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ABSTRACT

Any finite abstract group can be realized as the automorphism group of a graph. The purpose of this memoir is to find the realization, for each finite abelian group, with the least number of vertices possible.

First, the cycle structure of permutation groups is investigated, and necessary conditions are established for a permutation to represent an automorphism in a graph with abelian group. Candidates are constructed for minimal graphs with given cyclic group (representing the unpublished results of Meriwether in a new format), and the above conditions are then utilized to show that these candidates are indeed minimal.

These methods are then used to find minimal graphs for abelian p-groups, first with two generators and then (using induction) with an arbitrary number of generators. In general, these are suggested by the minimal graphs of the cyclic groups associated with the generators. However, exceptions do occur, and they are thoroughly investigated.

Finally, the results are extended to all finite abelian groups. Thus a complete classification is provided for minimal graphs with given finite abelian automorphism group.

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DEDICATION

To Roberto Frucht, whose 1938 work began the subject,

to Frank Harary, who introduced the problem to me,

and foremost in memory of Geert Prins, who read and criticized my efforts, and whose advice and counsel are deeply missed

INTRODUCTION

The purpose of this memoir is to develop the construction of minimal graphs having prescribed abelian automorphism group and to completely determine the permutation group structure of that group in these minimal cases. To do this, the abelian nature of the desired group must be exploited to assure that the associated graph attains sufficient size. Lemmas for this purpose are developed in Chapter 3. Lemma 3.2a), which states that no automorphism of an abelian group can consist of a single cycle of length greater than 2, has long been known but rarely if ever explicitly stated. Certainly its implications and extensions, some of which comprise the remainder of Chapter 3 (and all of Appendix B), have not been sufficiently appreciated.

Chapters 4 through 6 explore the special case of cyclic groups first investigated successfully but never published by Meriwether [26]. The construction of his exceptional graphs is carried out in great detail, since they are not well-known and since the use of the F-diagrams of Chapter 2 (see also Frucht [11]) makes their description much clearer. As few readers are familiar with this area, and as no explicit proofs of minimality exist in the literature, examples are included for small cases even when unnecessary for the general results. The special situations involving primes less than 7 (and the exceptional role of the prime 3) lead to the two parts of Meriwether's Theorem (Theorem 5.4 and Theorem 6.4). The proofs are entirely new and make full use of the machinery of Chapter 3, whereas Meriwether relied on properties of doubly transitive groups in his approach. These theorems completely settle the minimality question for cyclic groups. Even the exact permutation structure of the appropriate automorphism groups is completely determined.

Chapters 7 and 8 extend the results to abelian groups. The number of vertices in a minimal graph is completely determined, and the associated permutation group structure is severely limited (being completely determined with the exception of two cases involving only groups of order 3, one involving groups of orders 3, 4, and 5, and one involving groups of orders 2, 3, and 4). Contrary to the expectations of both Sabidussi [34] and Meriwether [26], several new classes of exceptional graphs arise in the abelian case that could not have been foreseen from the results for cyclic groups alone. In some cases (Lemma 7.13 and Lemma 7.14) graphs with given 3-group have fewer vertices than would be expected from the Fundamental Theorem of Abelian Groups. Both 3-groups and 2-groups (Lemmas 7.15-7.18, 7.23-7.26) give rise to unusual permutation group structures.

Since the small primes (particularly 3) play such a special role, Chapter 7 concentrates on p-groups, first for $p \geq 7$ and then successively for p=5,3,2. Chapter 8 coordinates the results to construct an algorithm for determining the minimal number of vertices $\alpha(A)$ in a graph with abelian group A and to show that this algorithm in fact gives $\alpha(A)$. Many new graphs are constructed (this time with determination of their groups left to Appendix A), and special extensions of the lemmas of Chapter 3 (proved in Appendix B) are sometimes needed. New techniques are exhibited carefully the first time, but as the number of special cases is extraordinarily high, details are often omitted later.

