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ABSTRACT: In this memoir, we study the interrelations between the topological, measure theoretical and geometrical structures in certain classes of Banach spaces. The focus is on those spaces whose bounded subsets have arbitrarily norm-small convex combinations of slices. This class contains spaces with the Radon-Nikodym property as well as B-convex Banach spaces. The topological analysis leads to the concept of "first class functions around sets". This extension of the classical notion of Baire-1 functions is developed in a general non-linear setting. The study of bounded linear operators from L^1 into these spaces leads to the measure-theoretical analysis of those subsets of L^∞ with "small" or "regular oscillation". Various geometrical properties of such spaces are established: A Krein-Milman type result as well as the existence of some geometrically distinguished points. The special cases of Banach lattices and C^*-algebras are also considered.

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Contents pages

0. Introduction: In the last twenty years, several remarkable results have been established in the context of infinite dimensional Banach space theory. Most of these results emphasize the interplay between the topological, geometrical and measure theoretical structures of a Banach space. Here are two well known prototypes of such interrelations. They are due to the combined efforts of several authors and we refer the reader to the books [8] and [14] for a detailed account of their history and for the notions involved in their statements.

Theorem (A): Let X be a Banach space and let D be a closed convex bounded subset of X. The following properties are then equivalent:

(i) All D-valued martingales norm converge almost surely.

(ii) Every non-empty subset of D has slices of arbitrarily small diameter.

Theorem (B): Let Y be a separable Banach space and let D be a w^*-compact convex subset of Y^*. The following properties are then equivalent:

(i) All D-valued martingales converge in the Pettis-norm.

(ii) Every functional in Y^{**} is the pointwise limit on D of a bounded sequence of elements in Y.

A set verifying the conditions of Theorem (A) is then said to have the Radon-Nikodym property (R.N.P) while a set verifying those of Theorem (B) is said to have the Weak Radon-Nikodym property (W.R.N.P). In both cases, the set is then the norm closed convex hull of its extreme points.

The concept of a Radon-Nikodym set turned out to be central in the study of extremal structures in convex sets, integral representations and problems involving non-linear optimization ([8], [14], [19]). On the other hand, the weak Radon-Nikodym property (for w^*-compact sets) is closely related to the classical Baire theory of functions and its relatively recent resurgence with the deep theorems of Bourgain-Fremlin-Talagrand (B.F.T [5]) following the pioneering work of Rosenthal [38] and Odell-Rosenthal [33].

However, even though these two concepts are now well understood

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and their theories well developed, a crucial link between them was still missing. Indeed, for various technical reasons that we shall discuss below, all results on the weak Radon-Nikodym property are proved in the case of w*-compact sets. Therefore, unless we are in this compact case, these results do not apply even for the R.N.P subsets of dual spaces. Actually, several recent counterexamples show that "geometrically well behaved" closed convex bounded sets do not necessarily embed in w*-compact sets which are as "geometrically regular". (See the discussion in section VI.A).

One of the reasons for which the w*-compactness condition is commonly imposed on a W.R.N.P set D, is to insure that the Pettis measurable limit of the D-valued martingale remains in the set. Already, Talagrand in his memoir [50], had noticed the irrelevancy of this constraint and obtained several interesting results about what he called the W^*.R.N.P sets.

The second reason, which is more relevant, has to do with Theorem (B)(ii). Note that this assertion means that any functional in Y^{**} is a Baire-1 function on the set D once equipped with the w*-topology. Since D is w*-compact, a large supply of theorems concerning Baire-1 functions on such spaces is available. However, in general D need not be a reasonable topological space for the w*-topology even when it is norm closed. To deal with this problem we are led to study the notion of a first class function around a subset of a compact space.

