Combinatorics of Finite Sets

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Preface

The past quarter century has seen the remarkable rise of combinatorics as a distinctive and important area of mathematics. Combinatorial topics have found a place in many university degree courses, and, since the founding of the *Journal of Combinatorial Theory* in 1966, there has been a flood of publications in the combinatorial area. Some branches of combinatorics are already well established, with their own unified body of theory and applications: examples are graph theory, with its topological flavour, coding theory and design theory, with their algebraic connections, and enumeration theory, concerned with the techniques of counting. These areas are already supplied with many textbooks at both undergraduate and postgraduate levels. The purpose of the present work, however, is to advertise another area of combinatorics, where a body of theory, at one time very scattered and *ad hoc*, is gradually being moulded into an elegant unity. Without attempting to be exhaustive, the book is intended to be a readable introduction to what is, for the author, a fascinating subject.

The origins of the theory can perhaps be traced back to 1928, when Sperner published a simple theorem which has had repercussions far beyond his wildest dreams. Sperner's theorem simply asserts that if you want to find as many subsets of an n-element set as possible, subject to the condition that no subset is contained in another, then you cannot do better than to choose all the subsets of size $[\frac{1}{2}n]$. This theorem has been reproved and generalized to such an extent that it has given rise to a whole branch of the theory of partially ordered sets (or posets) called Sperner theory. Although we shall go some way down this path, we shall not restrict ourselves to this area. Instead, we shall use Sperner's theorem as a springboard and a signpost, discovering where the ideas involved in its various proofs lead to. We shall be led to consider the structure of the set of subsets of a finite set viewed as a poset, its chain decompositions and its antichains, its rank levels, and the inclusion relations between these levels. This will take us, for example, to the famous Kruskal–Katona theorem which answers the following question: given r subsets of S, all of size k, what is the least possible number of sets of size $k-1$ contained in them? The answer to this question involves a nice interplay between two different orderings, namely the partial ordering of the subsets of S by inclusion, and the total ordering of the subsets of S of a given size by what we shall call the squashed ordering, which is a variation on the more familiar lexicographic or alphabetical ordering.

Anyone writing a book in this area is faced right at the start by a fundamental problem. Many results for the poset of subsets of a set can be extended to more general posets. Sometimes one of the available proofs for sets easily extends to more general posets; sometimes, however, the nicest proof for sets does not generalize. So should the results be presented in their most general form (thereby sometimes losing out on clarity) or should they be presented for subsets of a set (thereby losing out on generality but perhaps gaining in clarity)? An example of this problem arises in connection with the Kruskal–Katona theorem. Several proofs are available, but a more difficult proof due to Clements and Lindström establishes the result in the more general context of the poset of divisors of a number (or subsets of a multiset). Since we include the Clements–Lindström theorem in Chapter 9, there is strictly speaking no need to include a separate proof of the Kruskal–Katona theorem. However, in the simpler context of sets, the Kruskal–Katona theorem has such an elegant theory surrounding it that it would be almost criminal to omit the simpler case. So we present both proofs. On many other occasions we prove results in more than one way because the different proofs illustrate different ideas and different techniques. On the whole, I have taken the view that I should present results in their simplest forms, concentrating mainly on sets and multisets. Accordingly, the reader will not find a discussion of, say, geometric lattices, although their 'prototype', the poset of partitions of a set, is discussed. In a few places we look at posets more generally; the final chapter, for example, discusses extensions of the theorem of Dilworth concerning chain decompositions of a general poset.

In searching out the material for this book I was greatly helped by several survey articles, two of which deserve special mention: the first is that by Greene and Kleitman (1978), and the second is the more recent one by D. B. West (1982). As will be seen from a glance at the extensive list of references at the end of the book, a number of more recent results have been included. Inevitably some of the results presented here will have been improved upon by the time this book appears in print, but in a sense this does not matter for the aim of the book is not to provide an exhaustive survey but to present some of the ideas and techniques which go to make up the subject. Inevitably, also, my choice of material will not meet with the approval of all, but a number of interesting results not in the text have been included in the exercises at the end of each chapter with hints or outlines of their solutions at the end of the book.

A number of people have helped and encouraged me in the writing of this book. In particular, I gladly acknowledge the helpful comments of Professor George Clements and Dr Hazel Perfect. I should also like to thank the Oxford University Press for encouraging me to write, and the University of Glasgow for granting me a period of study leave during which the final compilation of the book was accomplished.

