Geometric Measure Theory and the Calculus of Variations

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Geometric Measure Theory and the Calculus of Variations

William K. Allard and
Frederick J. Almgren, Jr., Editors

AMERICAN MATHEMATICAL SOCIETY
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Geometric Measure Theory and the Calculus of Variations

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Preface

The papers in these Proceedings are associated with lectures delivered at the Thirty-Second Summer Research Institute of the American Mathematical Society devoted to Geometric Measure Theory and the Calculus of Variations. The Institute was held at Humboldt State University in Arcata, California from July 16 to August 3, 1984 and was financed by the National Science Foundation.

The Institute served as a forum for the large collection of new (and older) ideas and techniques comprising modern geometric measure theory and its applications in the calculus of variations. A major theme of the Institute was the introduction and application of multiple-valued function techniques as a basic new tool in geometric analysis; Almgren's basic paper on this subject appears in these Proceedings. In addition, major new results were announced and discussed during the Institute. Allard announced his integrality and regularity theorems for surfaces which are stationary with respect to general elliptic integrands. Scheffer announced the first example of a singular solution to the Navier–Stokes equations for fluid flow with an opposing force; this follows his earlier pioneering work on the possible size of such singular sets. Hutchinson introduced his new definition of the second fundamental form of a general varifold and gave various applications including a multiple-valued function regularity theorem. Papers on these topics are included in these Proceedings.

A session on open problems was also held during the Institute and a list of eighty such problems appears in these Proceedings.

The organizing committee for the Institute consisted of: William K. Allard, Frederick J. Almgren, Jr. (co-chairmen); Enrico Bombieri, Robert M. Hardt, H. Blaine Lawson, Jr., Jon T. Pitts, Richard Schoen, and William P. Ziemer.

The editors would like to thank the many people who cooperated to make the Institute and this volume possible. Of special direct help were Dottie Smith (Conference Secretary), Marie D. Byrnes, Susan Hautala and Elaine Wellman Becker.

William K. Allard
Durham, North Carolina
May 6, 1985

List of Speakers and Titles of Their Talks

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Hausdorff measures and area and co-area formulas, Besicovitch type covering theorems, first variation of area and an isoperimetric inequality, by Enrico Bombieri

Lipschitz multiple-valued functions and approximations of currents, by Frederick J. Almgren, Jr.

Compactness theorems for rectifiable currents from multiple-valued approximations, by Frederick J. Almgren, Jr.

Applications of isoperimetric inequalities, continuity and lower semicontinuity of elliptic parametric functionals, positive currents, "monotonicity", by Enrico Bombieri

First variation distributions, mean curvature, varifolds, and formulas for first and second variations, by William K. Allard

Coflat forms and flat currents, by F. Reese Harvey

(2) Regularity Theory (5 lectures)

Single-valued regularity theory for elliptic variational problems with constraints, by Frederick J. Almgren, Jr.

Q-valued functions minimizing Dirichlet's integral, by Frederick J. Almgren, Jr.

Existence of minimal surfaces on Riemannian manifolds, by Jon T. Pitts

Regularity of minimal hypersurfaces on Riemannian manifolds, by Jon T. Pitts

Strong approximation of nearly flat area minimizing currents, by Frederick J. Almgren, Jr.

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Some regularity results in plasticity, by Robert M. Hardt

Crystals, by Jean E. Taylor

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(4) Partial Differential Equations and Geometric Measure Theory (5 lectures)

Compactification of minimal submanifolds in E^n by the Gauss map, by Michael T. Anderson

PDE methods in the study of geometric variational problems, by Leon Simon (2 lectures)

Regularity properties of quasi-minima, by William P. Ziemer

Geometry of minimal surfaces, by Hyeong Choi

(5) Riemannian Curvature and Minimal Surfaces (1 lecture)

Riemannian curvature and minimal surfaces, by William K. Allard

(6) Calibrated Geometries and the Geometry of the Unit Co-mass Ball (2 lectures)

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(7) Least Area Mappings and Spaces of Minimal Surfaces (1 lecture)

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(8) Geometric Measure Theory over the Complex Numbers (2 lectures)

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Basic techniques, by Enrico Bombieri

