, the coordinates of any zero of the Cayley-Chow form of a variety of dimension d are simply the coefficients of $d+1$ hyperplanes having a common point on the variety.

Since potential users of Cayley-Chow forms sometimes seem inhibited by foundational uncertainties, we will be rather complete in our exposition of the basic concepts in Section II below. We state clearly the basic tools and give an indication of the most straightforward applications in Sections III and V. The scheme of proof of Philippon's criterion is more intricate and, besides, has already been given in the terminology adopted here in [B6].

Yu.V. Nesterenko has shown in a series of papers [N2]-[N4] that the Cayley-Chow form responds to many questions of effective elimination which could formerly be approached only through the classical resultant in low(est) dimension. In particular he realized that one could define the valuation of a Cayley-Chow form at a point of an ambient space, and he proved that any smallness would be preserved during elimination corresponding to hypersurface intersection.

Nesterenko initially applied these ideas in [N2] to bound the order of zero of a polynomial in a solution of a system of linear first-order differential equations. This important bound partially effectivized the crucial Shidlovsky lemma in the Siegel-Shidlovsky theory for algebraic independence.

In his thesis [P2], P. Philippon used the Cayley-Chow form to perfom elimination for the elliptic Lindemann theorem. Then in [P3],[P4] he adopted and adapted Nesterenko's constructions and proved two more very important properties of Cayley-Chow forms. These properties were used to obtain his deep generalization of Gelfond's criterion and ensuing results on algebraic independence.

Later the present author was able to employ the ideas of Nesterenko and Philippon to obtain a local generalization of Liouville's inequality [B5] giving a somewhat more direct approach [B6],[BT] to the applications to independence of Philippon. A variant of this inequality then provided the basis for the (analytic!) proof [B2] of the existence of coefficients in the Nullstellensatz satisfying essentially optimal bounds on their degrees, as was kindly pointed out to the author by C. Berenstein and A. Yger.

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II. Definition and Properties of Cayley-Chow Forms

A. Planes through a Point

Because the Cayley-Chow form is used to express the condition that hyperplanes have a point in common on a given variety, we make a preliminary elementary remark on the representation of an arbitrary hyperplane in projective space $\mathbb{P}_n(k)$ through a given point $\underline{x} = [x_0:\ldots:x_n]$. First of all, note that all linear relations $\underline{u}\cdot\underline{x} = \Sigma_i\, u_i x_i = 0$ on the coordinates of $\underline{x}$ are generated by the relations between the various pairs of coordinates:

$$\underline{u}_{jk} = (\ldots 0 \ldots \underset{j}{x_k} \ldots 0 \ldots \underset{k}{-x_j} \ldots 0) = \underbrace{\begin{array}{c} \vdots \\ j \\ \vdots \\ k \\ \vdots \end{array}\begin{bmatrix} & * & & 1 & \\ & & & & \\ & -1 & & * & \\ & \ldots j & \ldots & k \ldots & \end{bmatrix}}_{\sigma_{jk}} \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix},$$

$0 \le j < k \le n$. Since a generic skew symmetric matrix S has the form

$$S = \sum_{j<k} s_{jk}\sigma_{jk}$$

for indeterminants s_{jk}, the coefficients $\underline{u}$ of a generic hyperplane passing through $\underline{x}$ are given by $\underline{u} = S\underline{x}$.

B. Cayley-Chow Forms

Let R be a unique factorization domain with field of quotients k of characteristic zero. Let V be a variety (irreducible) in $\mathbb{P}_n$ of dimension d corresponding to a homogeneous prime ideal $\mathfrak{P}$ in $k[\underline{x}] = k[x_0,\ldots,x_n]$. For $j = 0,\ldots,d$ let $\underline{u}_j = (u_{j0},\ldots,u_{jn})$ be an n+1-tuple of new variables and denote by H_j the hyperplane

