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Min Ho Lee

Mixed Automorphic Forms, Torus Bundles, and Jacobi Forms

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To Virginia, Jenny, and Katie

Preface

This book is concerned with various topics that center around equivariant holomorphic maps of Hermitian symmetric domains and is intended for specialists in number theory and algebraic geometry. In particular, it contains a comprehensive exposition of mixed automorphic forms that has never appeared in book form.

The period map $\omega : \mathcal{H} \to \mathcal{H}$ of an elliptic surface E over a Riemann surface X is a holomorphic map of the Poincaré upper half plane $\mathcal{H}$ into itself that is equivariant with respect to the monodromy representation $\chi : \Gamma \to SL(2, \mathbb{R})$ of the discrete subgroup $\Gamma \subset SL(2, \mathbb{R})$ determined by X. If ω is the identity map and χ is the inclusion map, then holomorphic 2-forms on E can be considered as an automorphic form for Γ of weight three. In general, however, such holomorphic forms are mixed automorphic forms of type $(2, 1)$ that are defined by using the product of the usual weight two automorphy factor and a weight one automorphy factor involving ω and χ. Given a positive integer m, the elliptic variety E^m can be constructed by essentially taking the fiber product of m copies of E over X, and holomorphic $(m+1)$-forms on E^m may be regarded as mixed automorphic forms of type $(2, m)$. The generic fiber of E^m is the product of m elliptic curves and is therefore an abelian variety, or a complex torus. Thus the elliptic variety E^m is a complex torus bundle over the Riemann surface X.

An equivariant holomorphic map $\tau : \mathcal{D} \to \mathcal{D}'$ of more general Hermitian symmetric domains $\mathcal{D}$ and $\mathcal{D}'$ can be used to define mixed automorphic forms on $\mathcal{D}$. When $\mathcal{D}'$ is a Siegel upper half space, the map τ determines a complex torus bundle over a locally symmetric space $\Gamma \backslash \mathcal{D}$ for some discrete subgroup Γ of the semisimple Lie group G associated to $\mathcal{D}$. Such torus bundles are often families of polarized abelian varieties, and they are closely related to various topics in number theory and algebraic geometry. Holomorphic forms of the highest degree on such a torus bundle can be identified with mixed automorphic forms on $\mathcal{D}$ of certain type. Mixed automorphic forms can also be used to construct an embedding of the same torus bundle into a complex projective space. On the other hand, sections of a certain line bundle over this torus bundle can be regarded as Jacobi forms on the Hermitian symmetric domain $\mathcal{D}$.

The main goal of this book is to explore connections among complex torus bundles, mixed automorphic forms, and Jacobi forms of the type described above. Both number-theoretic and algebro-geometric aspects of such connections and related topics are discussed.

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Cedar Falls, Iowa, April 5, 2004 *Min Ho Lee*

Contents

Introduction

Let E be an elliptic surface in the sense of Kodaira [52]. Thus E is a compact smooth surface over $\mathbb{C}$, and it is the total space of an elliptic fibration $\pi : E \to X$ over a compact Riemann surface X whose generic fiber is an elliptic curve. Let $\Gamma \subset SL(2, \mathbb{R})$ be a Fuchsian group of the first kind acting on the Poincaré upper half plane $\mathcal{H}$ by linear fractional transformations such that the base space for the fibration π is given by $X = \Gamma \backslash \mathcal{H}^*$, where $\mathcal{H}^*$ is the union of $\mathcal{H}$ and the set of cusps for Γ. Given $z \in X_0 = \Gamma \backslash \mathcal{H}$, let Φ be a holomorphic 1-form on the fiber $E_z = \pi^{-1}(z)$, and choose an ordered basis $\{\gamma_1(z), \gamma_2(z)\}$ for $H_1(E_z, \mathbb{Z})$ that depends on the parameter z in a continuous manner. Consider the periods ω_1 and ω_2 of E given by

$$\omega_1(z) = \int_{\gamma_1(z)} \Phi, \quad \omega_2(z) = \int_{\gamma_2(z)} \Phi.$$