SHORT TALKS

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Jost, Juergen	Embedded minimal surfaces with free boundaries
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Mattila, Pertti	Hausdorff dimension and capacities of sets in n-space
Michelsohn, Marie-Louise	A new class of metrics on complex manifolds
Miranda, Mario	MACSYMA and minimal surfaces
Morgan, Frank	On finiteness of the number of stable minimal hypersurfaces
Nance, Dana	Multiplicity theorems for projections of 2- and 3-dimensional hypersurfaces
Nishikawa, Seiki	Deformation of Riemannian metrics and manifolds with bounded curvature ratios
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Taylor, Jean E.	Catalog of F-minimizing embedded F-crystalline cones
Wei, S. Walter	Strongly unstable manifolds and weakly stable currents
White, Brian	Regularity of minimizing hypersurfaces mod p

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Contents

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An Integrality Theorem and a Regularity Theorem for Surfaces whose First Variation with respect to a Parametric Elliptic Integrand is Controlled

WILLIAM K. ALLARD

0. Introduction. The following question is fundamental in the higher dimensional parametric calculus of variations: *Suppose* $C = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^N: |x| < 2\}$ *and* V *is an* n*-dimensional integral varifold in* C *with small tilt excess which has small first variation in* C *with respect to a parametric integrand which is nearly translation invariant. Is it true that, given an a priori bound on the* n*-area of* V, *the* n*-area of* V *inside* $\{(x, y): |x| < 1\}$ *is nearly an integer times the area of the unit* n*-disc*? We answer this question affirmatively in this paper. In fact, we show in 2.2 that *if* W *is the varifold projection on* $\mathbf{R}^n$ *of* V *inside* $\{(x, y): |x| < 1\}$, *the total variation of* W *minus the varifold associated to an integer multiple of the unit disc in* $\mathbf{R}^n$ *is small.* I call such a result an *integrality theorem*. It holds for any parametric integrand; ellipticity is not required. The key lemma used in the proof of the above integrality theorem may be roughly stated as follows: *Suppose* u *is a nonnegative measure on* $\{x \in \mathbf{R}^n: |x| < 2\}$ *whose total mass is not too large and the flat norm of the gradient of which is small. Then there is a nonnegative constant* c *such that the total variation of* u *minus* c *times Lebesgue measure is small on* $\{x: |x| < 1\}$. The proof of this lemma depends on the weak $(1, 1)$-inequality for singular integrals as stated, for example, on p. 33 of [**ES**]. I use these results in 2.6 to prove a Lipschitz approximation theorem for general integrands which may be considered a generalization of a similar theorem in [**AW1**, 8.12]. This approximation theorem is the basis for the Regularity Theorem of 3.6 which extends the regularity results of [**AW1**] to "nearly all" elliptic integrands in codimension one. We feel that our methods should yield regularity results for all elliptic integrands in codimension one and for a large and naturally defined class of integrands in

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higher codimension. In particular, we believe the conditions 3.6(1)–(2) are unnecessary. In order to use the method of attack in [**AW1**] to prove regularity theorems it seems necessary to establish a *height bound* as in 3.5. In the present paper this is accomplished by using barriers and this forces upon us the conditions 3.6(1)–(2). We feel one ought to be able to establish such a height bound without the use of barriers. It would suffice to establish a lower bound for mass ratios as in [**AW1**, 5.1]. These remain outstanding problems.

Conversations with F. J. Almgren, Jr. were helpful in carrying out this work.

0.1. *Notation.* Throughout this paper, n and N are positive integers and $n \geqslant 2$. We adopt the notation of [**FE**]; note the detailed glossary in the back of this book. We adopt the notation of [**AW1**]; note the definitions of the various varifold spaces in [**AW1**, 3]. In particular, $\mathbf{G}(n+N, n)$ is the Grassmann manifold of n-dimensional linear subspaces of $\mathbf{R}^{n+N}$; as is done in [**AW1**], we frequently identify $S \in \mathbf{G}(n+N, n)$ with orthogonal projection of $\mathbf{R}^{n+N}$ onto S. We let $\mathbf{P}$ be the projection of $\mathbf{R}^{n+N} \times \mathbf{G}(n+N, n)$ on its first factor $\mathbf{R}^{n+N}$. We let $\mathbf{p}$ be projection of $\mathbf{R}^n \times \mathbf{R}^N$ on its first factor $\mathbf{R}^n$. We will frequently identify $\mathbf{R}^{n+N}$ with $\mathbf{R}^n \times \mathbf{R}^N$, writing (x, y) for a variable point in $\mathbf{R}^n \times \mathbf{R}^N$ and writing z for a variable point in $\mathbf{R}^{n+N}$. We let $\mathbf{U}(a, r) = \{x \in \mathbf{R}^n : |x - a| < r\}$ whenever $a \in \mathbf{R}^n$ and $0 < r < \infty$. We let $e_1, \ldots, e_{n+N}$ be the standard basis for $\mathbf{R}^{n+N}$ and we let $e^1, \ldots, e^{n+N}$ be the standard basis for the dual of $\mathbf{R}^{n+N}$.

