The Carus Mathematical Monographs

NUMBER FOUR

PROJECTIVE GEOMETRY

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Published for
THE MATHEMATICAL ASSOCIATION OF AMERICA
by
THE OPEN COURT PUBLISHING COMPANY
CHICAGO, ILLINOIS

PREFACE

Projective Geometry may be approached by various routes: postulational or intuitive, synthetic or analytic, metric or purely projective. In a monograph which is to give a first approach to the subject it has seemed to me that the treatment should be based on intuition, should develop the subject by synthetic methods, and should keep projective properties sharply distinguished from the metric specializations. The reader will accordingly find in the first five chapters a systematic and thoroughly elementary treatment of the most fundamental propositions of projective geometry, culminating in the theorems of Pascal and Brianchon and the polar system of a conic. My purpose in these chapters has been to develop on an intuitive basis the concepts and the properties of projective space, without any admixture of metric ideas. Only in this way, I believe, can the reader gain a clear impression of what the word projective implies.

A monograph on projective geometry, however, which aims at some degree of comprehensiveness can not stop there. Much of the beauty and value of the subject lies in its relation to metric geometries, and the foundation for the use of analytic methods should at least be laid. Accordingly, I devote the remaining chapters to such additional aspects of our subject in order to fill in and round out the picture. Chapter VI, devoted to a first introduction to the metric specialization of projective theorems, is still thoroughly elementary. Beginning with Chapter VII, however, the treatment will make somewhat greater demands on the reader's mathematical maturity, since it is based on the group concept. After a preliminary Chapter (VII), Chapter VIII lays the foundation for the use of analytic methods and Chapter IX discusses metric properties from the more general standpoint of the group to which a geometry belongs.

In writing this monograph I have, of course, made free use of the text by Professor Veblen and myself, *Projective Geometry*, two volumes, Ginn & Company (the second volume by Professor Veblen alone). I have also found Professor Severi's *Geometria Proiettiva* very useful in certain parts of my work. I am greatly indebted to the other members of the editorial committee of the Carus Monographs, Professors Slaught, Bliss, Curtis and Kempner, for many valuable criticisms and suggestions resulting from their careful reading of the manuscript and the proof sheets. Especially must I express my gratitude and appreciation to Professor Slaught for the large amount of painstaking and time-consuming work which he put on the task of seeing the little book through the press, especially in its earlier stages when I was abroad. If this monograph proves to be a worthy companion for the earlier members of the monograph family, it will be very largely due to the unselfish efforts of these friends.

J. W. Young

Hanover, New Hampshire
November, 1929

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CHAPTER I

INTRODUCTORY CONCEPTIONS

1. Perspective drawing. Projective geometry like many another mathematical discipline has its historical origin in a practical problem: How to draw a picture in a plane that shall represent a three-dimensional object in such a way that the various portions of the picture, in their mutual relations, present the same aspect as do the corresponding visible portions of the object. Among the first to consider this problem of perspective drawing in a scientific way was Leonardo da Vinci (1452–1519), whose fame is perhaps greatest as a painter, but who according to more recent research must also be classed as a great pioneer in the domain of science.

The geometric formulation of the problem, as conceived by Leonardo, is as follows: From every visible point of a given three-dimensional object rays of light enter the eye of the observer. If a transparent plate be inserted between the eye and the object, each of these rays pierces the plate in a definite point, which is the image of the corresponding point of the object. The aggregate of all these points on the plate constitutes the desired picture. The problem consists of finding out how to draw the picture without the intervention of the transparent plate. We may note in passing that the photographic camera accomplishes precisely this feat when it collects the rays from the object in its lens, the "eye of the camera," and projects them on the sensitized plate. That the plate is in this case behind the "eye," rather than in front of it, is obviously an unimportant difference.