$$H_j:\quad u_{j0}x_0 + \cdots + u_{jn}x_n = 0.$$

Then the hyperplanes $H_0,\ldots,H_{d-1}$ intersect V in $\deg \mathfrak{P}$ points (this is one standard definition of $\deg \mathfrak{P}$), say $\underline{\alpha}_k = (\alpha_{k0}:\ldots:\alpha_{kn})$, $k = 1,\ldots,\deg \mathfrak{P}$. One can always normalize the choice of projective coordinates by demanding that a certain non-zero one of them be equal to 1. These points can be considered to be "generic zeros" of $\mathfrak{P}$, and their coordinates are algebraic over $\mathfrak{R} = R[\underline{u}_0,\ldots,\underline{u}_{d-1}]$. Any automorphism of the algebraic closure of the field of quotients $\mathfrak{F}$ of $\mathfrak{R}$ leaves the equations for $V, H_0,\ldots,H_{d-1}$ invariant and therefore permutes the $\underline{\alpha}_k$. In fact it can be shown (Lemma 2, [N3]) that the $\underline{\alpha}_k$ are conjugate over $\mathfrak{F}$. Consequently the product $\Pi_k(\alpha_{k0}u_{d0}+\ldots+\alpha_{kn}u_{dn})$ is an irreducible element of $\mathfrak{F}[\underline{u}_d]$. We can therefore choose $a \in k(\underline{u}_0,\ldots,\underline{u}_{d-1})$ (and even in $R[\underline{u}_0,\ldots,\underline{u}_{d-1}]$) so that the polynomial

$$F(\underline{u}_0,\ldots,\underline{u}_d) = a\, \Pi_k\, (\alpha_{k0}u_{d0}+\ldots+\alpha_{kn}u_{dn}) \tag{2.1}$$

has coefficients in R, but no (non-unit) factors in $\mathfrak{R}$. We shall call it an (R-*integral*) *Cayley-Chow form of* V or $\mathfrak{P}$.

The condition

$$F(\underline{u}_0,\ldots,\underline{u}_d) = 0$$

is precisely the condition that H_d pass through a point of $V\cap H_0\cap\ldots\cap H_{d-1}$, i.e. that $H_0,\ldots,H_d$ intersect in at least one point of V. By the symmetry of this condition, the irreducibility of F as a polynomial in $\underline{u}_d$, and the lack of factors from $\mathfrak{R}$, we see that F is invariant up to a factor from R under permutations of $\underline{u}_0,\ldots,\underline{u}_d$. In particular

$$\text{(total) deg } F = (d+1)\deg \mathfrak{P}.$$

More generally we extend the notion of R-integral Cayley-Chow form to any product

$$F = F_1^{e_1}\ldots F_r^{e_r}$$

of R-integral Cayley-Chow forms F_i of homogeneous prime ideals of $k[\underline{x}]$ (of the same dimension for our purposes).

There is a way of assigning such a Cayley-Chow form to any unmixed homogeneous ideal of $k[\underline{x}]$ (see [N2]), but we shall not need that here. We remark that the more usual definitions took R to be a field [VdW] or a PID [N2],[N3],[P4], but that we plan to use the more general situation in a later paper. Another contrast to [P4] is that Philippon takes hypersurfaces of arbitrary degree for his H_i.

C. Degrees and Valuations of Cayley-Chow Forms

One can associate some invariants with such an F. For example there is always the partial degree $\delta(F) = \deg_{\underline{u}_i} F$, $i = 0,\ldots,d$. We will be interested in the cases $R = \mathbb{Z}$ and $R = \mathbb{C}[z]$ where R has a valuation. Then we have the height of F defined as $H(F) = \max |\text{coeff of } F|$ and coefficient degree $\deg_z F$, respectively.

If there were a canonical way of chosing a basis for ideals, we could also define the valuation of a homogeneous ideal $\mathfrak{J}$ at a projective point $\underline{z}$ (with coordinates in a field extension K of k with valuation extending one on k) simply by

$$\|\mathfrak{J}\|_{\underline{z}} = \max |B(\underline{z})|/\|\underline{z}\|^{\deg B},$$

where B runs over the canonical basis of $\mathfrak{J}$ and the denominator is introduced to ensure definition on projective space. Here we mean that $\|\underline{z}\| = \max |z_i|$ under the valuation on K (in particular in $\mathbb{C}\langle\langle z\rangle\rangle$, $|f| = \exp(-\text{ord } f)$, where ord denotes the "order of zero" of the power series).