1. The Strong Constancy Lemma. For $u \in \mathscr{D}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$ we set

$$Gu(x) = \begin{cases} \dfrac{1}{n\alpha(n)} \displaystyle\int \log|x - y| u(y)\, d\mathscr{L}^n y & \text{if } n = 2, \\ \dfrac{2-n}{\alpha(n)} \displaystyle\int |x - y|^{2-n} u(y)\, d\mathscr{L}^n y & \text{if } n > 2; \end{cases}$$

thus, G is a linear map from $\mathscr{D}(\mathbf{R}^n)$ to $\mathscr{E}(\mathbf{R}^n)$. As is well known,

(1) $$G\Delta u = u \quad \text{for } u \in \mathscr{D}(\mathbf{R}^n)$$

where $\Delta = \sum_{j=1}^n D_j D_j$ is the classical Laplacian. We set

$$Q_j u = GD_j u \quad \text{and} \quad P_{jk} u = GD_j D_k u$$

for $u \in \mathscr{D}(\mathbf{R}^n)$ and $j, k \in \{1, \ldots, n\}$.

Suppose $j \in \{1, \ldots, n\}$ and consider Q_j. We have

(2) $$Q_j u = GD_j u = D_j Gu = s_j * u \quad \text{for } u \in \mathscr{D}(\mathbf{R}^n),$$

where

$$s_j(x) = \begin{cases} \dfrac{1}{n\alpha(n)} \dfrac{x^j}{|x|^2} & \text{if } n = 2, \\ \dfrac{(2-n)^2}{n\alpha(n)} \dfrac{x^j}{|x|^n} & \text{if } n > 2, \end{cases}$$

for $x \in \mathbf{R}^n$. For any $\varepsilon > 0$ we may write $s_j = a_j + b_j$ where $a_j \in \mathscr{E}(\mathbf{R}^n)$ and $\int |b_j|\, d\mathscr{L}^n \leqslant \varepsilon$. It follows that if $u_1, u_2, u_3, \ldots$ is a sequence in $\mathscr{D}(\mathbf{R}^n)$ such that $\sup\{\int |u_\nu|\, d\mathscr{L}^n : \nu = 1, 2, 3, \ldots\} < \infty$ and K is a compact subset of $\mathbf{R}^n$ then there is an $\mathscr{L}^n$-summable $\mathbf{R}$-valued function l on K such that

$$\liminf_{\nu \to \infty} \int_K |Q_j u_\nu - l|\, d\mathscr{L}^n = 0.$$

Suppose $j, k \in \{1, \ldots, n\}$ and consider P_{jk}. We assert that

$$\mathscr{L}^n\left\{x \in \mathbf{R}^n : \left|P_{jk} u(x)\right| \geqslant \alpha\right\} \leqslant \frac{C}{\alpha} \int |u|\, d\mathscr{L}^n \tag{3}$$

whenever $u \in \mathscr{D}(\mathbf{R}^n)$ and $0 < \alpha < \infty$, and where C depends only on n. For the proof of this highly nontrivial fact we refer to [SE] as follows. On p. 75 of [SE] we see that

$$P_{jk} u(x) = c u(x) + \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbf{R}^n : |y| \geqslant \varepsilon\}} \frac{\Omega(y)}{|y|^n} u(x - y)\, d\mathscr{L}^n y$$

whenever $u \in \mathscr{D}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$, and where $c \in \mathbf{R}$ and $\Omega : \mathbf{R}^n \sim \{0\} \to \mathbf{R}$ is smooth and positively homogeneous of degree 0 and has mean value zero on $\mathbf{S}^{n-1}$. An estimate like (3) is made on p. 33, line 3 under the hypotheses of Theorem 1 on p. 29. The present Ω does not satisfy the hypotheses of Theorem 1. However, as Stein points out in the two paragraphs after the statement of Theorem 1 on p. 29, the part of the hypotheses that the present Ω does not satisfy is not important and is made for short-term technical convenience. To complete the proof of (3) look at Theorem 2 on p. 35, especially at 3.4 on pp. 37–38.

