FUNCTIONS OF A COMPLEX VARIABLE

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PREFACE

One reason for the existence of this preface is to provide a place where I can acknowledge the help I received and thank the helpers, my colleagues and good friends, Mr. Lewis Feldman and Dr. Frank Lane. I repeatedly called on them and brought with me what must have been some ten thousand awkward sentences—my best attempt at an English manuscript. Several of their evenings were spoiled before I was able to obtain a readable manuscript. Incidentally, some of my mathematics was improved too. The awkward expressions and the dubious mathematics still scattered around are the original contribution of my own stubbornness.

The book itself finds its justification in my experience both as a teacher and as an applied mathematician. I have taught in schools of engineering and physics, and my current work deals for the great part with the application of mathematical techniques to the investigation of engineering problems. Therefore, I naturally regard mathematics as a tool for solving physics problems.

A good tool stimulates the ingenuity and the creativeness of the user. Different expressions of art become attainable by use of different painting tools. Different industrial designs are suggested by different power tools. And it is well known that many chords and melodies have been inspired by the arrangement of the notes on a keyboard.

A tool becomes a Tool when the user masters its technique. The handling is then unconscious, and the mind is left free to deal with applications. For example, a language learned as vocabulary and grammar is a foreign language; it becomes a tool when one can joke and appreciate jokes in it.

With these ideas in mind, the question remains, "Is this mastery the result of a natural gift or can it be largely achieved through good training?" Personally, I am inclined to think that the latter is the case, though a natural disposition is obviously required. I saw many potentially talented people deterred by poor teaching. A good teacher is one that combines the spirit of an eager student with the knowledge of an accomplished scientist. It can be observed that often the earlier works of famous authors, artists, and composers are extremely appealing and expressive, though sometimes lacking in sophistication, while later works of these men are more obscure and profound. As the man matures, he tends to become more interested in himself and his own research. He loses interest in communicating with less sophisticated minds, especially in discussing facts and ideas that to him are so commonplace as to be trivial, not to say primitive. These attitudes are bear traps for teachers and I trust that I avoided them.

In writing this book, I reviewed my student notes, and I did my best to recall my doubts, dilemmas, and misunderstandings in an effort to help others to overcome them as I have.

I would be most happy if I succeeded.

Gino Moretti

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1 INTRODUCTION

1.1. FUNCTIONS

In this introductory chapter, some definitions and concepts usually learned in courses on calculus will be recalled and re-examined. The first of these, which appears in the very title of the book, is the concept of "function."

We take for granted that the reader knows what a formula like

$$y = f(x) \tag{1.01}$$

means. However, in this text we shall define functions in many different ways; it may be interesting to review them briefly.

Usually, by (1.01), we mean that a number of prescribed arithmetical operations must be performed on x to obtain the corresponding value of y. For example,

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \tag{1.02}$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants, gives y as the result of a finite number of products and sums;

$$y = \frac{ax + b}{cx + d} \tag{1.03}$$

where a, b, c, and d are constants, requires two products, two sums, and a division to get y;

$$y = \sqrt{1 - x^2} \tag{1.04}$$

implies the computation of one product, one difference, and one square root.

When a function is defined by the method described above, the range of values of x in which a corresponding y can be computed is easily found. In case (1.02), any real value of x provides a value of y. In case (1.03), any real value of x provides a value of y, except where $x = -d/c$, which makes the denominator vanish and the division meaningless. In case (1.04) only values of x contained between -1 and 1 can provide a corresponding value of y because, out of that interval, the radicand is negative and no real number can be the square root of a negative one.

We shall say that the function in (1.02) is *defined* for every real value of x, the function in (1.03) is defined for every real value of x, except $-d/c$, and the function in (1.04) is defined for every real value of x between -1 and 1.

Sometimes, the function is expressed by a formula in an interval of values of x and by another formula in another interval. Here is an example:

(1.05)
$$y = \begin{cases} \sqrt{1 - x^2} & (-1 \le x \le 1) \\ 0 & (|x| > 1) \end{cases}$$

The first definition of the function is the same as that of (1.04), but a second definition has been given for the values of x outside the range of validity of (1.04). Now y is defined again for every real value of x.