Therefore before we proceed, we want to consider how closely the Cayley-Chow form F of a prime homogeneous ideal $\mathfrak{P}$ (still of dimension d). determines $\mathfrak{P}$. Let $S^{(0)},\ldots,S^{(d)}$ be d+1 generic skew symmetric matrices, $S^{(i)} = (s_{jk}^{(i)})$, and write

$$F(S^{(0)}\underline{x},\ldots,S^{(d)}\underline{x}) = \Sigma\, p_\sigma(x)\sigma, \qquad (2.2)$$

where now σ runs through the monomials in the $s^{(i)}_{jk}$, $0 \le j < k \le n$, which are homogeneous of degree $\delta(F)$ for each $i = 0,\ldots,d$. Let $z \in \mathbb{P}_n(K)$, where K is an extension field of k. Then since $S^{(0)}\underline{z},\ldots,S^{(d)}\underline{z}$ are $d+1$ generic hyperplanes through $\underline{z}$,

$$F(S^{(0)}\underline{z},\ldots,S^{(d)}\underline{z}) = 0 \iff \underline{z} \text{ is a zero of } \mathfrak{P}.$$

But clearly also in terms of (2.2),

$$F(S^{(0)}\underline{z},\ldots,S^{(d)}\underline{z}) = 0 \iff \underline{z} \text{ is a zero of all } p_\sigma(x);$$

in other words, $\mathfrak{P} = \text{radical}\langle ..p_\sigma(x)..\rangle$. In fact it is a theorem of Krull (Lemma 11 of [N2]) that

$$\langle ..p_\sigma(\underline{x})..\rangle = \mathfrak{P}\cap\mathfrak{E},$$

where $\mathfrak{E} = k[\underline{x}]$ or else $\mathfrak{E}$ is embedded in $\mathfrak{P}$. Thus the polynomials p_σ are tantamount to a canonical basis of $\mathfrak{P}$. Now we can apply a slight modification of the original strategy for defining the absolute value of an ideal.

For any Cayley-Chow form F we make the sustitution of equation (2.2) and set

$$\|F\|_{\underline{z}} = \frac{\max |p_\sigma(\underline{z})|}{\|\underline{z}\|^{\deg F}}.$$

Philippon [P4] takes the Mahler measure of the analog of $F(S^{(0)}\underline{x},\ldots,S^{(d)}\underline{x})$ when working in $\mathbb{C}$ or $\mathbb{C}_p$ to establish counterparts of all the remaining results of this section.

D. Principal Ideals

1. Chow Forms of Principal Ideals

Let $P(\underline{x}) \in R[\underline{x}]$ be homogeneous and irreducible of degree d and consider the n generic hyperplanes

$$\begin{array}{ccc} u_{00}x_0 + \cdots + u_{0n}x_n & = & 0 \\ \vdots & & \vdots \\ u_{n-1,0}x_0 + \cdots + u_{n-1,n}x_n & = & 0, \end{array}$$

i.e. hyperplanes with undetermined coefficients u_{ij}. According to Cramer's rule, they meet in a unique point whose projective coordinates can be taken as the cofactors Δ_i of the x_i in

$$\begin{bmatrix} x_0, & \cdot & \cdot & \cdot & \cdot, & x_n \\ u_{00}, & \cdot & \cdot & \cdot & \cdot, & u_{0n} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ u_{n-1,0}, & \cdot & \cdot & \cdot & , & u_{n-1,n} \end{bmatrix}.$$

Therefore the condition that this point lies on the hypersurface $P = 0$ is that $P(\Delta_0,\ldots,\Delta_n) = 0$. Thus the form

$$F(\underline{u}_0,\ldots,\underline{u}_{n-1}) = P(\Delta_0,\ldots,\Delta_{n-1})$$

is a Cayley-Chow form of the principal ideal (P) multiplied by a GCD of the coefficients of P itself. Clearly $\delta(F) = \deg_{x_i} P$, $i = 0,\ldots,n$ and

$$H(F) \le H(P)((n+1)!)^{\delta(F)} \quad \text{when} \quad R = \mathbb{Z}, \quad \deg_z F \le \deg_z F \quad \text{when} \quad R = \mathbb{C}[z].$$