Then the imaginary part of the quotient $\omega_1(z)/\omega_2(z)$ is nonzero for each z, and therefore we may assume that $\omega_1(z)/\omega_2(z) \in \mathcal{H}$. In fact, ω_1/ω_2 is a many-valued holomorphic function on X_0, and the period map $\omega : \mathcal{H} \to \mathcal{H}$ is obtained by lifting the map $\omega_1/\omega_2 : X_0 \to \mathcal{H}$ from X_0 to its universal covering space $\mathcal{H}$. If Γ is identified with the fundamental group of X_0, the natural connection on E_0 determines the monodromy representation $\chi : \Gamma \to SL(2, \mathbf{R})$ of Γ, and the period map is equivariant with respect to χ, that is, it satisfies

$$\omega(\gamma z) = \chi(\gamma)\omega(z)$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$. Given nonnegative integers k and ℓ, we consider a holomorphc function f on $\mathcal{H}$ satisfying

$$f(\gamma z) = (cz + d)^k (c_\chi \omega(z) + d_\chi)^\ell f(z) \tag{0.1}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $\chi(\gamma) = \left(\begin{smallmatrix} a_\chi & b_\chi \\ c_\chi & d_\chi \end{smallmatrix}\right) \in SL(2, \mathbf{R})$. Such a function becomes a mixed automorphic or cusp form for Γ of type (k, ℓ) if in addition it satisfies an appropriate cusp condition. It was Hunt and Meyer [43] who observed that a holomorphic form of degree two on the elliptic surface E can be interpreted as a mixed cusp form for Γ of type $(2, 1)$ associated to ω and χ. If χ is the inclusion map of Γ into $SL(2, \mathbb{R})$ and if ω is the identity map on $\mathcal{H}$, then E is called an elliptic modular surface. The observation of

Hunt and Meyer [43] in fact generalizes the result of Shioda [115] who showed that a holomorphic 2-form on an elliptic modular surface is a cusp form of weight three. Given a positive integer m, the elliptic variety E^m associated to the elliptic fibration $\pi : E \to X$ can be obtained by essentially taking the fiber product of m copies of E over X (see Section 2.2 for details), and holomorphic $(m+1)$-forms on E^m provide examples of mixed automorphic forms of higher weights (cf. [18, 68]). Note that the generic fiber of E^m is an abelian variety, and therefore a complex torus, obtained by the product of elliptic curves. Thus the elliptic variety E^m can be regarded as a family of abelian varieties parametrized by the Riemann surface X or as a complex torus bundle over X.

Another source of examples of mixed automorphic forms comes from the theory of linear ordinary differential equations on a Riemann surface (see Section 1.4). Let $\Gamma \subset SL(2, \mathbb{R})$ be a Fuchsian group of the first kind as before. Then the corresponding compact Riemann surface $X = \Gamma \backslash \mathcal{H}^*$ can be regarded as a smooth algebraic curve over $\mathbb{C}$. We consider a second order linear differential equation $\Lambda_X^2 f = 0$ with

$$\Lambda_X^2 = \frac{d^2}{dx^2} + P_X(x) \frac{d}{dx} + Q_X(x), \tag{0.2}$$

where $P_X(x)$ and $Q_X(x)$ are rational functions on X. Let ω_1 and ω_2 be linearly independent solutions of $\Lambda_X^2 f = 0$, and for each positive integer m let $S^m(\Lambda_X^2)$ be the linear ordinary differential operator of order $m+1$ such that the $m+1$ functions

$$\omega_1^m, \omega_1^{m-1}\omega_2, \dots, \omega_1\omega_2^{m-1}, \omega_2^m$$

are linearly independent solutions of the corresponding homogeneous equation $S^m(\Lambda_X^2)f = 0$. By pulling back the operator in (0.2) via the natural projection map $\mathcal{H}^* \to X = \Gamma \backslash \mathcal{H}^*$ we obtain a differential operator