Throughout this memoir, theorems are numbered consecutively within each chapter (e.g, Theorem 7.1, Lemma 7.2, ...). If lemmas or corollaries refer only to a particular theorem, an additional decimal point shows the relationship (e.g., Lemma 6.4.2 is used in the proof of Theorem 6.4). In general, notation follows Harary [18] for graphs and Hall [17] for groups, although Aut G rather than $\Gamma(G)$ will be used for the automorphism group of G, and circuit will be used for a graph cycle to distinguish it from a cycle of a permutation group. An edge of a graph G with endpoints x and y will be denoted [x,y]; the greatest integer contained in z will be denoted [z]; and the greatest common divisor of two integers a and b will be denoted (a,b).

Suppose Φ is an automorphism of a graph with a given labeling. Then Φ will act on the symbols of the labeling, and the notation $x\Phi=y$ will mean that x is replaced by y under the action of Φ. Any convention leading to right-hand notation would do as well; for a thorough discussion of this issue, giving all combinations which lead to left-hand or right-hand notation, see Singmaster [35, pp.5-6].

Other notation is introduced as needed, and there is an index to that notation which is special to this memoir immediately preceding the references.

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CHAPTER 1

A HISTORICAL PERSPECTIVE

In 1938 Roberto Frucht [9] proved that any abstract (finite) group is the automorphism group of some graph. For some time, attention then centered on whether the graph having prescribed group could be required to have other properties. Frucht [10] proved in 1949 that the graph could be required to be cubic (every vertex of degree 3). In 1957 Sabidussi [33] was able to show that there were infinitely many non-homeomorphic possibilities for the graph, all of them connected and fixed-point free, and having either chromatic number (≥ 2) or regularity (≥ 3) arbitrarily chosen. Izbicki [21] was able to show that both chromatic number and degree of regularity can be chosen arbitrarily at the same time.

But eventually a more natural question arose. How small a graph could be chosen? How few vertices (edges) were possible in a graph with group G? Let $\alpha(G)$ be the minimum number of vertices possible; let $e(G,n)$ be the minimum number of edges possible in such a graph with n vertices. Clearly $\alpha(S_n)=n$, since the complete graph K_n has Aut $G=S_n$. Babai [3] proved in 1974 that $\alpha(G) \leq 2|G|$ for $|G| \geq 6$. Since $\alpha(G)=2|G|$ for G cyclic of prime order $p \geq 7$, this is the most general theorem of this type possible.

In 1959 Sabidussi [34] claimed to have discovered $\alpha(G)$ for G cyclic. He stated that $\alpha(2)=2$, $\alpha(Z_m)=3m$ if $3\leq m\leq 5$, $\alpha(Z_m)=2m$ if $m=p^e\geq 7$. He then claimed that the Fundamental Theorem of Abelian Groups would complete the cylic case (by assuring $\alpha(Z_m) = \sum_{i=1}^{n} \alpha(Z_{p_i^{a_i}})$ if $m=p_1^{a_1}\ldots p_n^{a_n}$). But his results were true only for primes (the key error in his proof was the assertion that if $m=p^e$, $e>1$, $\phi \in$ Aut $G \simeq Z_m$, then $y, y\phi, \ldots, y\phi^{m-1}$ are all distinct for $y \in V(G)$).

Writing of this paper, Meriwether [26] said, "This line of reasoning was taken up by Sabidussi in a short paper ... in which a number of incorrect statements appeared. These must, however, be regarded more as conjectures than theorems, since only the bare rudiments of proof were

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offered to substantiate them." Meriwether discovered both the error of the prime power results (e.g., $\alpha(Z_{49})=56$, not 98) and the error of the Fundamental Theorem assertion (e.g., $\alpha(Z_3)=9$, $\alpha(Z_5)=15$, but $\alpha(Z_{15})=21<24$). He then made the same mistake of assuming the Fundamental Theorem would extend the cyclic result to abelian groups. Even worse, Meriwether's results remain unpublished (they are reproduced in a different format and with entirely different proofs in this memoir).

Indeed, Meriwether's results were almost completely unknown until 1963, when Harary and Palmer [19] exhibited a graph with automorphism group Z_3, 9 vertices, and 15 edges, 3 fewer edges than Sabidussi had found in 1959 (see Figure 4.3). In his review of that article (MR 33 (2563)), Sabidussi made known Meriwether's results. But the fact that they were never published seems to have inhibited research in this area, for even the few isolated results [15,16,23] that have been published since then have been developed as a by-product of developing e(G,n).