Glasgow I.A.
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Some notation used in the text

$\binom{n}{r}$	binomial coefficient (n-choose-r)
$\Delta\mathcal{A}$	the shadow of $\mathcal{A}$
$\Delta^{(k)}\mathcal{A}$	the shadow of $\mathcal{A}$ at level k
$\nabla\mathcal{A}$	the shade of $\mathcal{A}$
$C\mathcal{A}$	the compression of $\mathcal{A}$
$\mathcal{A}'$	the set of complements of members of $\mathcal{A}$
$(k)S$	the set of k-subsets of S
N_i	the number of members of rank i
$<_s$	the squashed ordering
$F_k(n)$	the first n k-sets (or k-vectors)
$\mathfrak{A}(P)$	the set of antichains of the poset P
$\varnothing$	the empty set
$\tau(m)$	the number of divisors of m
$d_k(P)$	the size of the largest k-union in P

Contents

1 Introduction and Sperner's theorem

1.1 A simple intersection result

The main theme of this book is the study of collections $\mathscr{A}$ of subsets of a finite set S, where $\mathscr{A}$ is described in terms of intersection, union, or inclusion conditions. An amazing richness and variety of results will be discovered, developed, and extended in various directions. Although our main initial theme will be a study of a theorem of Sperner which could be said to be the inspiration of all that follows, we get into training by first of all asking what must surely be one of the simplest questions possible.

Problem Let $\mathscr{A}$ be a collection of subsets of an n-element set S (or an *n-set* S) such that $A_i \cap A_j \neq \varnothing$ for each pair i, j. How big can $|\mathscr{A}|$ be? The answer, and more besides, is given by the following theorem.

Theorem 1.1.1 If $\mathscr{A}$ is a collection of distinct subsets of the n-set S such that $A_i \cap A_j \neq \varnothing$ for all $A_i, A_j \in \mathscr{A}$, then $|\mathscr{A}| \leq 2^{n-1}$. Further, if $|\mathscr{A}| < 2^{n-1}$, $\mathscr{A}$ can be extended to a collection of 2^{n-1} subsets also satisfying the given intersection property.

Proof If $A \in \mathscr{A}$ then the complement $A' = S - A$ is certainly not in $\mathscr{A}$, since $A \cap A' = \varnothing$. So we immediately obtain $|\mathscr{A}| \leq \frac{1}{2}2^n = 2^{n-1}$. This bound cannot be improved upon since the collection of all subsets of $\{1, \ldots, n\}$ containing 1 satisfies the intersection condition and has 2^{n-1} members.

Now suppose $|\mathscr{A}| < 2^{n-1}$. Then there must be a subset A with $A \notin \mathscr{A}$ and also $A' \notin \mathscr{A}$. We can then add A to the collection $\mathscr{A}$ unless there exists $B \in \mathscr{A}$ such that $A \cap B = \varnothing$. But then $B \subseteq A'$ and so we could add A' to $\mathscr{A}$. If the resulting collection has fewer than 2^{n-1} members, repeat the process. □

This example pinpoints some key questions. Given a property involving union, intersection, and inclusion, how large can a collection $\mathscr{A}$ of subsets of S be if $\mathscr{A}$ satisfies the property? Can we characterize those collections which maximize $|\mathscr{A}|$? These are the sort of questions which we shall study.

1.2 Sperner's theorem

We now consider the property: if $A_i, A_j \in \mathscr{A}$, then $A_i \nsubseteq A_j$. A collection of subsets of S with this property is called a collection of *incomparable* sets, or an *antichain*, or sometimes a *clutter*. It is an antichain in the sense that its property is the other extreme from that of a chain in which every pair of sets is comparable.

Theorem 1.2.1 (Sperner 1928) Let $\mathscr{A}$ be an antichain of subsets of an n-set S. Then

$$|\mathscr{A}| \leq \binom{n}{[n/2]}.$$

This result is clearly best possible since the subsets of size $[n/2]$ form an antichain. The original proof given by Sperner will be analysed in Chapter 2, but we start here by giving a simple elegant proof due to Lubell which is pregnant with generalizations and extensions. Altogether we shall give three different proofs, not just because they exist, but because each in its own way presents us with ideas which can be developed to suit a wider range of ordered structures.