In section I we isolate the topological setting that is suitable for our study. Here is the needed concept:

Let D be a subset of a compact Hausdorff space K and let f be a real valued function on K. We shall say that f is in the first class around D if for each non-empty subset F of D and any $\varepsilon > 0$, there exists an open set O such that $O \cap F \neq \emptyset$ and the oscillation of f on $O \cap \bar{F}$ is less than ε. We denote by $B_1(K,D)$ the class of such functions endowed with the topology of pointwise convergence on K. Simple examples show that an element f in $B_1(K,D)$ is more than just a function in the first Baire class on the topological space D. However, it falls short of being in the first Baire class on K even when D is dense in K. On the other hand, we show in section I.A that the restrictions of functions in $B_1(K,D)$ to D are actually "extendable" to Baire-1 functions on K from which we can prove that they are pointwise limits on D of a sequence of continuous functions on K, thus extending the Baire characterization theorem to our setting.

In section I.B, we prove some results about compact subsets of $B_1(K,D)$ that extend the deep theorems of B.F.T [5] on compact subsets of $B_1(K) = B_1(K,K)$. However, in spite of the characterization of functions in $B_1(K,D)$ mentioned above, our results do not seem to follow directly from those of B.F.T and some refinements of their methods were needed.

In section II, we study the properties of an affine function in $B_1(K,D)$ where D is a subset of a convex compact K in some locally convex topological vector space. We show, for instance that such functions have points of continuity that are also extreme points relative to the closure of any convex subset of D. We also obtain that they are pointwise limits on D of sequences of continuous and affine functions.

In section III we isolate the geometrical concepts that are needed to give a unified treatment of some aspects of R.N.P sets and w*-compact W.R.N.P sets. Recall first that a slice of a subset D of a Banach space X is a non-empty intersection of D with an open half-space of X. The following notion was implicit in the work of Bourgain [3]: The set D is said to be strongly regular if every non-empty convex subset of D has convex combination of slices of arbitrarily small diameter. The following weaker concept also turned out to be relevant for our study. Roughly speaking, the set D is said to be regular if once regarded as a subset of X^{**}, every non-empty convex subset of D has convex combinations of w*-slices which are "arbitrarily small in the weak topology of X^{**}". (Here the w*-slices are meant to be intersections of open-half spaces in X^{**} determined by functionals in X^* with the w*-closures of the subset of D in question.)

We then compare these concepts to other well known geometrical notions in Banach space theory: dentability, huskability, etc. We also define the corresponding properties for operators between general Banach spaces in order to describe them from the point of view of the theory of "operator ideals". This section is essentially a warm-up for the following ones.

In section IV, we restrict our attention to operators whose domain is $L^1[0,1]$, and we are interested in relating the topological properties of these operators to the measure-theoretical structure of the range of their adjoints in L^∞. For instance, it was well known to Grothendieck that an operator $T : L^1 \to X$ is representable by a Bochner density (or dentable in our terminology) if and only if $T^*(\text{Ball}(X^*))$ is equimeasurable in L^∞. We give similar characterizations for strongly regular (resp. regular operators) in terms of sets of small oscillation (resp. sets of regular oscillation) in L^∞. This allows us to make direct contact with the important work of Fremlin [17] and Talagrand [50] on stable subsets of L^∞. Another interesting feature of strongly regular (resp. regular) operators on L^1 is that they are completely characterized as the ones that are weak to norm (resp. T^{**} is weak* to weak) continuous on the positive sphere of L^1 (resp. $(L^1)^{**}$).

Section V is devoted to the proof of the following theorem: A Banach space X is strongly regular if and only if every operator from L^1 into X is strongly regular. This result is parallel to the well known characterization of Banach spaces X with the Radon-Nikodym property in terms of the representability of X-valued operators on L^1 (Theorem A).

In section VI, we study the concept of w*-regularity with respect to a given duality as opposed to what we have done in section III where the duality was given by the adjoint space. The main result of this section in that every bounded sequence in a Banach space Y has a subsequence that pointwise converges (to a point in Y^{**}) on any given w*-regular subset of Y^*. This provides a satisfactory answer to our initial query since the assumption of w*-compactness of the set in question is not needed.