$$\Lambda^2 = \frac{d^2}{dz^2} + P(z) \frac{d}{dz} + Q(z) \tag{0.3}$$

such that $P(z)$ and $Q(z)$ are meromorphic functions on $\mathcal{H}^*$. Let $\omega_1(z)$ and $\omega_2(z)$ for $z \in \mathcal{H}$ be the two linearly independent solutions of $\Lambda^2 f = 0$ corresponding to ω_1 and ω_2 above. Then the monodromy representation for the differential equation $\Lambda^2 f = 0$ is the group homomorphism $\chi : \Gamma \to GL(2, \mathbb{C})$ which can be defined as follows. Given elements $\gamma \in \Gamma$ and $z \in \mathcal{H}$, we assume that the elements $\omega_1(\gamma z)$ and $\omega_2(\gamma z)$ can be written in the form

$$\omega_1(\gamma z) = a_\chi \omega_1(z) + b_\chi \omega_2(z), \quad \omega_2(\gamma z) = c_\chi \omega_1(z) + d_\chi \omega_2(z).$$

Then the image of $\gamma \in \Gamma$ under the monodromy representation χ is given by

$$\chi(\gamma) = \begin{pmatrix} a_\chi & b_\chi \\ c_\chi & d_\chi \end{pmatrix} \in GL(2, \mathbb{C}). \tag{0.4}$$

We assume that $\chi(\Gamma) \subset SL(2, \mathbb{R})$ and that

$$\omega(z) = \omega_1(z)/\omega_2(z) \in \mathcal{H}$$

for all $z \in \mathcal{H}$. Then the resulting map $\omega : \mathcal{H} \to \mathcal{H}$ satisfies

$$\omega(\gamma z) = \frac{a_\chi \omega(z) + b_\chi}{c_\chi \omega(z) + d_\chi} = \chi(\gamma)\omega(z)$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$. Thus the map ω is equivariant with respect to χ, and we may consider the associated meromorphic mixed automorphic or cusp forms as meromorphic functions satisfying the transformation formula in (0.1) and a certain cusp condition. If $S^m(\Lambda^2)$ is the differential operator acting on the functions on $\mathcal{H}$ obtained by pulling back $S^m(\Lambda^2_X)$ via the projection map $\mathcal{H}^* \to X$, then the solutions of the equation $S^m(\Lambda^2)f = 0$ are of the form

$$\sum_{i=0}^{m} c_i \omega_1(z)^{m-i} \omega_2(z)^i$$

for some constants $c_0, \ldots, c_m$. Let ψ be a meromorphic function on $\mathcal{H}^*$ corresponding to an element ψ_X in $K(X)$, and let f^ψ be a solution of the nonhomogeneous equation

$$S^m(\Lambda^2)f = \psi.$$

If k is a nonnegative integer k, then it can be shown the function

$$\Phi_k^\psi(z) = \omega'(z)^k \frac{d^{m+1}}{d\omega(z)^{m+1}} \left(\frac{f^\psi(z)}{\omega_2(z)^m} \right)$$

for $z \in \mathcal{H}$ is independent of the choice of the solution f^ψ and is a mixed automorphic form of type $(2k, m-2k+2)$ associated to Γ, ω and the monodromy representation χ.

If f is a cusp form of weight w for a Fuchsian group $\Gamma \subset SL(2, \mathbb{R})$, the periods of f are given by the integrals

$$\int_0^{i\infty} f(z) z^k dz$$

with $0 \le k \le w-2$, and it is well-known that such periods of cusp forms are closely related to the values at the integer points in the critical strip of the Hecke L-series. In [22] Eichler discovered certain relations among the periods of cusp forms, which were extended later by Shimura [112]; these relations are called Eichler-Shimura relations. More explicit connections between the Eichler-Shimura relations and the Fourier coefficients of cusp forms were found by Manin [91]. If f is a mixed cusp form of type $(2, m)$ associated to Γ and an equivariant pair (ω, χ), then the periods of f can be defined by the integrals

$$\int_0^{i\infty} f(z)\omega(z)^k dz$$

with $0 \leq k \leq m$. The interpretation of mixed automorphic forms as holomorphic forms on an elliptic variety described earlier can be used to obtain a relation among such periods, which may be regarded as the Eichler-Shimura relation for mixed cusp forms (see Section 2.4).