Proof of Theorem 1.2.1 (Lubell 1966) First of all note that there are $n!$ permutations of the elements of S. We shall say that a permutation π of the elements of S *begins with* A if the first $|A|$ members of π are the elements of A in some order. Now the number of permutations beginning with A must be $|A|!(n-|A|)!$. Also, no permutation can begin with two different sets in $\mathscr{A}$, since one of these sets would contain the other; therefore permutations beginning with different sets in $\mathscr{A}$ are distinct. Thus

$$\sum_{A \in \mathscr{A}} |A|!(n-|A|)! \leq n!$$

If we let p_k denote the number of members of $\mathscr{A}$ of size k, we have

$$\sum_k k!(n-k)!p_k \leq n!$$

whence

$$\sum_k \frac{p_k}{\binom{n}{k}} \leq 1. \tag{1.1}$$

Thus

$$|\mathscr{A}| = \sum_k p_k = \binom{n}{[n/2]} \sum_k \frac{p_k}{\binom{n}{[n/2]}} \leqslant \binom{n}{[n/2]} \sum_k \frac{p_k}{\binom{n}{k}} \leqslant \binom{n}{[n/2]}. \quad (1.2)$$

□

Note that Lubell's proof actually gives a stronger result than Sperner's theorem. The inequality (1.1) is called a *LYM inequality* after Lubell, Yamamoto (1954), and Meschalkin (1963), whose work includes similar results. Instead of giving an upper bound for $\sum_k p_k$, the inequality gives an upper bound for the weighted sum $\sum_k p_k \Big/ \binom{n}{k}$. Note that $p_k \Big/ \binom{n}{k}$ is the proportion of all those subsets of S of size k which are in $\mathscr{A}$; the LYM inequality asserts that the sum of these proportions is at most unity. The special case where $p_k = 1$ if $k = [n/2]$, $p_k = 0$ otherwise, shows that the bound can be attained.

We have thus shown that the maximum size of an antichain of subsets of an n-set S is $\binom{n}{[n/2]}$. Can we identify all the antichains which are as big as this? The first inequality in (1.2) shows that we can attain the bound $\binom{n}{[n/2]}$ only if $p_k = 0$ whenever $\binom{n}{k} < \binom{n}{[n/2]}$. Therefore, if n is even, there is only one maximum-sized antichain, namely the collection of all $n/2$-subsets. If n is odd, all sets in a maximum-sized antichain must be of size $\frac{1}{2}(n-1)$ or $\frac{1}{2}(n+1)$. We now show that there can in fact be no mixture of sizes; a maximum-sized antichain consists either of all subsets of size $\frac{1}{2}(n-1)$ or of all subsets of size $\frac{1}{2}(n+1)$.

Theorem 1.2.2 If n is even, the only antichain consisting of $\binom{n}{[n/2]}$ subsets of the n-set S is made up of all the $n/2$-subsets of S. If n is odd, an antichain of size $\binom{n}{[n/2]}$ consists of either all the $\frac{1}{2}(n-1)$-subsets or all the $\frac{1}{2}(n+1)$-subsets.

Proof (Lovász 1979) The case of even n has been dealt with. Suppose now that $n = 2m+1$ and that $\mathscr{A}$ is an antichain of size $\binom{n}{m}$. Note that in Lubell's proof the only way of finishing up with equality in (1.2) is to have equality at each stage, so in particular every permutation must contribute a member of $\mathscr{A}$ with which it begins.

Now $\mathscr{A}$ must consist only of sets of size m or $m+1$. Suppose that X, Y are subsets of size m, $m+1$ respectively and that $X \subset Y$. If $X = \{x_1, \ldots, x_m\}$ and $Y = \{x_1, \ldots, x_m, x_{m+1}\}$ then since any permutation beginning with $x_1, \ldots, x_{m+1}$ must begin with a member of $\mathscr{A}$, X or Y must be in $\mathscr{A}$.