In section VII, we deal with the structure of regular Banach spaces. We obtain, for instance, that they are unique isometric preduals of their adjoints. Moreover, their convex bounded subsets are contained in the norm closed convex hull of the extreme points of their w*-closures. We also show that a regular bounded subset of a Banach space X actually has the Radon-Nikodym property in the following two special cases:

(i) If X is a Banach lattice not containing a copy of c_0.

(ii) If X is a predual of a Von Neuman algebra.

We note that an important result in this direction was recently established by the fourth-named author [42]: Namely that a strongly regular Banach space has the Radon-Nikodym property provided it has the Krein-Milman property.

In section VIII we give a supply of examples that might illuminate the ideas behind the general results obtained throughout the paper. We also give a few counterexamples to some questions that arise naturally from our study.

This memoir essentially consists of a combination of the unpublished manuscript of Ghoussoub-Godefroy-Maurey [23] and the subsequent unpublished paper of Schachermayer [43]. The first one dealt with the topological and geometrical structures of sets while the second developed the same concepts from the "operator theoretical" point a view. The decision to combine these two papers stems from our desire to give a readable account of this part of infinite dimensional Banach space theory.

We are grateful to G. Debs, G. Mokobodzki and J. St. Raymond of the University of Paris VI for very fruitful conversations during the preparation of this paper.

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September 1986

I. FIRST CLASS FUNCTIONS AROUND A SUBSET OF A POLISH SPACE

A. Characterization of first class functions around sets:

Let D be a subset of a completely regular topological space K and let f be a real-valued function defined on K. We shall say that f is in the first class around D if for every non-empty subset F of D and any $\varepsilon > 0$, there exists an open set 0 in K such that $0 \cap F \neq \emptyset$ and the oscillation of f on $0 \cap \bar{F}$ is less than ε. We shall denote by $B_1(K,D)$ the family of such functions endowed with the topology of pointwise convergence on K. Note that if K is Polish $B_1(K) = B_1(K,K)$ is the well known class of Baire-1 functions on K.

Recall that a filter $\mathcal{U}$ on the set of real-valued functions on K is said to converge quasi-uniformly on K to the function f if for all $\varepsilon > 0$ and each non-empty closed subset F of K, there exists an open subset 0 of K with $0 \cap F \neq \emptyset$ and a set A in $\mathcal{U}$ such that $\sup_{x \in 0 \cap F, g \in A} |f(x) - g(x)| \leq \varepsilon$.

Remark: This notion of convergence is behind the remarkable "subsequence principles" obtained in [5] for subsets of $B_1(K)$. Note that if K is a Polish space, then a sequence of continuous functions (f_n) pointwise converges to f on K if and only if it converges quasi-uniformly on K to f. Indeed, for each $\varepsilon>0$ and every closed subset F of K we have $F = \bigcup_n F_n^\varepsilon$ where F_n^ε is the closed set $\bigcap_{m \geq n} \{x \in F; |f_n(x) - f_m(x)| \leq \varepsilon\}$. By the Baire category theorem, at least one of the F_n^ε, has a non-empty interior. This clearly proves our claim.

The notion of convergence that is compatible with the class $B_1(K,D)$ is the following:

We shall say that a filter $\mathcal{U}$ on the set of real-valued functions on K converges quasi-uniformly around D to the function f if for all $\varepsilon > 0$ and for any non-empty subset F of D, there exists an open subset 0 of K with $0 \cap F \neq \emptyset$ and a set A in $\mathcal{U}$ such that $\sup_{x \in 0 \cap \bar{F}, g \in A} |f(x) - g(x)| \leq \varepsilon$.

We denote by $\operatorname{osc}(f|L)$ the oscillation of f on a subset L of K and for each x in K we will write

$$\operatorname{osc}(f)(x) = \inf\{\operatorname{osc}(f|U); U \text{ open in } K \text{ containing } x\}.$$

The set $C(f) = \{x \in K; \operatorname{osc}(f)(x) = 0\}$ is then the set of points of continuity of f on K.