Connections between the cohomology of a discrete subgroup Γ of $SL(2,\mathbb{R})$ and automorphic forms for Γ were made by Eichler [22] and Shimura [112] decades ago. Indeed, they established an isomorphism between the space of cusp forms of weight $m+2$ for Γ and the parabolic cohomology space of Γ with coefficients in the space of homogeneous polynomials of degree m in two variables over $\mathbb{R}$. To be more precise, let $\mathrm{Sym}^m(\mathbb{C}^2)$ denote the m-th symmetric power of $\mathbb{C}^2$, and let $H^1_P(\Gamma, \mathrm{Sym}^m(\mathbb{C}^2))$ be the associated parabolic cohomology of Γ, where the Γ-module structure of $\mathrm{Sym}^m(\mathbb{C}^2)$ is induced by the standard representation of $\Gamma \subset SL(2,\mathbb{R})$ on $\mathbb{C}^2$. Then the Eichler-Shimura isomorphism can be written in the form

$$H^1_P(\Gamma, \mathrm{Sym}^m(\mathbb{C}^2)) = S_{m+2}(\Gamma) \oplus \overline{S_{m+2}(\Gamma)},$$

where $S_{m+2}(\Gamma)$ is the space of cusp forms of weight $m+2$ for Γ (cf. [22, 112]). In particular, there is a canonical embedding of the space of cusp forms into the parabolic cohomology space. The Eichler-Shimura isomorphism can also be viewed as a Hodge structure on the parabolic cohomology (see e.g. [6]). If (ω, χ) is an equivariant pair considered earlier, we may consider another action of Γ on $\mathrm{Sym}^m(\mathbb{C}^2)$ which is determined by the composition of the homomorphism $\chi : \Gamma \to SL(2,\mathbb{R})$ with the standard representation of $SL(2,\mathbb{R})$ in $\mathrm{Sym}^m(\mathbb{C}^2)$. If we denote the resulting Γ-module by $\mathrm{Sym}^m_\chi(\mathbb{C}^2)$, the associated parabolic cohomology $H^1_P(\Gamma, \mathrm{Sym}^m_\chi(\mathbb{C}^2))$ is linked to mixed automorphic forms for Γ associated to the equivariant pair (ω, χ). Indeed, the space of certain mixed cusp forms can be embedded into such parabolic cohomology space, and a Hodge structure on $H^1_P(\Gamma, \mathrm{Sym}^m_\chi(\mathbb{C}^2))$ can be determined by an isomorphism of the form

$$H^1_P(\Gamma, \mathrm{Sym}^m_\chi(\mathbb{C}^2)) \cong S_{2,m}(\Gamma, \omega, \chi) \oplus W \oplus \overline{S_{2,m}(\Gamma, \omega, \chi)}, \tag{0.5}$$

where W is a certain subspace of $H^1_P(\Gamma, \mathrm{Sym}^m_\chi(\mathbb{C}^2))$ and $S_{2,m}(\Gamma, \omega, \chi)$ is the space of mixed cusp forms of type $(2, m)$ associated to Γ, ω and χ (see Chapter 3). The space W in (0.5) is not trivial in general as can be seen in [20, Section 3], where mixed cusp forms of type $(0, 3)$ were studied in connection with elliptic surfaces. The isomorphism in (0.5) may be regarded as a generalized Eichler-Shimura isomorphism.