2. Valuation of a Cayley-Chow Form of a Principal Ideal

Lemma. $[x_0:\ldots:x_n] = [\Delta_0':\ldots:\Delta_n']$, *where* Δ_i' *is the cofactor of* x_i *in*

$$\begin{bmatrix} \underline{x} \\ S^{(0)}\underline{x} \\ \cdot \\ \cdot \\ \cdot \\ S^{(n-1)}\underline{x} \end{bmatrix} = \begin{bmatrix} x_0 & \cdots & x_n \\ \Sigma_j s_{0j}^{(0)} x_j & \cdots & \Sigma_j s_{nj}^{(0)} x_j \\ & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ \Sigma_j s_{0j}^{(n-1)} x_j & \cdots & \Sigma_j s_{nj}^{(n-1)} x_j \end{bmatrix}.$$

Proof. By the remarks in the first section, the vectors $S^{(0)}\underline{x},\ldots,S^{(n-1)}\underline{x}$ form a basis for the dual space of $\mathbb{C}^{(n+1)}\cdot\underline{x}$. Thus

$$\det \begin{bmatrix} 0\ldots0 & \overset{i}{x_j} & 0\ldots0 & \overset{j}{-x_i} & 0\ldots0 \\ & & S^{(0)}\underline{x} & & \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ & & S^{(n-1)}\underline{x} & & \end{bmatrix} = 0,$$

and therefore $x_j\Delta_i' - x_i\Delta_j' = 0$. Consequently

$$F(S^{(0)}\underline{x},\ldots,S^{(n-1)}\underline{x}) = Q(\Delta_0',\ldots,\Delta_n') = \left(\frac{\Delta_i'}{x_i}\right)^{\deg Q} Q(x_0,\ldots,x_n).$$

Thus when $R = \mathbb{Z}$, we see that

$$\|F\|_{\underline{z}} \le \|Q\|_{\underline{z}} n^{\deg Q},$$

where $\|Q\|_{\underline{z}} = |Q(\underline{z})|/(\max |z_i|)^{\deg Q}$. In the non-archimedean case, $\|F\|_{\underline{z}} \le \|Q\|_{\underline{z}}$.

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E. Resultants

Let $Q \in R[\underline{x}]$ be homogeneous and F the R-integral Cayley-Chow form appearing in (2.1). Then we define the *resultant* $R(F,Q)$ to be the product

$$R(F,Q) = a^{\deg Q} \prod_k Q(\alpha_{k0},\ldots,\alpha_{kn}).$$

Nesterenko established the following properties of the resultant:

1. *If* $d > 0$, *then* $R(F,Q)$ *is a product of Cayley-Chow forms associated to the minimal prime components of* $(\mathfrak{P},Q)$.
2. *If* $d > 0$, *then* $\delta(R(F,Q)) = \delta(F)\cdot\deg Q$.
3. *If* $R = \mathbb{C}[z]$, *then*

$$\deg_z R(F,Q) \le \deg_z F\cdot\deg Q + \delta(F)\deg_z Q,$$
$$\operatorname{ord} R(F,Q) \ge \min\{\operatorname{ord} F,\ \operatorname{ord} Q\}.$$

4. *If* $R = \mathbb{Z}$, *then for an explicit constant* c, *depending on* n,

$$H(R(F,Q)) \le H(F,Q) = H(F)^{\deg Q} H(Q)^{\delta(F)} \exp\{\delta(F)\deg Q\},$$
$$\|R(F,Q))\|_{\underline{z}} \le H(F,Q)\max\{\|F\|_{\underline{z}},\|Q\|_{\underline{z}}\}.$$

The proofs are given in Lemma 4 [N3], Lemmas 5 and 6 and equation (28) of [N4]. There only an inequality is asserted for (2), but, as remarked in [B2], the proof gives the equality which had been noticed earlier in [P6].