Obviously, the combination of two (or more) definitions is arbitrary but permissible. It is commonly used to avoid complications; an interesting example will be shown in Section 1.34.

1.11. FUNCTIONS DEFINED BY SERIES EXPANSIONS

So far, we have seen examples of functions defined by a *finite* number of arithmetical operations. In mathematics and in physics, however, much more interesting functions are defined by an *infinite* number of arithmetical operations, such as *series* and *integrals*. Following are three examples:

(1.06)
$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(1.07)
$$y = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

(1.08)
$$y = \left(\frac{x}{2}\right)^p \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,(n+p)!} \left(\frac{x}{2}\right)^{2n}$$

The reader is asked to ignore, for a moment, what he already knows about series and put himself in the place of one who sees the symbol

$$\sum_{n=1}^{\infty} u_n$$

for the first time. A definition is needed to give this symbol a meaning.

A sum of a finite number of terms can be computed in many different ways, according to the ordinary rules of arithmetic. But, dealing with a sum of infinitely many terms, one is naturally inclined to begin summing some of them, say the first p terms, and then to keep *adding-on* the following terms, one after the other, observing the trend of these *partial sums*. Here the concept of *limit* enters the picture. If the partial sums appear to accumulate in the neighborhood of a certain value, we will accept it as the sum of the infinitely many terms, without trying to perform the whole computation, which would require an infinite time. The symbol given above is thus defined as

$$\sum_{n=1}^{\infty} u_n = \lim_{p\to\infty} \sum_{n=1}^{p} u_n = \lim_{p\to\infty} s_p \tag{1.09}$$

where s_p is the partial sum of order p, that is, the sum of the first p terms in the series.

The series is said to *converge* to S if a finite number

$$S = \lim_{p\to\infty} s_p$$

exists.

We want to make very clear that this definition is one among many possible definitions of the sum of a series, although it is perhaps the simplest, most natural, and most commonly used. Later on, we will need other, more involved, but more powerful definitions (Section 12.2).

The three examples given above, (1.06), (1.07), and (1.08), are *power series* (series whose terms are powers of x times a constant). Any series of functions of x defines a function of x only at those values of x at which the series converges. If the definition (1.09) is accepted, the series (1.06) and (1.08) converge at every real value of x, whereas (1.07) converges only when $-1 < x \leq 1$. Other series, whose terms do not contain powers but more complicated functions, can also be considered (see Sections 1.34 and 1.5, Chapter 12 and 14, etc.).

If a function defined by a series is frequently used, it is convenient to give it a name and a special symbol. For example, the function defined by (1.06) is called the *exponential function* and is indicated by the symbol

$$y = \exp x \quad \text{or} \quad y = e^x$$

The function defined by (1.07) is called the *natural logarithm* and is indicated by the symbol

$$y = \ln(1 + x)$$

The function defined by (1.08) is called the *Bessel function of the first kind and order p* and is indicated by the symbol

$$y = J_p(x)$$

1.12. FUNCTIONS DEFINED BY INTEGRALS

Functions defined by integrals appear naturally when a function is sought, the derivative of which is one of the functions mentioned in Section 1.1, and the function itself is not any of those functions or any combination of a finite number of them. For example,

$$y = \int_1^x \frac{dt}{t} \tag{1.10}$$

$$y = \int_0^x \frac{dt}{1 + t^2} \tag{1.11}$$

In the first case the function can be defined only for positive values of x because the integral becomes infinitely large at $x = 0$, and thus integrating through the origin has no meaning. It can also be proved that, when x belongs to the interval between 0 and 2, the values of the function coincide with those of (1.07) when $-1 < x \leq 1$. Therefore, (1.10) can be interpreted as a generalization of (1.07) to the whole positive set of numbers and is called the *natural logarithm* of x:

$$y = \ln x$$

The functions defined by (1.11) at every value of x is called the *arctangent* or the *inverse tangent* and is indicated by the symbols

$$y = \arctan x = \tan^{-1} x$$

The first, $y = \arctan x$, is preferable to avoid the confusion which might arise from using a negative power.

A more complicated case occurs when the integrand itself is defined by a series or an integral. For example, the function *logarithmic integral* is defined by

$$y = \mathrm{li}\, x = \int_0^x \frac{dt}{\ln t}$$