The correspondence between holomorphic forms of the highest degree on an elliptic variety and mixed automorphic forms of one variable described above can be extended to the case of several variables by introducing mixed Hilbert and mixed Siegel modular forms. For the Hilbert modular case we consider a totally real number field F of degree n over $\mathbb{Q}$, so that $SL(2, F)$ can be embedded in $SL(2,\mathbb{R})^n$. Given a subgroup Γ of $SL(2, F)$ whose embedded image in $SL(2,\mathbb{R})^n$ is a discrete subgroup, we can consider the associated Hilbert modular variety $\Gamma\backslash\mathcal{H}^n$ obtained by the quotient of the n-fold product $\mathcal{H}^n$ of the Poincaré upper half plane $\mathcal{H}$ by the action of Γ given by linear fractional transformations. If $\omega : \mathcal{H}^n \to \mathcal{H}^n$ is a holomorphic map equivariant with respect to a homomorphism $\chi : \Gamma \to SL(2, F)$, then the equivariant pair (ω, χ) can be used to define mixed Hilbert modular forms, which can be regarded as mixed automorphic forms of n variables. On the other hand, the same equivariant pair also determines a family of abelian varieties parametrized by $\Gamma\backslash\mathcal{H}^n$. Then holomorphic forms of the highest degree on such a family correspond to mixed Hilbert modular forms of certain type (see Section 4.2). Another type of mixed automorphic forms of several variables can be obtained by generalizing Siegel modular forms (see Section 4.3). Let $\mathcal{H}_m$ be the Siegel upper half space of degree m on which the symplectic group $Sp(m,\mathbb{R})$ acts as usual, and let Γ_0 be a discrete subgroup of $Sp(m,\mathbb{R})$. If $\tau : \mathcal{H}_m \to \mathcal{H}_{m'}$ is a holomorphic map of $\mathcal{H}_m$ into another Siegel upper half space $\mathcal{H}_{m'}$ that is equivariant with respect to a homomorphism $\rho : \Gamma_0 \to Sp(m',\mathbb{R})$, then the equivariant pair (τ, ρ) can be used to define mixed Siegel modular forms. The same pair can also be used to construct a family of abelian varieties parametrized by the Siegel modular variety $\Gamma\backslash\mathcal{H}_m$ such that holomorphic forms of the highest degree on the family correspond to mixed Siegel modular forms (see Section 4.3).

A further generalization of mixed automorphic forms can be considered by using holomorphic functions on more general Hermitian symmetric domains which include the Poincaré upper half plane or Siegel upper half spaces. Let G and G' be semisimple Lie groups of Hermitian type, so that the associated Riemannian symmetric spaces $\mathcal{D}$ and $\mathcal{D}'$, respectively, are Hermitian symmetric domains. We consider a holomorphic map $\tau : \mathcal{D} \to \mathcal{D}'$, and assume that it is equivariant with respect to a homomorphism $\rho : G \to G'$. Let Γ be a discrete subgroup of G. Note that, unlike in the earlier cases, we are assuming that τ is equivariant with respect to a homomorphism ρ defined on the group G itself rather than on the subgroup Γ. This provides us with more flexibility in studying associated mixed automorphic forms. Various aspects of such equivariant holomorphic maps were studied extensively by Satake in [108]. Given complex vector spaces V and V' and automorphy factors $J : G \times \mathcal{D} \to GL(V)$ and $J' : G' \times \mathcal{D}' \to GL(V')$, a mixed automorphic form on $\mathcal{D}$ for Γ is a holomorphic function $f : \mathcal{D} \to V \otimes V'$ satisfying

$$f(\gamma z) = J(\gamma, z) \otimes J'(\rho(\gamma), \tau(z))f(z)$$

for all $z \in \mathcal{D}$ and $\gamma \in \Gamma$ (see Section 5.1). Another advantage of considering an equivariant pair (τ, ρ) with ρ defined on G instead of Γ is that it allows us to introduce a representation-theoretic description of mixed automorphic forms. Such interpretation includes not only the holomorphic mixed automorphic forms described above but also nonholomorphic ones. Given a semisimple Lie group G, a maximal compact subgroup K, and a discrete subgroup Γ of finite covolume, automorphic forms on G can be described as follows. Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra of the complexification $\mathfrak{g}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ of G, and let V be a finite-dimensional complex vector space. A slowly increasing analytic function $f : G \to V$ is an automorphic form for Γ if it is left Γ-invariant, right K-finite, and $Z(\mathfrak{g})$-finite. Let G' be another semisimple Lie group with the corresponding objects K', Γ' and V', and let $\varphi : G \to G'$ be a homomorphism such that $\varphi(K) \subset K'$ and $\varphi(\Gamma) \subset \Gamma'$. Then the associated mixed automorphic forms may be described as linear combinations of functions of the form $f \otimes (f' \circ \varphi) : G \to V \otimes V'$, where $f : G \to V$ is an automorphic form for Γ and $f' : G' \to V'$ is an automorphic form for Γ' (see Section 5.2.