Lemma I.0: Let D be a dense subset of a Baire completely regular topological space K. Let f be a function defined on K. The following properties are then equivalent:

a) The set of points of continuity of f on K is a dense G_δ in K.

b) For any relatively open set W in D and any $\varepsilon > 0$, there exists an open set O in K such that $O \cap W \neq \emptyset$ and $\operatorname{osc}(f|O \cap \bar{W}) \leq \varepsilon$.

Proof: a) => b) is immediate. To prove b) => a), it is enough to show - in view of the Baire category theorem - that for any non-empty open subset U in K and any $\varepsilon > 0$, there exists a non-empty open set $U' \subset U$ such that $\operatorname{osc}(f|U') \leq \varepsilon$. To do that consider the non-empty relatively open set $W = U \cap D$ in D and find an open set O in K with $O \cap U \cap D \neq \emptyset$ and $\operatorname{osc}(f|O \cap \overline{U \cap D}) \leq \varepsilon$. But this implies that $\operatorname{osc}(f|(O \cap U) \cap \bar{D}) = \operatorname{osc}(f|O \cap U) \leq \varepsilon$. Q.E.D.

By localizing the above argument to each subset of D we obtain the following:

Lemma I.1: Let D be a subset of a completely regular topological space K such that the closure of any subset of D is a Baire space for the induced topology. Let f be a function defined on K. The following properties are then equivalent:

a) f belongs to $B_1(K,D)$.

b) For any non-empty subset F of D, the set of points of continuity of f relative to $\bar{F}$ is a dense G_δ in $\bar{F}$.

The following is the analogue of Baire's characterization theorem for functions in $B_1(K)$. We are indebted to J. St. Raymond for assertion (d).

Theorem I.2: Let D be a subset of a Polish space K. Let f be a bounded real-valued function defined on K. The following properties are then equivalent:

a) f belongs to $B_1(K,D)$

b) There exists a G_δ-subset G of K containing D such that f belongs to $B_1(K,G)$ and for any x in G there exists (x_n) in D with $\lim_n x_n = x$ and $\lim_n f(x_n) = f(x)$.

c) For any $\varepsilon > 0$, there exists an F_σ-G_δ subset B_ε of K

containing D and a function f_ε in $B_1(K)$ such that $|f - f_\varepsilon| \leq \varepsilon$ on B_ε.

d) There exist two functions g_1 and g_2 in $B_1(K)$ such that $g_2 \leq f \leq g_1$ and $g_1 = g_2$ on D.

e) There exist a sequence $(f_n)_n$ of continuous functions on K that converges quasi-uniformly around D to the function f.

Remark: Condition d) of the above theorem is clearly equivalent to the following:

d') There exists functions g and φ in $B_1(K)$ such that $|f-g| \leq \varphi$ on K and $\varphi = 0$ on D.

This reformulation suggests a possible extension of the above results to the vector-valued case. Moreover, it immediately implies that the sum of two functions in $B_1(K,D)$ is also in $B_1(K,D)$. See also the proof of Lemma I.12 for another stability result (for such functions) that is easily seen through the d') criterium.

Proof: a) => b) Let Δ be a complete metric that induces the topology on K. For each $\varepsilon > 0$, we construct by transfinite induction a

decreasing family $(F_\alpha^\varepsilon)_\alpha$ of subsets of D in the following manner:

i) $F_0^\varepsilon = D$ and if $\alpha = \beta+1$ and $F_\beta^\varepsilon \neq \emptyset$, find an open set O_β^ε in K such that $F_\beta^\varepsilon \cap O_\beta^\varepsilon \neq \emptyset$, Δ-diam $(\bar{F}_\beta^\varepsilon \cap O_\beta^\varepsilon) \leq \varepsilon$ and $\operatorname{osc}(f|O_\beta^\varepsilon \cap \bar{F}_\beta^\varepsilon) \leq \varepsilon$. We then set $F_\alpha^\varepsilon = F_\beta^\varepsilon \setminus O_\beta^\varepsilon$.

ii) If α is a limit ordinal, we let $F_\alpha^\varepsilon = \bigcap_{\beta<\alpha} F_\beta^\varepsilon$.