This result is the principal tool for our applications. It allows one to reduce the dimension while controlling size. Philippon established two further powerful inequalities, which we will come to later.

III. Zero Estimates for Linear Differential Equations

A. The Siegel-Shidlovsky Setting

In 1928 C.L. Siegel began the investigation of the algebraic independence of values of E-function solutions of systems of linear first order differential equations (see [Sh] for a complete exposition of the results mentioned in this section)

$$\begin{aligned} y_1' &= a_{11}y_1 + \cdots + a_{1n}y_n \\ &\cdots\cdots \\ y_n' &= a_{n1}y_1 + \cdots + a_{nn}y_n, \end{aligned} \qquad (3.1)$$

$a_{ij} \in L(z)$, $[L:\mathbb{Q}] < \infty$. A prototypical E-function is represented by the generalized hypergeometric function

$${}_pF_{q-1}(z^{q-p}) = \sum \frac{(\mu_1)_n\cdots(\mu_p)_n}{(\nu_1)_n\cdots(\nu_{q-1})_n}\,\frac{z^{(q-p)n}}{n!}$$

$\mu_i, \nu_j \in \mathbb{Q}$, $-\nu_j \notin \mathbb{Z}_{\geq 0}$.

Although Siegel could completely treat the cases $n = 1,2$ of (3.1) and systems which decomposed into such subsystems, the general case was settled in a certain sense by A.B. Shidlovsky in 1955 [Sh]. Shidlovsky showed that the values of an E-function solution $(f_1, \ldots, f_n)$ of (3.1) at an algebraic non-singular point α of the system would be algebraically independent exactly when the functions themselves were algebraically independent over $L(z)$. The crucial advance established a general lower bound on the rank of a certain matrix in terms of the order of zero of an auxiliary function; this step has become known as Shidlovsky's Lemma.

B. Zero Estimates

Nesterenko realized that to obtain effective dependence on degree in a quantitative analogue of Shidlovsky's independence results, it would suffice to be able to bound the order of zero at the origin of an arbitrary polynomial expression $P(z, f_1, \ldots, f_n)$. To this end, he introduced the operator

$$D = t\Sigma_i (a_{i1}x_1 + \ldots + a_{in}x_n)\frac{\partial}{\partial x_i} + t\frac{\partial}{\partial z},$$

where t is the least common denominator for the a_{ij}, on the ring $T = \mathbb{C}[z, x_1, \ldots, x_n]$ to mirror differentiation in $\mathbb{C}[z, f_1, \ldots, f_n]$:

$$DP|_{\underline{x}=\underline{f}} = t\frac{d}{dz} P(z, f_1, \ldots, f_n).$$

He defined $\text{ord}\, P = \text{ord}_{z=0} P(z, f_1, \ldots, f_n)$ and for a prime ideal $\mathfrak{P}$ in T, we have the definition from Section II above of $\text{ord}\, \mathfrak{P}$ via the Chow form of its homogenization in $\mathbb{C}[z, x_0, \ldots, x_n]$. The following is a major result of [N1]:

THEOREM. (Nesterenko) *If* $f_1, \ldots, f_n$ *are algebraically independent (over* $\mathbb{C}(z)$*) solutions of* (3.1), *then there is a constant* c^* *depending on* (3.1), *and* $f_1, \ldots, f_n$ *such that*

$$D\mathfrak{P} \subset \mathfrak{P} \Rightarrow \text{ord}\, \mathfrak{P} \leq c^*.$$

The zero estimate now attainable using results in the literature is the following:

THEOREM. *There is a constant* c, *depending effectively on* c^* *and on* (3.1) *such that if* $P \in \mathbb{C}[z, x_1, \ldots, x_n]$ *is non-zero, then*

$$\text{ord}_{z=0} P(z, f_1, \ldots, f_n) \leq c\delta_0\delta^n + c^*\delta^{2n},$$

where $\delta_0 = \deg_z P$, $\delta = \deg_{\underline{x}} P$.