(4) THE STRONG CONSTANCY LEMMA. *Suppose*

(a) $a \in \mathbf{R}^n$ *and* $0 < r < \infty$;

(b) $u \in \mathscr{D}'(\mathbf{U}(a, r))$, $0 \leqslant M < \infty$ *and*

$$0 \leqslant u(\varphi) \leqslant M r^n \sup\{\varphi(x) : x \in \mathbf{U}(a, r)\}$$

whenever $\varphi \in \mathscr{D}(\mathbf{U}(a, r))$ *and* $\varphi \geqslant 0$;

(c) $f_j, X_j^i \in \mathscr{D}'(\mathbf{U}(a, r))$ *for* $i, j \in \{1, \ldots, n\}$, $0 \leqslant \delta < \infty$ *and*

$$\left|X_j^i(\varphi)\right| + \left|r f_j(\varphi)\right| \leqslant \delta r^n \sup\{|\varphi(x)| : x \in \mathbf{U}(a, r)\}$$

whenever $\varphi \in \mathscr{D}(\mathbf{U}(a, r))$;

(d) $X_j = (X_j^1, \ldots, X_j^n)$ *for* $j \in \{1, \ldots, n\}$;

(e) $D_j u = \operatorname{div} X_j + f_j$ *for* $j \in \{1, \ldots, n\}$ *and*

(f) $0 < \lambda < 1$ *and* $0 < \varepsilon < \infty$.

There is $\delta_1 = \delta_1(M, \lambda, \varepsilon)$ *such that if* $\delta \leqslant \delta_1$ *then there is* c *with* $0 \leqslant c < \infty$ *such that*

(g) $r^{-n}|u(\varphi) - c\int_{\mathbf{U}(a,r)} \varphi\, d\mathscr{L}^n| \leqslant \varepsilon \sup\{|\varphi(x)| : x \in \mathbf{U}(a, r)\}$ *whenever* $\varphi \in \mathscr{D}(\mathbf{U}(a, r))$ *and* $\operatorname{spt} \varphi \subset \mathbf{U}(a, \lambda r)$.

PROOF. We may assume $a = 0$ and $r = 1$. Regularizing if necessary by non-negative approximations to the identity with small supports, we see that we may assume u, X_j and f_j, $j \in \{1, \ldots, n\}$, are smooth.

Were the lemma false, there would be M, λ and ε such that $0 \leqslant M < \infty$, $0 < \lambda < 1$ and $0 < \varepsilon < \infty$; a sequence of positive real numbers $\mu_1, \mu_2, \mu_3, \ldots$ with limit 0; and sequences

$$u_\nu \in \mathscr{E}(\mathbf{U}(0,1)), \quad X_{\nu,j} \in \mathscr{E}(\mathbf{U}(0,1)) \quad \text{and} \quad f_{\nu,j} \in \mathscr{E}(\mathbf{U}(0,1)),$$

$\nu = 1, 2, 3, \ldots$, $j \in \{1, \ldots, n\}$, such that

$$u_\nu \geqslant 0 \quad \text{and} \quad \int_{\mathbf{U}(0,1)} u_\nu \, d\mathscr{L}^n \leqslant M, \tag{5}$$

$$\int_{\mathbf{U}(0,1)} |X_{\nu,j}| + |f_{\nu,j}| d\mathscr{L}^n \leqslant \mu_\nu \tag{6}$$

and

$$D_j u_\nu = \operatorname{div} X_{\nu,j} + f_{\nu,j}, \tag{7}$$

whenever $\nu = 1, 2, 3, \ldots$ and $j \in \{1, \ldots, n\}$ but such that for each c with $0 \leqslant c < \infty$ and each $\nu = 1, 2, 3, \ldots$ we have

$$\int_{\mathbf{U}(0,\lambda)} |u_\nu - c| d\mathscr{L}^n > \varepsilon. \tag{8}$$

Let $\chi \in \mathscr{D}(\mathbf{U}(0,1))$ be such that $0 \leqslant \chi \leqslant 1$ and $\chi(x) = 1$ for $x \in \mathbf{U}(0, \lambda)$. For $\nu = 1, 2, 3, \ldots$ and $j \in \{1, \ldots, n\}$, let

$$v_\nu = \chi u_\nu, \qquad Y_{\nu,j} = \chi X_{\nu,j}$$

and

$$g_{\nu,j} = (D_j \chi) u_\nu - \operatorname{grad} \chi \cdot X_{\nu,j} + \chi f_{\nu,j}$$

on $\mathbf{U}(0,1)$ and let them be zero elsewhere. It follows from (7) that

$$D_j v_\nu = \operatorname{div} Y_{\nu,j} + g_{\nu,j}, \qquad j \in \{1, \ldots, n\},$$

so that

$$\Delta v_\nu = \sum_{j=1}^n D_j \operatorname{div} Y_{\nu,j} + \sum_{j=1}^n D_j g_{\nu,j} \tag{9}$$