In fact, even today knowledge of Meriwether's results is not great. Gewirtz, Hill, and Quintas [13] know it, as does Czerniakiewicz [7], as she attacked the same problem for color-groups. But, as recently as 1978, Capobianco and Molluzzo [6, p. 105] reaffirmed Sabidussi's result as the correct one, complete with the original "proof."

With the minimal vertex problem stalled, most recent work has been aimed at determining e(G,n). Quintas was the first to make great strides on this problem, solving it first in 1967 for asymmetric graphs [30] and then in 1968 for symmetric groups [31]. In 1970 Frucht, Gewirtz, and Quintas [12] found $e(Z_3,n)$ for all $n\geq 9$. Here it is clear that knowledge of $\alpha(Z_3)$ was needed. But it is also clear that Meriwether's work has not been exploited, for other cyclic groups have not been similarly investigated. Dihedral groups have been investigated; Haggard [15] found e(D,n) in 1973. Unfortunately, his results were incomplete, since $\alpha(D)$ was not determined correctly for all dihedral groups until McCarthy did it in 1979 [22]. Haggard, McCarthy, and Wohlgemuth [16] had to determine $\alpha(H)$ for H a hyperoctahedral group before they could determine e(H,n).

Actually, McCarthy and Quintas [23] have made considerable progress on the general problem of calculating e(G,n). While e(G,n) appears to behave erratically for small n, McCarthy and Quintas have shown that eventually e(G,n) is obtained from the union of a fixed graph M with automorphism group G and a forest with identity group. They outline the progress made thus far in [24].

Given the history of the subject, it seems clear that any attempt to deal with the minimal vertex problem for all finite abelian groups must present a great amount of detail. No general techniques appear in the literature; in fact, there is not even a single proof that any single graph is the minimal one with any given group. Seldom is it even proved that a given graph has a given group. For this reason, and because so much of what has been done remains unpublished, this memoir tries to present a wide variety of techniques in some detail, in the hope that not only the results but also the techniques will be useful for research in the future. (Historically, areas dealing with finite group theory have benefited as much from presentation of new techniques as from presentation of results, as illustrated in the works of Feit and Thompson [8], Gorenstein and Harada [14], and Thompson [36].)

Looking toward that future, it can be seen that the minimality problem could be considered for extensions of graphs, for graphs with other graph-theoretical properties, or for special classes of graphs in which analogues of Frucht's Theorem hold. Miller [27] considers minimum simplicial complexes with given group, obtaining upper bounds in all cases and exact results when the complex has dimension ≥ 4 and the group has no factors Z_{p^α} with $p^\alpha < 17$. Tournaments, strongly regular graphs, and Latin square graphs are three classes of graphs in which analogues of Frucht's Theorem do hold. Since no automorphism of a tournament has order 2 (Reid and Beineke [32]), every tournament has automorphism group of odd order, and Moon [28] has shown that every group of odd order occurs. More recently, Mendelsohn [25] has shown that every finite group is the group of a strongly regular graph, and Phelps [29] has done the same for Latin square graphs.

CHAPTER 2

F-DIAGRAMS OF GRAPHS

Graphs with automorphism groups of large order tend to have a high degree of symmetry. They also often have enough points that drawing a picture of the graph is difficult. In 1970 Frucht [11] developed a method of describing such graphs more efficiently. Unfortunately, the article in which he described it was not widely distributed, and only Bouwer and Frucht [5] appear to have used his notation. Also, his notation was basically designed only to describe edges joining sets of n points to other sets of n points. There is some loss of clarity if edges from an n-set to an m-set (where $m|n$) are described. Thus it appears that a refinement of his notation would be useful without causing undue confusion in reading the literature. The diagrams developed in this chapter to represent graphs will be called F-diagrams. (The reader seeking motivation for this name should not find the search fruchtless.)