2. Projection and section. Correspondence. By considering in more detail the nature of the process just described, we shall become familiar with one of the fundamental processes of projective geometry, and shall also get a glimpse of some of the characteristics of this geometry which differentiate it from the more familiar, so-called metric, geometry of our school days.

The purely geometric description of the process suggested by Leonardo consists of two parts: From a point O lines are drawn to every point of a geometric figure F; these lines issuing from O are cut by a plane ω. We may now make our first definition.

The set of lines joining a point O to the points of a figure F is called the *projection of F from O*. If a set of lines issuing from a point O is cut by a plane ω, the set of points in which the plane ω cuts the lines through O is called the *section* of the lines through O by the plane ω.

This process of projection and section is fundamental in projective geometry. By means of it, to every point of the figure F is made to correspond a definite line through O, and in general, to every line through O is made to correspond a definite point on ω. Certain exceptions to this statement which may arise will be considered presently. The concept thus suggested of a correspondence between the elements of two figures is of fundamental importance. We shall, therefore, give a formal definition of it.

The elements of two geometric figures are said to be in *reciprocally one-to-one correspondence*, by some definite process (as for example, by the process of projection and section just described), if to every element of one figure is made to correspond a uniquely determined element of the other, and, vice-versa, if every element of the second figure is the correspondent of a uniquely determined element of the first.

If now we return to our problem of perspective drawing, we see that the picture on the transparent plate

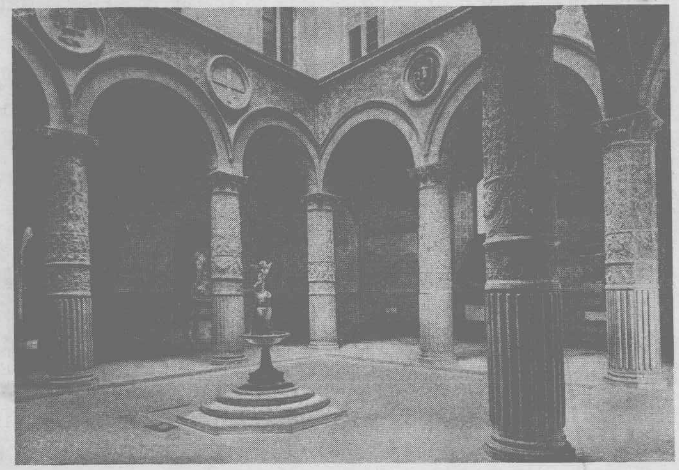

FIG. 1.

between the object and the eye of the artist is the figure obtained by means of a projection from the eye of the observer and a section (of this projection) by the plane of the transparent plate. If the reader will consider the nature of any perspective representation, for example the adjoined photograph of a court in the Palazzo Vecchio in Florence, he will recognize the following characteristics: A straight line in the original object is represented in the photograph by a straight line. This must be the case, since all the lines through the center of projection and the points of a given line lie in a plane and the section of this plane by the plane of section must be a straight line. The intersection of two straight lines, that is a point, of the original is represented by the intersection of the corresponding lines in the photograph. An angle of the original will be represented by an angle in the photograph, but *not* in general *by an angle of the same size.* A right angle in the original, for example, may be represented by either an acute or an obtuse angle in the picture. The reader will observe a number of examples in the adjoined photograph. He should, however, not content himself by merely observing the fact, but should make clear to himself the reason for it, by considering the nature of the process of projection and section. The same remark applies to the observations that follow.

Two parallel lines in the original will not in general be represented by parallel lines in the picture; equal distances in the original do not in general correspond to equal distances in the picture; the perspective representation of a circle is usually an ellipse; etc.

It is clear then that the perspective representation of an object involves a very considerable distortion, but always such that points are represented by points and straight lines by straight lines (except when a line of the original figure passes through O).