The equivariant pair (τ, ρ) considered in the previous paragraph also determines a family of abelian varieties parametrized by a locally symmetric space if G' is a symplectic group. Let $\mathcal{H}_n$ be the Siegel upper half space of degree n on which the symplectic group $Sp(n, \mathbb{R})$ acts as usual. Then the semidirect product $Sp(n, \mathbb{R}) \ltimes \mathbb{R}^{2n}$ operates on the space $\mathcal{H}_n \times \mathbb{C}^n$ by

$$\left(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right), (\mu, \nu) \right) \cdot (Z, \zeta) = ((AZ + B)(CZ + D)^{-1}, (\zeta + \mu Z + \nu)(CZ + D)^{-1})$$

for $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in Sp(n, \mathbb{R})$, $(\mu, \nu) \in \mathbb{R}^{2n}$ and $(Z, \zeta) \in \mathcal{H}_n \times \mathbb{C}^n$, where elements of $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ are considered as row vectors. We consider the discrete subgroup $\Gamma_0 = Sp(n, \mathbb{Z})$ of $Sp(n, \mathbb{R})$, and set

$$X_0 = \Gamma_0 \backslash \mathcal{H}_n, \quad Y_0 = \Gamma_0 \ltimes \mathbb{Z}^{2n} \backslash \mathcal{H}_n \times \mathbb{C}^n.$$

Then the map $\pi_0 : Y_0 \to X_0$ induced by the natural projection map $\mathcal{H}_n \times \mathbb{C}^n \to \mathcal{H}_n$ has the structure of a fiber bundle over the Siegel modular space X_0 whose fibers are complex tori of dimension n. In fact, each fiber of this bundle has the structure of a principally polarized abelian variety, and therefore the Siegel modular variety $X_0 = \Gamma_0 \backslash \mathcal{H}_n$ can be regarded as the parameter space of the family of principally polarized abelian varieties (cf. [63]). In order to consider a more general family of abelian varieties, we need to consider an equivariant holomorphic map of a Hermitian symmetric domain into a Siegel upper half space. Let G be a semisimple Lie group of Hermitian type, and let $\mathcal{D}$ be the associated Hermitian symmetric domain, which can be identified with the quotient G/K of G by a maximal compact subgroup K. We assume that there are a homomorphism $\rho : G \to Sp(n, \mathbb{R})$ of Lie groups and a holomorphic map $\tau : \mathcal{D} \to \mathcal{H}_n$ that is equivariant with respect to ρ. If Γ is a torsion-free discrete subgroup of G with $\rho(\Gamma) \subset \Gamma_0$ and if we set $X = \Gamma \backslash \mathcal{D}$, then τ induces a map $\tau_X : X \to X_0$ of the locally symmetric space X into the Siegel modular variety X_0. By pulling the bundle $\pi_0 : Y_0 \to X_0$ back via τ_X we obtain a fiber bundle $\pi : Y \to X$ over X whose fibers are n-dimensional complex tori. As in the case of π_0, each fiber is a polarized abelian variety, so that the total space Y of the bundle may be regarded as a family of abelian varieties parametrized by the locally symmetric space X. Such a family Y is known as a Kuga fiber variety, and various arithmetic and geometric aspects of Kuga fiber varieties have been studied in numerous papers over the years (see e.g. [1, 2, 31, 61, 62, 69, 74, 84, 96, 108, 113]). A Kuga fiber variety is also an example of a mixed Shimura variety in more modern language (cf. [94]). Holomorphic forms of the highest degree on the Kuga fiber variety Y can be identified with mixed automorphic forms on the symmetric domain $\mathcal{D}$ (see Section 6.3).