Note that this result is sharp except for the δ^{2n} term. Nesterenko's original result [N2] had $c\delta_0\delta^{(n+1)^{(n+1)}}$ on the right, which however for transcendence applications is practically as good. In the meantime somewhat more general results have been established by C.F. Osgood [O] and by D. Bertrand and F. Beukers [BB] based on an approach of G.V. Chudnovsky. Very recently Nesterenko has announced an essentially optimal result [N9] even in the very general case that the functions satisfy only a system of differential equations in which the polynomials on the right hand side of (3.1) can be polynomials of higher degree.

Outline of Proof: We prove inductively that if ord P is large enough, then there exist polynomials $L_1(=P), L_2, \ldots, L_n$ such that each $L_i \in \mathbb{Z}P+\mathbb{Z}DP+\ldots+\mathbb{Z}D^{e_i}P$, $e_i = \delta+\ldots+\delta^{i-1}$ and a Cayley-Chow form F_i with the following properties:

Each F_i is a power product of the Cayley-Chow forms of the the isolated prime components (of dimension n-i over $\mathbb{C}(z)$) of $I_i = ({}^hL_1, \ldots, {}^hL_i)$ having ord > c* (assumed > 0 for simplicity) and

i) $\operatorname{ord} F_i \geq \operatorname{ord} P - c^*(\delta+\delta^2+\ldots+\delta^i)$,

ii) $\deg_z F_i \leq i\delta_0\delta^{i-1} + \tau((i-1)\delta^i+(i-2)\delta^{i+1}+\ldots+\delta^{2(i-1)})$, where $\tau = \max\{\deg t, \deg ta_{ij}\}$,

iii) $\delta(F) \leq \delta^i$ if $i \leq n$.

Case i = 1.

The assertions all follow from the remarks on the Cayley-Chow form of a principal ideal once we delete all factors Q of P with $\operatorname{ord} Q \leq c^*\deg_{\underline{x}}Q$.

Induction Step.

The induction relies on two basic facts, the first of which follows from putting together ideas from [N1], [N2], [BM]:

a) If $\mathfrak{Q}$ is an isolated $\mathfrak{P}$-primary component of the $\underline{x}$-homogeneous ideal $\mathfrak{A}$ of T, then either $D\mathfrak{P} \subset \mathfrak{P}$ or else $D^e\mathfrak{A} \not\subset \mathfrak{P}$ for some $e \leq \exp(\mathfrak{Q})$.

The second result that we need is a consequence of the development of bounds on the exponent of primary components whose generators have degree satisfying known bounds [BM], [MW], [P5], [B7]:

b) If $\mathfrak{Q}$ is an isolated $\mathfrak{P}$-primary component of I_i with $z \notin \mathfrak{P}$, then $\exp(\mathfrak{Q}) \leq \delta^i$. Furthermore there are at most δ^i such components.

Let us assume now that the claim has been established for some $i \leq n$. By a) if $\mathfrak{Q}$ is an isolated $\mathfrak{P}$-primary component of I_i, then some $D^jD^{e_k}P \notin \mathfrak{P}$ for some $1 \leq j \leq \delta^i$ and $0 \leq k \leq i$, i.e. $D^\ell P \notin \mathfrak{P}$ for $\ell \leq \delta+\ldots+\delta^i = e_{i+1}$. By the technique of taking sufficiently general $\mathbb{Z}$-linear combinations (see [BM], [MW]), we find $L_{i+1} \in \mathbb{Z}P+\mathbb{Z}DP+\ldots+\mathbb{Z}D^{e_{i+1}}P$ not lying in any underlying prime ideal of F_i.

Now when $i < n$, produce F_{i+1} by removing all irreducible factors from the resultant $R(F,L_{i+1})$ having ord $\le c^*$. Using the fundamental inequalities for resultants, we see that properties i)-iii) follow by induction. On the other hand, when $i = n$, we see that $F_{i+1} = F_{n+1} \in \mathbb{C}[z]$ and so

$$\text{ord } F_{n+1} \le \deg F_{n+1} \le (n+1)\delta_0\delta^n + t(n\delta^{n+1} + \ldots + 2\delta^{2n-1} + \delta^{2n}).$$

Combining this inequality with part i) gives the inequality claimed.□

The original result of Nesterenko was extended to the case that the coordinates of $\underline{f}$ are not necessarily algebraically independent over $\mathbb{C}(z)$ by N.G. Tai and by the author (unpublished). A.B. Shidlovsky implicitly and S. Lang explicitly gave measures of algebraic independence in the general Siegel-Shidlovsky setting in terms of the height of the polynomials. Nesterenko's work [N1],[N2] established the dependence on the degree as well.