Now let $K_\alpha^\varepsilon = \bar{F}_\alpha^\varepsilon$ and note that the family of closed sets $(K_\alpha^\varepsilon)_\alpha$ is strictly decreasing whenever they are not empty. Since K is separable, there exists $\gamma_\varepsilon < \Omega$ (the first uncountable ordinal) such that $(K_\alpha^\varepsilon)_\alpha$ becomes stationary after γ_ε, hence $K_{\gamma_\varepsilon}^\varepsilon = \emptyset$. Note that now $D \subset \bigcap_{\alpha \leq \gamma_\varepsilon} (K_\alpha^\varepsilon \cup \bigcup_{\beta<\alpha} O_\beta^\varepsilon) = B_\varepsilon \subset K$. It is clear that B_ε is a G_δ-subset of K. We shall prove that $G = \bigcap_n B_{1/n}$ verifies the claimed properties in b). Let F be a non-empty subset of G and take $\varepsilon > 0$. Since $F \subset B_\varepsilon$, let α_0 be the first ordinal such that $F \not\subset K_{\alpha_0}^\varepsilon$. This means that $F \subset K_\beta^\varepsilon$ for each $\beta < \alpha_0$ and that $F \cap (\bigcup_{\beta<\alpha_0} O_\beta^\varepsilon) \neq \emptyset$; that is, there is $\beta < \alpha_0$ so that $F \cap O_\beta^\varepsilon \neq \emptyset$ and

$\bar{F} \cap O_\beta^\varepsilon \subset K_\beta^\varepsilon \cap O_\beta^\varepsilon$, hence $\operatorname{osc}(f|\bar{F} \cap O_\beta^\varepsilon) \leq \varepsilon$ and f then belongs to $B_1(K,G)$. Now let x in G. As above there exists for each $\varepsilon > 0$ an ordinal β such that $x \in K_\beta^\varepsilon \cap O_\beta^\varepsilon$. Since $\Delta\text{-diam}(K_\beta^\varepsilon \cap O_\beta^\varepsilon) \leq \varepsilon$ and $\operatorname{osc}(f|K_\beta \cap O_\beta^\varepsilon) \leq \varepsilon$, there exists x_ε in $F_\beta^\varepsilon \subset D$ such that $\Delta(x,x_\varepsilon) \leq \varepsilon$ and $|f(x) - f(x_\varepsilon)| \leq \varepsilon$.

b) => c) Use the same construction as above to get B_ε and note that $B_\varepsilon = \bigcup_{\alpha \leq \gamma_\varepsilon} (K_\alpha^\varepsilon \cap O_\alpha^\varepsilon)$, hence it is also an F_σ in K. Now let P_α be a real number such that $\sup_{x \in K_\alpha^\varepsilon \cap O_\alpha^\varepsilon} f(x) - P_\alpha \leq \varepsilon$. The function on K defined by $f_\varepsilon(x) = \begin{cases} P_\alpha & \text{if } x \in K_\alpha^\varepsilon \cap O_\alpha^\varepsilon \\ 0 & \text{if } x \notin B_\varepsilon \end{cases}$ is in $B_1(K)$ and $\sup_{x \in B_\varepsilon} |f(x) - f_\varepsilon(x)| \leq \varepsilon$.