for $\nu = 1, 2, 3, \ldots$. Setting

$$Z_\nu = \sum_{j=1}^n G D_j \operatorname{div} Y_{\nu,j} \quad \text{and} \quad h_\nu = \sum_{j=1}^n G D_j g_{\nu,j}$$

for $\nu = 1, 2, 3, \ldots$, we infer from (1) and (9) that

$$v_\nu = Z_\nu + h_\nu, \qquad \nu = 1, 2, 3, \ldots. \tag{10}$$

Passing to a subsequence if necessary, we infer from (5)–(7) and the constancy theorem for distributions that there is a c such that $0 \leqslant c < \infty$ and such that

$$\lim_{\nu \to \infty} \int_{\mathbf{U}(0,1)} u_\nu \varphi \, d\mathscr{L}^n = c \int_{\mathbf{U}(0,1)} \varphi \, d\mathscr{L}^n \quad \text{for each } \varphi \in \mathscr{D}(\mathbf{U}(0,1)). \tag{11}$$

It follows from (6) that

$$\lim_{\nu\to\infty}\int \varphi Z_\nu\, d\mathscr{L}^n = \lim_{\nu\to\infty} -\sum_{j=1}^{n}\int \operatorname{grad} D_j G\varphi \cdot Y_{\nu,j}\, d\mathscr{L}^n \tag{12}$$
$$= 0 \quad \text{whenever } \varphi \in \mathscr{D}(\mathbf{R}^n).$$

Using (5) and our observations about the operators Q_j, $j \in \{1,\ldots,n\}$, and passing again to a subsequence if necessary, we obtain an $\mathscr{L}^n$-summable real-valued function l on spt χ such that

$$\lim_{\nu\to\infty}\int_{\operatorname{spt}\chi} |h_\nu - l|\, d\mathscr{L}^n = 0. \tag{13}$$

It follows from (11)–(13) that

$$l(x) = c\chi(x) \quad \text{for } \mathscr{L}^n \text{ almost all } x \in \operatorname{spt}\chi. \tag{14}$$

Most importantly, we infer from (3) and (6) that

$$\mathscr{L}^n\{x \in \mathbf{R}^n : |Z_\nu(x)| \geqslant \alpha\} \leqslant \frac{Cn^2}{\alpha}\sum_{j=1}^{n}\int_{\mathbf{U}(0,1)} |X_{\nu,j}|\, d\mathscr{L}^n \tag{15}$$
$$\leqslant \frac{Cn^2}{\alpha} n\mu_\nu$$

for $\nu = 1, 2, 3, \ldots$ and $0 < \alpha < \infty$.

Now suppose $0 < \eta < \infty$. For each $\nu = 1, 2, 3, \ldots$ let $G_\nu = \{x \in \mathbf{R}^n : |Z_\nu(x)| < \eta\}$ and let $B_\nu = \{x \in \mathbf{R}^n : |Z_\nu(x)| \geqslant \eta\}$. From (15) we infer that

$$\lim_{\nu\to\infty} \mathscr{L}^n(B_\nu) = 0. \tag{16}$$

We have

$$\int_{B_\nu} v_\nu\, d\mathscr{L}^n = \int \chi(u_\nu - c)\, d\mathscr{L}^n - \int_{G_\nu} \chi(u_\nu - c)\, d\mathscr{L}^n + c\int_{B_\nu} \chi\, d\mathscr{L}^n.$$

Using (11), (16) and the fact that $v_\nu \geqslant 0$ we infer that

$$\limsup_{\nu\to\infty}\int_{B_\nu} |v_\nu| \leqslant \eta\mathscr{L}^n(\operatorname{spt}\chi). \tag{17}$$

Finally, we estimate

$$\int_{\mathbf{U}(0,1)} |v_\nu - c\chi|\, d\mathscr{L}^n \leqslant \int_{G_\nu} |Z_\nu| + |h_\nu - c\chi|\, d\mathscr{L}^n + \int_{B_\nu} |v_\nu| + c|\chi|\, d\mathscr{L}^n$$

so that, by virtue of (13), (14) and (16),

$$\limsup_{\nu\to\infty}\int_{\mathbf{U}(0,1)} |v_\nu - c\chi|\, d\mathscr{L}^n \leqslant 2\eta\mathscr{L}^n(\operatorname{spt}\chi).$$

Taking η so that $2\eta\mathscr{L}^n(\operatorname{spt}\chi) < \varepsilon$ we contradict (8). □