The attentive reader will have noted, however, that the process described above, of representing a given three-dimensional figure on a plane does not in general establish a *reciprocally* one-to-one correspondence between the points of the given figure and those of the plane. For if two points of the former are on a line with O, they correspond to the same point on the latter, and if two lines of the former are on the same plane through O, they correspond to the same line of the latter. The process of projection and section does, however, in general give rise to a reciprocally one-to-one correspondence, *if the elements of the first figure all lie in one plane.* Such a correspondence is more precisely defined in the next section. It is illustrated in a photograph, if attention is confined to one plane of the scene depicted.

3. Projective transformations. Let F be any figure in a plane and let it be projected from a point O not in the plane of the figure. The section of the projection by a plane gives rise to a new figure F' and the correspondence between the elements of F and F' is called a *perspective correspondence* or a *perspective transformation.* If then F' be projected from a new center O' on to a third plane a new figure F'' results. The figure F'' is obtained from F by means of two perspective transformations, one performed after the other. Similarly, we may consider the result of a sequence of any number of perspective correspondences. This leads to the following definition:

The resultant of a sequence of perspective transformations is called a *projective transformation* or a *projective correspondence.*

The concept of a projective correspondence lies at the very foundation of projective geometry, as will be seen in what follows. Indeed, we may now characterize projective geometry as follows: *Projective geometry is* *concerned with those properties of figures which remain unchanged by projective transformations.*

It follows at once that the parallelism of straight lines, the equality of distances or of angles, can have no place in projective geometry, since these properties are all changed by projective transformations. Parallel lines may be transformed into intersecting lines, equal distances may be transformed into unequal distances, right angles may be transformed into acute or obtuse angles, etc. On the other hand, a point and a straight line are always transformed into a point and a straight line respectively by any projective transformation (no matter how many projections and sections may have had a part in the projective transformation); also, if a point A of F is on a line l of F, the point A' corresponding to A under any projective transformation will lie on the line l' corresponding to l; and if A is not on l, A' will not be on l'. Two intersecting straight lines will correspond to two intersecting straight lines, a triangle will correspond to a triangle, a quadrilateral to a quadrilateral, etc. Certain possible exceptions to some of the above statements which may occur to the critical reader will, as has been indicated, be considered presently. Enough has been said to show that properties concerning merely the *incidence* of points and lines are *projective properties*, i.e., properties which remain unchanged under projective transformations, while any properties concerned with measurement, i.e., *metric properties*, are foreign to projective geometry as such. We shall see later, however, how such metric properties may be obtained from projective properties by a process of specialization.

A triangle is, as has been indicated, a figure of projective geometry; but equilateral, or isosceles, or right triangles are not; because they involve properties which are not preserved under projective transformations. A quadrilateral is a figure of projective geometry; but a parallelogram, a rectangle, a square, etc. is not. It will be seen later that a conic section is a curve of projective geometry, but that the classification into hyperbola, ellipse, and parabola involves metric properties.

At first sight it may appear that the consideration of projective properties only would so greatly restrict the field of operations as to give little content to projective geometry. This, however, is not the case, as will soon become apparent enough. By confining ourselves to the consideration of projective properties the resulting geometry becomes structurally much simpler than one involving in addition a host of metric properties; but projective geometry is, nevertheless, very rich in content. Indeed, as has already been indicated, it contains, when its theorems are suitably specialized, the whole content of ordinary euclidean metric geometry and also the content of certain non-euclidean geometries. Such considerations, which will mean much more to the uninitiated reader when he has reached the end of this monograph than they can possibly mean now, led the English mathematician Cayley to exclaim: "Projective Geometry is all geometry." To make clear in what sense this famous dictum is true is one of the primary objects of this little book.