c) => d) Suppose $m \leq f \leq M$ on K; let $(B_n)_n$ be a sequence of G_δ-F_σ sets containing D and let $(f_n)_n$ be a sequence of functions in $B_1(K)$ such that $\sup_{x \in B_n} |f - f_n|(x) \leq \frac{1}{n}$. We can clearly assume the B_n's decreasing. Now let

$$g_1(x) = \begin{cases} f_n(x) + \frac{1}{n} & \text{if } x \in B_n \setminus B_{n+1} \\ M & \text{if } x \notin B_1 \\ f(x) & \text{if } x \in \bigcap_n B_n \end{cases} \quad \text{and } g_2(x) = \begin{cases} f_n(x) - \frac{1}{n} & \text{if } x \in B_n \setminus B_{n+1} \\ m & \text{if } x \notin B_1 . \\ f(x) & \text{if } x \in \bigcap_n B_n \end{cases}$$

It is clear that $g_2 \leq f \leq g_1$ and $g_1(x) = g_2(x) = f(x)$ if $x \in D$. On the other hand, g_1 (resp. g_2) are in $B_1(K)$ since they are the uniform limit on K of the functions

$$g_1^N(x) = \begin{cases} f_n(x) + \frac{1}{n} & \text{if } x \in B_n \setminus B_{n+1} \text{ and } n < N \\ f_N(x) + \frac{1}{N} & \text{if } x \in B_N \\ M & \text{if } x \notin B_1 \end{cases}$$

and $$g_2^N(x) = \begin{cases} f_n(x) - \frac{1}{n} & \text{if } x \in B_n \setminus B_{n+1} \text{ and } n < N \\ f_N(x) - \frac{1}{N} & \text{if } x \in B_N \\ m & \text{if } x \notin B_1 \end{cases}$$

respectively, which are clearly in $B_1(K)$.

d) => e) Suppose g_1 and g_2 are two functions in $B_1(K)$ such that $g_2 \leq f \leq g_1$ and $g_1 = g_2$ on D. Let (f_n) be a sequence of continuous functions on K that converge pointwise, and thus quasi-uniformly, to g_2 on K. For each $\varepsilon > 0$, let $0 < \varepsilon' < \frac{\varepsilon}{2}$ and note that the sets $A_{\varepsilon'} = \{x;(g_1-g_2)(x) \leq \varepsilon'\}$ and $C_\varepsilon = \{x;(g_1-g_2)(x) \geq \varepsilon/2\}$ are two disjoint G_δ-subsets of K. Hence, a classical separation theorem ([29] p. 485) gives that $A_{\varepsilon'}$ is contained in a F_σ-G_δ set B_ε such that $B_\varepsilon \cap C_\varepsilon = \emptyset$. Let F be any non-empty subset of D and note that $\bar{F} \cap B_\varepsilon$ is a dense G_δ-F_σ subset of $\bar{F}$, hence it has a dense interior in $\bar{F}$. That is, there exists an open set V in K such that $V \cap \bar{F} \cap B_\varepsilon = V \cap \bar{F}$ and $V \cap \bar{F}$ is dense in $\bar{F}$. Let W be an open set and let n be an integer such that $W \cap F \neq \emptyset$ and $\sup_{m\geq n, x\in W\cap\bar{F}} |f_m(x) - g_2(x)| \leq \frac{\varepsilon}{2}$. Note that $W \cap V \cap \bar{F} = W \cap V \cap \bar{F} \cap B_\varepsilon$ is non-empty. On the other hand, $\sup_{x\in B_\varepsilon} |f(x) - g_2(x)| \leq \frac{\varepsilon}{2}$. Hence, if we let $U = W \cap V$, we get that $\sup_{m\geq n, x \in U \cap \bar{F}} |f_m(x) - f(x)| \leq \varepsilon$ and (f_m) converges quasi-uniformly around D to f.

e) => a) follows from the following general lemma.

<u>Lemma I.3</u>: Let D be a subset of a completely regular topological space K. Let $\mathcal{U}$ be a filter on $B_1(K,D)$. If $\mathcal{U}$ converges quasi-uniformly around D to a function f, then f belongs to $B_1(K,D)$.