Our aim is to prove that $\mathscr{A}$ consists of all m-sets or all $(m+1)$-sets. Suppose that $\mathscr{A}$ contains some but not all of the $(m+1)$-sets. Then we can find sets E, F such that $|E| = |F| = m+1$, $E \in \mathscr{A}$, $F \notin \mathscr{A}$. By relabelling the elements of S if necessary we can suppose that $E = \{x_1, \ldots, x_{m+1}\}$ and $F = \{x_i, \ldots, x_{m+i}\}$ for some i. Since $E \in \mathscr{A}$ and $F \notin \mathscr{A}$ there must be a largest integer $j < i$ with $\{x_j, \ldots, x_{m+j}\} \in \mathscr{A}$. Then $E^* = \{x_j, \ldots, x_{m+j}\} \in \mathscr{A}$ and $F^* = \{x_{j+1}, \ldots, x_{m+j+1}\} \notin \mathscr{A}$. We now have an impossible situation. Since $E^* \cap F^* \subset E^*$ where $E^* \in \mathscr{A}$, we must have $E^* \cap F^* \notin \mathscr{A}$. However, $E^* \cap F^* \subset F^*$ where $|E^* \cap F^*| = m$ and $|F^*| = m+1$, so by an earlier part of the proof one of $E^* \cap F^*$ and F^* must be in $\mathscr{A}$. This contradiction shows that our assumption must have been false and so $\mathscr{A}$ indeed consists only of sets all of the same size. □

1.3 A theorem of Bollobás

As another example of how the permutation approach of Lubell's proof can be used to obtain elegant proofs of results obtained originally by more complicated arguments, we now give a generalization of Sperner's theorem due to Bollobás (1965). The result was also independently proved by Katona (1974), Tarjan (1975), and Griggs, Stahl, and Trotter (1984). This repeated discovery of results by authors working independently is a frequent occurrence in this area of mathematics!

Theorem 1.3.1 (Bollobás 1965) Let $A_1, \ldots, A_m$, $B_1, \ldots, B_m$ be subsets of an n-set S such that $A_i \cap B_j = \varnothing$ if and only if $i = j$. Let $a_i = |A_i|$ and $b_i = |B_i|$. Then

$$\sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \leq 1.$$

Proof Consider each of the $n!$ permutations of the elements of S, and say that a permutation π *contains A followed by B* if all the elements of A occur in π before all the elements of B. If a particular permutation π contains A_i followed by B_i and also contains A_j followed by B_j, then $A_i \cap B_j = \varnothing$ (if A_i ends before B_j begins) or $A_j \cap B_i = \varnothing$ (if A_i ends after B_j begins), and either of these contradicts the hypotheses. So, for each permutation π, there is at most one i for which π contains A_i followed by B_i. However, given i, the number of permutations containing A_i followed by B_i can be found as follows. Choose the $a_i + b_i$ positions to be filled by the elements of A_i and B_i; this can be done in $\binom{n}{a_i + b_i}$ ways. Then place the a_i members of A_i in some order in the first a_i of the chosen positions and then the b_i members of B_i in some order in the remaining b_i positions; this can all be done in $a_i!b_i!$ ways. Finally order the remaining $n - a_i - b_i$ elements of S and place then in the remaining places of the permutation; this can be done in $(n - a_i - b_i)!$ ways. Thus the number of permutations π containing A_i followed by B_i is

$$\binom{n}{a_i + b_i} a_i!b_i!(n - a_i - b_i)! = \frac{n!}{\binom{a_i + b_i}{a_i}}.$$

Summing over all i we now obtain

$$\sum_i \frac{n!}{\binom{a_i + b_i}{a_i}} \leqslant n!$$

as required. □

Note that Sperner's theorem follows on taking $B_i = A_i'$, the complement of A_i, for the condition $A_i \cap B_i = \varnothing$ becomes $A_i \cap A_i' = \varnothing$, the condition $A_i \cap B_j \neq \varnothing$ becomes $A_i \cap A_j' \neq \varnothing$, i.e. $A_i \nsubseteq A_j$, and the conclusion yields

$$\sum_k \frac{p_k}{\binom{n}{k}} = \sum_i \frac{1}{\binom{n}{a_i}} = \sum_i \frac{1}{\binom{a_i + b_i}{a_i}} \leqslant 1.$$

Theorem 1.3.1 has been generalized in a number of ways. Frankl (1982) and Kalai (1984) weakened the condition to $A_i \cap A_j \neq \varnothing$ for $1 \leqslant i < j \leqslant m$ and obtained the same conclusion. Lovász (1977) generalized the theorem to subspaces of a linear space. As a recent application, we now apply Theorem 1.3.1 to the following generalization of the Sperner situation. Suppose we are given m chains of subsets of an n-set S which are incomparable in the sense that no member of one chain is contained in a member of any other chain. How large can m be? In the case where all the chains have $k+1$ members, let $f(n, k)$ denote the largest possible value of m. Then Sperner's theorem corresponds to the case $k=0$ and asserts that $f(n, 0)=\binom{n}{[n/2]}$.