An F-diagram contains the following components:

1) An encircled number m represents m points (say $a_1,\ldots,a_m$).
2) A parenthesized letter k next to such an encircled number m represents the edges $[a_i, a_{i+k}]$, $1\leq i\leq m$, addition mod m. Thus, for example, the F-diagram of Figure 2.1a represents the graph of Figure 2.1b.

a b

FIGURE 2.1

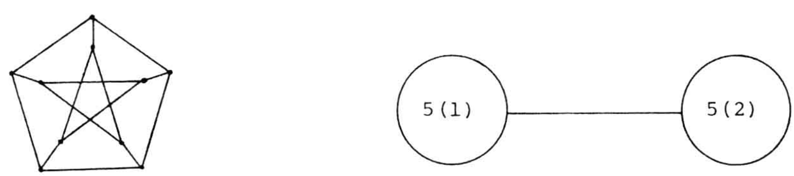
3) An undirected line between 2 encircled numbers m and n (defining) points a_i $(1\leq i\leq m)$ and b_j $(1\leq j\leq n)$ represents the edges $[a_i, b_j]$, where $i\equiv j \pmod{(m,n)}$.
4) A directed arrow $\xrightarrow{k}$ from encircled number m to encircled number n represents edges $[a_i, b_{j+k}]$, where $i\equiv j \pmod{(m,n)}$.

a) If more than one set of lines is needed between points a_i and b_j, several integers (positive, negative, or zero) may be written over the arrow.

b) The undirected line may be replaced by placing a zero over a directed arrow.

c) A series of lines may even be indicated by writing a symbol such as k (mod d) over an arrow.

Example 1 The Petersen graph (Figure 2.2a) may be represented by the F-diagram of Figure 2.2b.



a b

FIGURE 2.2

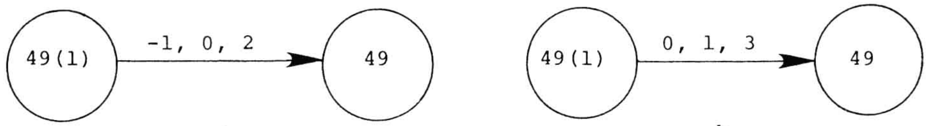
Example 2 Sabidussi [34, p. 126] exhibited a graph defined as follows as a candidate for the minimal graph G with automorphism group Z_{49}.

$V(G) = \{1, \ldots, 49; 1', \ldots, 49'\}$

$E(G) = \{[x, x+1] \mid 1 \le x \le 49, \text{ addition mod } 49\}$

$\cup \{[x, y'] \mid 1 \le x, y \le 49, y-x \equiv -1, 0, \text{ or } 2 \pmod{49}\}$

This graph may be represented by the F-diagram of Figure 2.3a; or, if only the non-negative numbers are allowed over an arrow, it may be equivalently represented by the F-diagram of Figure 2.3b.



a b

FIGURE 2.3

Example 3 A graph very similar to Sabidussi's actually provides a minimal graph with automorphism group Z_{49} (as exhibited in Chapter 4). Its F-diagram is exhibited in Figure 2.4. Incidentally, Frucht's original notation (which has not been used in the literature) would have required reversing the arrow in this situation.

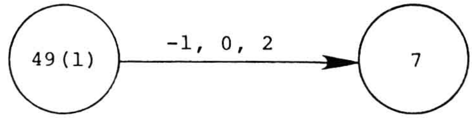


FIGURE 2.4

Example 4 Consider the three F-diagrams below.

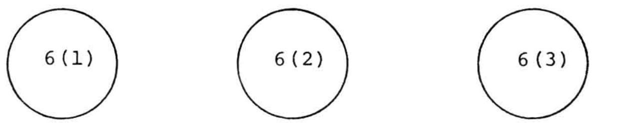


FIGURE 2.5

The first represents a hexagon, the second two triangles, and the third three lines.

From the above examples, it is easy to see the advantages of using F-diagrams as a shorthand method of describing highly symmetric graphs.

CHAPTER 3

COMMUTATIVITY LEMMAS

In future chapters it will be necessary to prove that certain graphs with given (abelian) automorphism group have a minimal number of vertices. The method used will be to establish that smaller graphs assumed to have the required automorphism group in fact have non-abelian group. Thus it will be helpful to provide some conditions under which an automorphism group can be shown to be non-abelian.

Notation:

1) If α is a cycle in a permutation, the orbit consisting of those letters moved by α will be denoted O_α. Similarly, the set of letters moved by a permutation g will be denoted O_g.