Moreover, if K is a Polish space then there exists a G_δ-set G containing D such that $\mathcal{U}$ converges quasi-uniformly to f around G.

<u>Proof</u>: Let F be a non-empty subset of D and let $\varepsilon > 0$. There exists $A \in \mathcal{U}$ and an open set U such that $U \cap F \neq \emptyset$ and $\sup_{g \in A, x \in U \cap \bar{F}} |g(x) - f(x)| \leq \frac{\varepsilon}{3}$. If $g_0 \in A$, there exists an open set W such that $W \cap U \cap F \neq \emptyset$ and $\operatorname{osc}(g_0|\overline{U \cap F \cap W}) \leq \frac{\varepsilon}{3}$. Note that $U \cap W \cap \bar{F} \neq \emptyset$, $U \cap W \cap \bar{F} \subset \overline{U \cap F} \cap W$ and $\operatorname{osc}(f|U \cap W \cap \bar{F}) < \varepsilon$.

Suppose that K is a Polish space. For each $\varepsilon > 0$, we define a decreasing family $(D_\alpha)_\alpha$ of subsets of D in the following manner:

$D_0^\varepsilon = D$ and if $\alpha = \beta+1$, let V_β^ε be an open set such that

$V_\beta^\varepsilon \cap D_\beta^\varepsilon \neq \emptyset$ and $\lim_{g\in\mathcal{U}} \sup\{|g(x) - f(x)|; x \in V_\beta^\varepsilon \cap \bar{D}_\beta^\varepsilon\} \leq \varepsilon$. Then let $D_\alpha^\varepsilon = D_\beta^\varepsilon \setminus V_\beta^\varepsilon$. If α is a limit ordinal, let $D_\alpha^\varepsilon = \bigcap_{\beta<\alpha} D_\beta^\varepsilon$. Now let $\gamma_\varepsilon < \Omega$ such that $D_{\gamma_\varepsilon} = \emptyset$. We get that $D \subset B_\varepsilon = \bigcup_{\alpha<\gamma_\varepsilon} \bar{D}_\alpha^\varepsilon \cap V_\alpha^\varepsilon$. Note that B_ε is a G_δ set. A reasoning similar to a) => b) of Theorem I.2 shows that for any $F \neq \emptyset$, $F \subset B_\varepsilon$, there exists $\beta < \gamma_\varepsilon$ such that $F \cap V_\beta^\varepsilon \neq \emptyset$ and $F \cap V_\beta^\varepsilon \subset \bar{D}_\beta^\varepsilon \cap V_\beta^\varepsilon$, hence $\lim_{g\in\mathcal{U}} \sup\{|g(x) - f(x)|; x \in \bar{F} \cap V_\beta^\varepsilon\} \leq \varepsilon$. It is now clear that the G_δ-set $G = \bigcap_{\varepsilon>0} B_\varepsilon$ verifies the claimed properties of the lemma.

B. <u>Spaces of continuous functions and convergence around sets</u>:

Let $C_b(K)$ be the space of bounded and continuous functions on a completely regular topological space K and let D be a subset of K. In this subsection, we shall study some properties of subsets $\mathcal{H}$ of $C_b(K)$ that are pointwise relatively compact in $B_1(K,D)$. The following concepts are dual to some topological notions that arise naturally in the context of Banach space theory (see section III). Since they can be defined in a "non-linear" context, we shall study them in full generality.

Let $\mathcal{H}$ be a uniformly bounded set of functions on K. We shall say that:

α) $\mathcal{H}$ is <u>equicontinuous around D</u> if for each $\varepsilon>0$ and every non-empty subset F of D, there exists an open set O such that $O \cap F \neq \emptyset$ and $\operatorname{osc}(f|O \cap \bar{F}) \leq \varepsilon$ for all f in $\mathcal{H}$.