Equivariant holomorphic maps of symmetric domains and Kuga fiber varieties are also closely linked to Jacobi forms of several variables. Jacobi forms on the Poincaré upper half plane $\mathcal{H}$, or on $SL(2,\mathbb{R})$, share properties in common with both elliptic functions and modular forms in one variable, and they were systematically developed by Eichler and Zagier in [23]. They are functions defined on the space $\mathcal{H} \times \mathbb{C}$ which satisfy certain transformation formulas with respect to the action of a discrete subgroup Γ of $SL(2,\mathbb{R})$, and important examples of Jacobi forms include theta functions and Fourier coefficients of Siegel modular forms. Numerous papers have been devoted to the study of such Jacobi forms in connection with various topics in number theory (see e.g. [7, 9, 54, 116]). In the mean time, Jacobi forms of several variables have been studied mostly for symplectic groups of the form $Sp(m,\mathbb{R})$, which are defined on the product of a Siegel upper half space and a complex vector space. Such Jacobi forms and their relations with Siegel modular forms and theta functions have also been studied extensively over the years (cf. [25, 49, 50, 59, 123, 124]). Jacobi forms for more general semisimple Lie groups were in fact considered more than three decades ago by Piatetskii-Shapiro in [102, Chapter 4]. Such Jacobi forms occur as coefficients of Fourier-Jacobi series of automorphic forms on symmetric domains. Since then, there have not been many investigations about such Jacobi forms. In recent years, however, a number of papers which deal with Jacobi forms for orthogonal groups have appeared, and one notable such paper was written by Borcherds [12] (see also [11, 55]). Borcherds gave a highly interesting construction of Jacobi forms and modular forms for an orthogonal group of the form $O(n+2,2)$ and investigated their connection with generalized Kac-Moody algebras. Such a Jacobi form appears as a denominator function for an affine Lie algebra and can be written as an infinite product. The denominator function for the fake monster Lie algebra on the other hand is a modular form for an orthogonal group, which can also be written as an infinite product. Thus many new examples of generalized Kac-Moody algebras may be constructed from modular or Jacobi forms for $O(n+2,2)$, and conversely examples of modular or Jacobi forms may be obtained from generalized Kac-Moody algebras. In this book we consider Jacobi forms associated to an equivariant holomorphic map of symmetric domains of the type that is used in the construction of a Kuga fiber variety (see Chapter 7). Such Jacobi forms can be used to construct an embedding of a Kuga fiber variety into a complex projective space. They can also be identified with sections of a certain line bundle on the corresponding Kuga fiber variety. Similar identifications have been studied by Kramer and Runge for $SL(2,\mathbb{R})$ and $Sp(n,\mathbb{R})$ (see [57, 58, 105]).

The construction of Kuga fiber varieties can be extended to the one of more general complex torus bundles by using certain cocycles of discrete groups. Let (τ, ρ) be the equivariant pair that was used above for the construction of a Kuga fiber variety. Thus $\tau : \mathcal{D} \to \mathcal{H}_n$ is a holomorphic map that is equivariant with respect to the homomorphism $\rho : G \to Sp(n,\mathbb{R})$ of Lie groups. Let L be a lattice in $\mathbb{R}^{2n}$, and let Γ be a torsion-free discrete subgroup of G such that $\ell \cdot \rho(\gamma) \in L$ for all $\ell \in L$ and $\gamma \in \Gamma$, where we regarded elements of L as row vectors. If L denotes the lattice $\mathbb{Z}^{2n}$ in $\mathbb{Z}^{2n}$, the multiplication operation for the semidirect product $\Gamma \ltimes L$ is given by

$$(\gamma_1, \ell_1) \cdot (\gamma_2, \ell_2) = (\gamma_1\gamma_2, \ell_1\rho(\gamma_2) + \ell_2) \tag{0.6}$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and $\ell_1, \ell_2 \in L$, and $\Gamma \ltimes L$ acts on $\mathcal{D} \times \mathbb{C}^n$ by

$$(\gamma, (\mu, \nu)) \cdot (z, w) = (\gamma z, (w + \mu\tau(z) + \nu)(C_\rho\tau(z) + D_\rho)^{-1}), \tag{0.7}$$

for all $(z, w) \in \mathcal{D} \times \mathbb{C}^n$, $(\mu, \nu) \in L \subset \mathbb{R}^n \times \mathbb{R}^n$ and $\gamma \in \Gamma$ with $\rho(\gamma) = \left(\begin{smallmatrix} A_\rho & B_\rho \\ C_\rho & D_\rho \end{smallmatrix}\right) \in Sp(n,\mathbb{R})$. Then the associated Kuga fiber variety is given by the quotient

$$Y = \Gamma \ltimes L \backslash \mathcal{D} \times \mathbb{C}^n,$$

which is a fiber bundle over the locally symmetric space $X = \Gamma\backslash\mathcal{D}$. We now consider a 2-cocycle $\psi : \Gamma \times \Gamma \to L$ define the generalized semidirect product $\Gamma \ltimes_\psi L$ by replacing the multiplication operation (0.6) with

$$(\gamma_1, \ell_1) \cdot (\gamma_2, \ell_2) = (\gamma_1\gamma_2, \ell_1\rho(\gamma_2) + \ell_2 + \psi(\gamma_1, \gamma_2)).$$