Theorem 1.3.2 (Griggs *et al.* 1984) Let $f(n, k)$ denote the largest value of m for which it is possible to find m chains of $k+1$ distinct subsets of an n-set S such that no member of any chain is a subset of a member of any other chain. Then

$$f(n, k)=\binom{n-k}{[(n-k)/2]}.$$

Proof Suppose that we have m chains

$$A_{i,0} \subset A_{i,1} \subset \ldots \subset A_{i,k} \quad (i=1, \ldots, m)$$

satisfying the conditions of the theorem. In Theorem 1.3.1 take $A_i=A_{i,0}$ and $B_i=S-A_{i,k}$. Then $a_i=|A_{i,0}|$ and $b_i=n-|A_{i,k}|$. It is clear that $|A_{i,k}| \geqslant a_i+k$, so $b_i \leqslant n-k-a_i$; thus

$$\binom{a_i+b_i}{a_i} \leqslant \binom{n-k}{a_i} \leqslant \binom{n-k}{[(n-k)/2]}.$$

Now we certainly have $A_i \cap B_i=\varnothing$. Also, if we had $A_i \cap B_j=\varnothing$ for some $i \neq j$ we would then have $A_{i,0} \subset A_{j,k}$, contradicting the hypotheses. So the sets A_i and B_i satisfy the conditions of Theorem 1.3.1, and we have

$$m=\sum_{i=1}^{m} 1 \leqslant \sum_{i=1}^{m}\binom{n-k}{[(n-k)/2]} \Big/ \binom{a_i+b_i}{a_i} \leqslant \binom{n-k}{[(n-k)/2]}.$$

Thus

$$f(n, k) \leqslant \binom{n-k}{[(n-k)/2]}.$$

To complete the proof we exhibit $\binom{n-k}{[(n-k)/2]}$ chains with the required properties. Consider the $[(n-k)/2]$-subsets X of

$\{k+1, \ldots, n\}$. There are $\binom{n-k}{[(n-k)/2]}$ such subsets. For each such X take the chain

$$X \subset X \cup \{1\} \subset X \cup \{1, 2\} \subset \ldots \subset X \cup \{1, 2, \ldots, k\}.$$

These chains have the required properties. □

Further generalizations of Sperner's theorem will be discussed later, particularly in Chapter 8 and in the study of the Littlewood–Offord problem in Chapter 11.

Exercises 1

1.1 Prove that if $A_1, \ldots, A_m$ are distinct subsets of an n-set S such that for each pair i, j, $A_i \cup A_j \neq S$ then $m \leqslant 2^{n-1}$.

1.2 Can we have equality in Theorem 1.1.1 without all the A_i having a common element?

1.3 Show that if $\mathscr{A}$ is an antichain of subsets of an n-set with $|A| \leqslant h \leqslant \frac{1}{2}n$ for all $A \in \mathscr{A}$, then $|\mathscr{A}| \leqslant \binom{n}{h}$.

1.4 How many antichains of subsets of S are there if (a) $|S| = 2$, (b) $|S| = 3$? (This will be followed up in Chapter 3.)

1.5 Show that the number of pairs X, Y of distinct subsets of an n-set S with $X \subset Y$ is $3^n - 2^n$.

1.6 A collection $\mathscr{B}$ of subsets of an n-set S is called a *cross-cut* if every subset of S is comparable with (i.e. contains or is contained in) at least one member of $\mathscr{B}$. Suppose that $\mathscr{B}$ is a minimal cross-cut (i.e. $\mathscr{B}$ is a cross-cut but no proper subset of $\mathscr{B}$ is a cross-cut). Show that $|\mathscr{B}| \leqslant \binom{n}{[n/2]}$.

1.7 Let $x_1, \ldots, x_n$ be real numbers, $|x_i| \geqslant 1$ for each i, and let I be any unit interval on the real line. Show that the number of linear combinations $\sum_{i=1}^{n} \varepsilon_i x_i$ with $\varepsilon_i = 0$ or 1 lying inside I is at most $\binom{n}{[n/2]}$. (Hint: associate with each sum the corresponding set of indices i for which $\varepsilon_i = 1$.) (Erdös 1945)

1.8 Let $A_1, \ldots, A_m, B_1, \ldots, B_m$ be subsets of an n-set S such that