β) $\mathcal{H}$ is <u>of small oscillation around D</u> if for each $\varepsilon>0$ and every non-empty subset F of D, there exists open sets $O_1, \ldots, O_n$ such that $O_i \cap F \neq \emptyset$ for each $1 \leq i \leq n$ and

$$\frac{1}{n}\sum_{i=1}^{n} \operatorname{osc}(f|O_i \cap \bar{F}) \leq \varepsilon \text{ for all } f \in \mathcal{H}.$$

γ) $\mathcal{H}$ verifies <u>Bourgain's condition around D</u> if for each $\varepsilon>0$ and every non-empty subset F of D, there exist open sets $O_1, \ldots, O_n$ with $O_i \cap F \neq \emptyset$ for each $1 \leq i \leq n$ and such that

$$\inf_{1\leq j\leq n} \operatorname{osc}(f|O_j \cap \bar{F}) \leq \varepsilon \text{ for all } f \text{ in } \mathcal{H}.$$

δ) $\mathcal{H}$ is <u>of regular oscillation around D</u> if the pointwise closure $\bar{\mathcal{H}}$ of $\mathcal{H}$ is contained in $B_1(K,D)$.

We shall now compare these topological notions in the case where K is a Polish space.

<u>Theorem I.4</u>: Let D be a subset of a Polish space K and let $\mathcal{H}$ be a uniformly bounded subset of $C_b(K)$. The following conditions are equivalent:

a) $\mathcal{H}$ verifies Bourgain's condition around D.

b) Every ultrafilter on $\mathcal{H}$ converges quasi-uniformly around D.

c) $\mathcal{H}$ is a set of regular oscillation around D.

In this case every sequence in $\mathcal{H}$ has a subsequence that converges quasi-uniformly around D.

<u>Proof</u>: a) => b) Let F be a non-empty subset of D and let $\varepsilon>0$. There exists open sets $O_1, \ldots, O_n$ such that $O_i \cap F \neq \emptyset$ and

$\mathcal{H} = \bigcup_{i=1}^{n} \{g\in\mathcal{H};\ \operatorname{osc}(g|\bar{F} \cap O_i) \leq \varepsilon)\}$. If $\mathcal{U}$ is an ultrafilter on $\mathcal{H}$,

there exists j $(1 \leq j \leq n)$ and $A \in \mathcal{U}$ such that

$\sup_{g\in A} \operatorname{osc}(g|\bar{F} \cap O_j) \leq \varepsilon$. This clearly implies that $\mathcal{U}$ converges quasi-uniformly around D.

b) => c) is immediate in view of Lemma I.3.

c) => a) Since $\bar{\mathcal{H}}$ is compact in the pointwise topology, it is enough to show that if F is a fixed non-empty subset of D and $\varepsilon>0$, then any f in $\bar{\mathcal{H}}$ has a neighborhood $\mathcal{V}$ such that there exist open sets $O_1, \ldots, O_n$ with $O_i \cap F \neq \emptyset$ with the following property:

$g \in \mathcal{V} \Rightarrow \exists j\ (1 \leq j \leq n)\ \operatorname{osc}(g|O_j \cap \bar{F}) \leq \varepsilon.$

We can clearly assume $\bar{D} = K$. Assume the claim false for some $\varepsilon > 0$ and some $F \subset D$. Since f is a first class function around D, there exists an open set V such that $\operatorname{osc}(f|V \cap \bar{F}) < \frac{\varepsilon}{8}$. Let $C = V \cap F$. We can assume - up to subtracting a constant off f - that we have the following situation:

i) $|f| < \frac{\varepsilon}{8}$ on $\bar{C}$

ii) For any neighborhood $\mathcal{V}$ of f (for the pointwise topology) and any open sets $O_1, \ldots, O_n$ with $O_i \cap C \neq \emptyset$ $(1 \leq i \leq n)$ there exists g in $\mathcal{V} \cap \mathcal{H}$ so that $\operatorname{osc}(g|O_i \cap \bar{C}) > \varepsilon$ for each $i=1, \ldots, n$.