We denote by $\mathcal{A}(\mathcal{D}, \mathbb{C}^n)$ the space of $\mathbb{C}^n$-valued holomorphic functions on $\mathcal{D}$, and let ξ be a 1-cochain for the cohomology of Γ with coefficients in $\mathcal{A}(\mathcal{D}, \mathbb{C}^n)$ satisfying

$$\delta\xi(\gamma_1, \gamma_2)(z) = \psi(\gamma_1, \gamma_2)\begin{pmatrix} \tau(z) \\ 1 \end{pmatrix}$$

for all $z \in \mathcal{D}$ and $\gamma_1, \gamma_2 \in \Gamma$, where δ is the coboundary operator on 1-cochains. Then an action of $\Gamma \ltimes_\psi L$ on $\mathcal{D} \times \mathbb{C}^n$ can be defined by replacing (0.7) with

$$(\gamma, (\mu, \nu)) \cdot (z, w) = (\gamma z, (w + \mu\tau(z) + \nu + \xi(\gamma)(z))(C_\rho\tau(z) + D_\rho)^{-1}).$$

If the quotient of $\mathcal{D} \times \mathbb{C}^n$ by $\Gamma \ltimes_\psi L$ with respect to this action is denoted by $Y_{\psi,\xi}$, the map $\pi : Y_{\psi,\xi} \to X = \Gamma\backslash\mathcal{D}$ induced by the natural projection $\mathcal{D} \times \mathbb{C}^n \to \mathcal{D}$ is a torus bundle over X which may be called a twisted torus bundle (see Chapter 8). As in the case of Kuga fiber varieties, holomorphic forms of the highest degree on $Y_{\psi,\xi}$ can also be identified with mixed automorphic forms for Γ of certain type.

This book is organized as follows. In Chapter 1 we discuss basic properties of mixed automorphic and cusp forms of one variable including the construction of Eisenstein and Poincaré series. We also study some cusp forms associated to mixed cusp forms and describe mixed automorphic forms associated to a certain class of linear ordinary differential equations. Geometric aspects of mixed automorphic forms of one variable are presented in Chapter 2. We construct elliptic varieties and interpret holomorphic forms of the highest degree on such a variety as mixed automorphic forms. Discussions of modular symbols and Eichler-Shimura relations for mixed automorphic forms are also included. In Chapter 3 we investigate connections between parabolic cohomology and mixed automorphic forms and discuss a generalization of the Eichler-Shimura isomorphism. In order to consider mixed automorphic forms of several variables we introduce mixed Hilbert modular forms and mixed Siegel modular forms in Chapter 4 and show that certain types of such forms occur as holomorphic forms on certain families of abelian varieties parametrized by Hilbert or Siegel modular varieties. In Chapter 5 we describe mixed automorphic forms on Hermitian symmetric domains associated to equivariant holomorphic maps of symmetric domains. We then introduce a representation-theoretic description of mixed automorphic forms on semisimple Lie groups and real reductive groups. We also construct the associated Poincaré and Eisenstein series as well as Whitaker vectors. In Chapter 6 we describe Kuga fiber varieties associated to an equivariant holomorphic map of a symmetric domain into a Siegel upper half space and show that holomorphic forms of the highest degree on a Kuga fiber variety can be identified with mixed automorphic forms on a symmetric domain. Jacobi forms on symmetric domains and their relations with bundles over Kuga fiber varieties are discussed in Chapter 7. In Chapter 8 we are concerned with complex torus bundles over a locally symmetric space which generalize Kuga fiber varieties. Such torus bundles are constructed by using certain 2-cocycles and 1-cochains of a discrete group. We discuss their connection with mixed automorphic forms and determine certain cohomology of such a bundle.