4. A projective theorem. The reader who is approaching the study of projective geometry for the first time will be curious as to the nature of a geometric theorem which involves no metric conceptions. The following proposition, known as Desargues' Theorem on perspective triangles, will prove to be fundamental in the systematic development of certain parts of projective geometry to which the later chapters are devoted:

If two triangles ABC and $A'B'C'$ are so related that the three points of intersection of the pairs of sides AB and $A'B'$, BC and $B'C'$, CA and $C'A'$ are on a straight line, the lines AA', BB', CC' joining corresponding vertices all pass through the same point (*or are parallel*). (The phrase in parentheses is necessary as long as we state the theorem in the so-called metric space of ordinary geometry; it becomes unnecessary in the projective space to be introduced in the next chapter, in which the theorem will, moreover, gain in content.)

It will be observed that we have here a theorem which involves in its statement only the incidence of points and lines; no metric notions are involved. A formal proof of the theorem will be given later (p. 34). At this point it is a good exercise for the spatial imagination to observe that, if the two triangles are in different planes, the theorem is almost self-evident. If the reader will exercise his imagination sufficiently to get a clear mental picture of two intersecting planes, in each of which is a triangle whose pairs of corresponding sides intersect on a line (the latter must be the intersection of the two planes), the conclusion of the theorem follows almost immediately. In fact, every pair of the lines AA', BB', CC' lies in a plane, and three planes must intersect in a point (or in parallel lines or in a single line). The reader may then think of one of the planes as rotating about its line of intersection with the other plane until it comes to coincide with the latter. This will make the theorem appear plausible, at least, even for the case when the two triangles are in the same plane.

Although the historical origin of projective geometry goes back to the latter part of the fifteenth century, and although isolated theorems of this form of geometry were proved by Desargues (1593–1662) and Pascal (1623–1662), projective geometry as a self-contained discipline was not developed until the great French mathematician Poncelet (1788–1867) published his classic "Traité des propriétés projectives des figures" in 1822. Since then the development of this branch of geometry has been rapid, so that it is now recognized as one of the truly fundamental disciplines of modern mathematics, not only on account of its varied and important contacts with mathematics as a whole, but also on account of the intrinsic beauty of its structure and of its results.

This beauty is largely due to its simplicity. The latter is due not merely to stripping geometry of the complexities of its various metric concepts, but in no small measure also to the introduction of a new conception of space. To describe this "projective" space is the object of the next chapter, after which we may begin the more systematic development of our subject to the extent that the small compass of this monograph will permit.

References. Besides the classic treatise of Poncelet already referred to, the reader interested in the historical development of the subject may consult Chasles, *Aperçu historique sur l'origine et le développement des méthodes en géométrie*, Paris, 1837; also, various parts of the *Source Book in Mathematics*, edited by David Eugene Smith, New York, 1929.

CHAPTER II

PROJECTIVE SPACE. THE PRINCIPLE OF DUALITY

5. Ideal elements. We had occasion in the preceding chapter to refer to the fact that the "one-to-one" character of a perspective correspondence is, in metric space, subject to certain exceptions. Our first task in the present chapter must be to examine these exceptions and to see how to remove them.

If the process of projection and section, described in the last chapter for three-dimensional space, is confined to a single plane, we obtain the idea of a perspective correspondence between the points of two lines in a plane. If u and u' in the adjoining figure are two such lines (in the future we use the word line always to mean straight line, unless otherwise specified), and if S is any point in the plane of the two lines, but not on either line, the lines joining S to points A, B, C, $\cdots$ of u will in general meet u' in definite points A', B', C', $\cdots$. To any point of u, say A, will correspond by means of this construction a uniquely determined point A'; to B will correspond B'; to C, C'; and so on. Vice-versa to every point of u' corresponds, in general, a uniquely determined point of u. The point of inter-

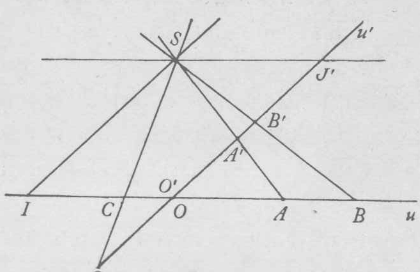

FIG. 2