2) Let $\alpha=(1...n)$ be a cycle.

a) ϕ_α is the mapping (1 n)(2 n-1)... defined by $i\phi_\alpha=n+1-i$, $1 \leq i \leq n$, $x\phi_\alpha=x$ otherwise.

b) f_α is the mapping (1 n-1)(2 n-2)... defined by $if_\alpha=n-i$, $1 \leq i \leq n$, $xf_\alpha=x$ otherwise.

Both mappings are of course dependent on the order in which α is written.

Note first that if $[x,y]\varepsilon E(G)$ and $\phi \varepsilon$ Aut G, then $[x\phi,y\phi] \varepsilon E(G)$. This suggests that the presence of and edge between vertices which are in different cycles of an element of Aut G forces the presence of many additional edges. Indeed, the fewer divisors the orders of the cycles share, the more adjacencies are forced.

Lemma 3.1 Suppose $g\varepsilon$Aut G contains an m-cycle α and an n-cycle β, where $(m,n)=d$. Then $[x,y]\varepsilon E(G)$ for $x\varepsilon O_\alpha$, $y\varepsilon O_\beta$ if and only if $[x',y']\varepsilon E(G)$ whenever $x'=x\alpha^{id}$, $y'=y\beta^{jd}$ (i, j arbitrary integers).

Proof: Suppose $x\varepsilon O_\alpha$, $y\varepsilon O_\beta$. Choose r, s such that rm+sn=d. Then $[x,y]g^{jrm}=[x\alpha^{jrm},y\beta^{jd-jsn}]=[x,y\beta^{jd}]$. Similarly $[x,y\beta^{jd}]g^{isn}=[x\alpha^{id},y\beta^{jd}]$. Thus $[x,y]\varepsilon E(G)$ if and only if $[x',y']\varepsilon E(G)$.

Corollary 3.1.1 If $(m,n)=1$, either every letter of α is adjacent to every letter of β or no letter of α is adjacent to a letter of β.

This last corollary suggests that cycles of relatively prime length have virtually no effect on each other. This leads to an attempt to produce conditions under which additional cycles of length not relatively prime to some specific number may be forced to exist (if Aut G is to be abelian). The basic idea leading to these results is that a given circuit of length $n>2$ may be reflected as well as rotated. Since reflection and rotation generally do not commute with each other, this situation can not exist in an abelian group. Corollary 3.1.1 then forces the presence of an additional number of points m, $(m,n) \neq 1$.

Lemma 3.2 Let G be a graph. Suppose $g \in$ Aut G contains a single cycle α such that one of the following conditions holds.

a) α is of length $n>2$, and all other cycles of Aut G have length relatively prime to n or are transpositions (let $c_1, \ldots, c_t$ be the transpositions).

b) α is of length $2n>4$, and all other cycles of Aut G have length relatively prime to n or are transpositions.

c) α is of length 3^n $(n \geq 1)$, Aut G contains another cycle β of length 3m $(m \geq 1, (m,3)=1)$, and all other cycles of Aut G have length relatively prime to 3m or are transpositions.

d) α is of length 3^n $(n \geq 1)$, Aut G contains another cycle β of length 6m $(m \geq 1, (m,6)=1)$, Aut G contains (possibly) cycles γ_i of length $2k_i$ $(k_i \geq 1, (k_i,3m)=1, 1 \leq i \leq t)$, and all other cycles of Aut G have length relatively prime to 6m and to each k_i.

e) α is of length 3n $(n \geq 1)$, Aut G contains another cycle β of length 3m $(m \geq 1, (m,3n)=1)$, and all other cycles of Aut G have length relatively prime to 3mn or are transpositions.

f) α is of length 3n $(n \geq 1)$, Aut G contains another cycle β of length 6m $(m \geq 1, (m,6n)=1)$, Aut G contains (possibly) cycles of length $2k_i$, $(k_i \geq 1, (k_i,3mn)=1, 1 \leq i \leq t)$, and all other cycles of Aut G have length relatively prime to 6mn and to each k_i.

g) α is of length 5^n $(n \geq 1)$, Aut G contains a cycle of length 5m $(m \geq 1, (m,5)=1)$, and all other cycles of Aut G have length relatively prime to 5m or are transpositions.

h) α is of length 5^n $(n \geq 1)$, Aut G contains a 10m-cycle $(m \geq 1,$