Very recent work of various authors promises to completely effectivize these results by different approaches.

IV. The Gelfond-Philippon Criterion

A. Gelfond's Criterion

The other classical method for algebraic independence was developed by A.O. Gelfond [G]. It used the upper bound for the number of zeros of exponential polynomials which reached its classical formulation in the theorem of R. Tijdeman [Tij]. That zero estimate was used to satisfy the hypothesis of Gelfond's criterion, which we give in the formulation of [B1].

Gelfond's Criterion. *Let* $\alpha \in \mathbb{C}$. *Suppose that* $a > 1$ *and that* $\{\delta_n\}$ *and* $\{\sigma_n\}$ *are two positive, strictly increasing unbounded sequences satisfying* $\delta_{n+1} \le a\delta_n$ *and* $\sigma_{n+1} \le a\sigma_n$. *If there is a sequence of non-zero polynomials* $P_n \in \mathbb{Z}[x]$ *with* $\deg P_n \le \delta_n$, $\deg P_n + \log \text{ht } P_n \le \sigma_n$ *and* $\log|P_n(\alpha)| \le -(2a+1)\delta_n\sigma_n$, *then each* $P_n(\alpha) = 0$.

To show transcendence degree ≥ 2, the Thue-Siegel Lemma (= Dirichlet Box Principle) is used to construct, for every large enough parameter N, an auxiliary function (exponential polynomial) whose values on one sector of a lattice will be polynomials in the numbers under consideration. If these numbers generated a field of transcendence degree one, say all algebraic over $\mathbb{Q}(\theta)$, then we would use Tijdeman's theorem to obtain a non-zero value of our function and take the norm down to $\mathbb{Q}(\theta)$ to obtain $P_N(\theta)$. We obtain a contradiction from Gelfond's criterion, and thus at least two of the numbers must be algebraically independent.

B. Philippon's Generalization

Philippon's generalization [P4] of Gelfond's criterion to higher dimensions was a major breakthrough for algebraic independence. In addition to furnishing the first sharp tools for independence of more than two numbers in the Gelfond-Schneider setting, it was a technical tour de force. Nesterenko had already shown [N4] that his development of the Cayley-Chow form was sufficient for a systematic proof of the results attacked by Chudnovsky in 1974 [C]. However by incorporating in an ingenious way two additional features, Philippon established a criterion which is sharp and which demonstrates lower bounds for transcendence degrees which are in general exponentially better that the previous ones (optimal to within a factor of 2).

We state the criterion for affine polynomials. So let us remark that for a polynomial $P \in R_a = \mathbb{Z}[x_1,\dots,x_n]$, *size* $P = \deg P + \log H(P)$. For an affine prime ideal $\mathfrak{P}$ of R_a, we mean by *size* $\mathfrak{P}$ the sum of the degree and log height of its homogenization ${}^h\mathfrak{P}$, i.e. size $\mathfrak{P}$ = size of Cayley-Chow form of ${}^h\mathfrak{P}$.

Theorem (Philippon). *Let* $\omega \in \mathbb{C}^n$. *Suppose that* $\mathfrak{P} < R_a$ *is a prime ideal of dimension* d *and size at most* $\sigma_d \geq 1$ *vanishing at* ω. *For* $a > 1$ *and* $N \geq N_0$, *let* $\{D_N\},\{S_N\}$ *denote monotonically increasing, unbounded sequences of positive integers such that* $D_{N+1} \leq aD_N$, $S_{N+1} \leq aS_N$. *Assume that* $C > 0$ *is sufficiently large and that for each* $N \geq N_0$ *there is an ideal* J_N *generated by homogeneous polynomials* $P_k \in R_a$ *such that*

i) J_N *has only finitely many zeros within the ball* $B_{\rho_N}(\omega)$ *of radius* $\rho_N = \exp(-CD_N{}^dS_N\sigma_d)$,

ii) $\deg P_k \leq D_N$, size $P_k \leq S_N$,

iii) $\log|P_k(\omega)| \leq C^d \log \rho_N$.

Then for all $N \geq N_1$, *the point* ω *is a zero of* J_N.

Note that we have chosen the formulation as in [B6] to more closely parallel our version of Gelfond's criterion. Philippon's proof is not as easy to sketch as the zero estimate of section 3 because the argument forks at every step in the reduction of dimension. Finally to treat the dimension zero case, an elaboration of Gelfond's proof is developed. We note that a detailed sketch for the case $D_N = S_N = N$ is given in [B6].

C. Two Useful Properties

Nesterenko's basic inequality above for resultants says roughly that if a Cayley-Chow form and an ordinary form are both small at a point of $\mathbb{P}_n$, then so is their resultant. Philippon discovered [P2] that the resultant is also small even if the form is only small compared to the distance to the zeros of a prime ideal underlying the Cayley-Chow form. To state this result more precisely, we define for representatives $\omega = [\omega_0:\dots:\omega_n]$, $\theta =$

$[\theta_0:\dots:\theta_n]$ of points in $\mathbb{P}_n$ the *projective distance* between them:

$$d(\omega,\theta) = \frac{\max_{i<j}\{|\omega_i\theta_j - \omega_j\theta_i|\}}{(\max|\omega_i|)(\max|\theta_i|)}$$

Proposition (Philippon). *If* F *is the Cayley-Chow form of a homogeneous prime ideal of* $\mathbb{Z}[x_0,\dots,x_n]$ *intersecting* $\mathbb{Z}$ *only in* 0 *and if for each of its zeros* β,

$$\|F\|_\omega \le d(\omega,\beta)^\mu,$$

where $0 < \mu \le 1$, *then*

$$\|R(F,Q)\|_\omega \le \|F\|^\mu H(F)^{\deg Q} H(Q)^{\delta(F)} \exp(8n(\deg F)(\deg Q)).$$

By continuity, it is clear that a near-by zero forces a Cayley-Chow form to be small. Philippon noticed [P2] that in a certain sense the converse is also true.

Proposition (Philippon). *If* F *is a Cayley-Chow form of a homogeneous prime ideal of* $\mathbb{C}[x_0,\dots,x_n]$ *of dimension* $d \ge 0$, *then for every* $\omega \in \mathbb{P}_n(\mathbb{C})$, *there is a zero* $\beta \in \mathbb{P}_n(\mathbb{C})$ *of* $\mathfrak{P}$ *such that*

$$d(\omega,\beta)^{\deg F} \le \|F\|_\omega \exp(3n^2 \deg F).$$

In other words, the Cayley-Chow form of a prime ideal is small only near zeros of the prime ideal. It seems that the exponent $\deg F$ on the left-hand side can be replaced by $\delta(F)$, which we plan to incorporate in a future note.

D. Applications.

1. $N\varepsilon$-independence

In [Ca], J.W.S. Cassels showed a general result which implied in particular that *for any* $\varepsilon > 0$ *and* $n \ge 3$, *there are algebraically independent* $\xi_1,\dots,\xi_n \in \mathbb{R}$ *such that for each* $N \ge N_0$ *there are* $\nu_1,\dots,\nu_n \in \mathbb{Z}$ *with* $0 < \max|\nu_i| < N$ *satisfying*

$$\log|\Sigma\nu_i\xi_i| < -N^\varepsilon.$$

Of course this is not typical for n-tuples of real numbers. In fact, to the best of my knowledge, not a single explicit such n-tuple is known. We summarize the fact that $\xi_1,\dots,\xi_n$ satisfy the inequality of Cassels' result by saying that the numbers $\xi_1,\dots,\xi_n$ are $N\varepsilon$*-dependent*. This terminology is chosen to call to mind both the exponent N^ε of the strong inequality being satisfied and the notion that such an inequality means that the numbers are quantitatively "Nεarly dependent."

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