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VOLUME I

PROGRESS IN THE THEORY OF FINITE ELEMENT APPROXIMATIONS OF PROBLEMS IN NONLINEAR ELASTICITY

J.T. Oden[(I)] and C.T. Reddy[(II)]

Summary

We summarize the state of the art in existence theory in nonlinear elastostatics and cite new results which have direct bearing on the construction of an approximation theory.

1. General Discussion

In this paper, we give a brief report on the current status of the mathematical theory of finite element approximations of the nonlinear equations of finite elastostatics. We also outline some of our recent results in this area.

Some significant barriers have stood in the way of the development of such a theory. Perhaps the most significant barrier has been the absence of an existence theory for elasticity problems, since the availability of such a theory is usually an intrinsic part of a viable theory of approximation. While the origins of the theory of elasticity date back over 300 years, some progress toward the development of an existence theory for one-, two-, and three-dimensional problems has been made only recently--a general theory still seems to be well outside the existing theory of nonlinear partial differential equations. The picture is further complicated by lack of agreement on reasonable constraints on the forms of the constitutive equations so as to produce problems with physically reasonable solutions. Truesdell's "main unsolved problem in the theory of finite elastic strain" [1] is, thus, still unsolved. A comprehensive survey of various proposals for "solutions" of the "main problem" can be found in Wang and Truesdell [2].

One hypothesis that has received much attention is the strong ellipticity or Legendre-Hadamard condition which asserts that

$$\frac{\partial^2 \sigma}{\partial w^i_{,\alpha}\, \partial w^j_{,\beta}} \lambda^i \lambda^j \mu_\alpha \mu_\beta \geq 0 \qquad \forall\ \underset{\sim}{\lambda}, \underset{\sim}{\mu} \in \mathbb{R}^3 \tag{1}$$

where σ is the strain energy function and $w^i_{,\alpha}$ are components of the displacement gradient tensor.

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Antman (e.g., [3-5]) has used this assumption to develop existence theories for elastic rods. However, the one-dimensional version of (1) is equivalent to the condition that σ be a convex function of deformation measures. Ordinarily, this would be physically unacceptable for arbitrary deformations because it would lead to strongly monotone operators, uniqueness of solutions, and, therefore, be incompatible with the theory of elastic stability. It is a peculiarity of the one dimensional theories of rods, however, that the governing equations of equilibrium contain lower-order terms. For instance, various kinematical conditions (e.g., the Kirchhoff hypotheses) lead to equations which are monotone in the highest derivatives but which always contain terms involving derivatives of lower order. These lower order terms (not necessarily present in two- or three-dimensional theories) make nonunique solutions possible. Hence, it is doubtful that techniques which have proved to be successful for one-dimensional problems can be extended to problems of higher dimension.

Of course, there are two- and three-dimensional problems in elasticity for which the operators are monotone on a restricted class of deformations. If "strong stability" is assumed, one can easily construct an existence theory, and this tact has been explored by Beju [6] and Oden [7]. In the latter paper, this rather severe restriction was also introduced to make tractable the analysis of finite element approximations, and error estimates were obaained for problems of this type. Unfortunately, such studies are of limited interest in elasticity theory as they, again, pertain to problems exhibiting unique solutions for each choice of data (excluding possibly rigid motions). The most interesting features of nonlinear elastostatics concern nonuniqueness of solutions, bifurcations, and other aspects of stability theory. A satisfactory theory should include these features.

A fairly complete theory of finite element approximations of two classes of one-dimensional nonlinear elasticity problems has been given by Oden and Reddy [8] and Oden and Nicolau [9,10]. In [8], monotone operator theory is used to study cases in which the energy is dominated by polynomial terms in the principal invariants of the deformation tensor. Error estimates in the $W^{1,p}$ -Sobolev and L^p -norms are given. In [9] and [10], a nonlinear system of equations corresponding to a vector-valued displacement defined on a one-dimensional domain is analyzed. The operators are not monotone, solutions are nonunique, and strong ellipticity is assumed. The necessary lower-order terms are introduced through a (somewhat contrived) gravitational potential energy. However, a complete existence theory is derived together with error estimates and convergence criteria.

A study of convergence behavior of one-dimensional nonlinear elasticity problems has also been contributed by Wellford [11]. His numerical experiments indicate that optimal rates of convergence can be obtained when the solutions are sufficiently smooth (a fact also confirmed in [8]) and that optimal rates on solution paths after bifurcations are also obtainable (also confirmed in [9,10]). Wellford's experiments also point to a signficiant decrease in convergence rates near bifurcation points, and this phenomena has not been explained theoretically.

When the nonlinear boundary-value problem under investigation falls within the framework of Sobolev spaces, the usual error analysis techniques based on monotone operator theory (e.g., Browder [12]) or the elliptic theory of Višik [13] or Lions [14], leads to estimates of the form

$$\|u - u_h\|_{W^{1,p}(\Omega)} = O(h^{k/(p-1)}) \tag{2}$$

where u is the solution, u_h its finite element approximation, $\|\cdot\|_{W^{1,p}(\Omega)}$ the usual Sobolev norm, $2 \leq p < \infty$, h the mesh parameter, and k the degree of the piecewise polynomial used in constructing u_h. Results such as this have been obtained by Glowinski and Marroco [15] for the operator $-\nabla(|\nabla u|^{p-2} \nabla u)$, by Babuška [16] for the second-order strongly elliptic, quasi-linear operators of Višik [13], and, for one-dimensional nonlinear elasticity problems by Wellford [11,17] and Oden and Reddy [8]. As noted earlier, these results are in direct conflict with numerical experiments whenever the solution is smooth. Thus, for general nonlinear elliptic problems, the problem of obtaining optimal error estimates is still open.

A measure of progress has been obtained in this area for a restricted class of problems. In [8], Oden and Reddy obtained a slight improvement in a $W^{1,p}$ -bound of $O(h^{1/2+1/p})$ for $k = 1$, which is obviously still not optimal. More recently, Reddy and Oden [18] obtained optimal error estimates for piecewise linear approximations of a class of one-dimensional problems involving monotone operators. Their analysis made heavy appeal to the assumed smoothness of the solution and of the monotonicity of the operators involved. Unfortunately, their analysis is not applicable to most elasticity problems since monotonicity generally does not exist. Very recently, Fix [19] has reportedly resolved the question of optimal error estimates for Dirichlet's problem for the operator $-\nabla(|\nabla u|^{p-2} \nabla u)$.

In the work of Antman (e.g., [3-5]), the role of the local invertibility condition in the construction of physically reasonable boundary-value problems in continuum mechanics has been emphasized. If $\underset{\sim}{F}(\underset{\sim}{X})$ is the deformation gradient at particle $\underset{\sim}{X}$, then the invertibility condition demands that the motion be locally invertible and orientation preserving at $\underset{\sim}{X}$, and this is guaranteed if and only if

$$\det \underset{\sim}{F}(\underset{\sim}{X}) > 0 \tag{3}$$

This condition imposes a constraint on the classes of admissible solutions of problems in nonlinear elasticity and leads to serious mathematical difficulties.

In addition, if the constitutive equation for the material (for instance the form of the strain energy σ) is to exhibit physically reasonable response to large compressive deformations, Antman points out that σ should exhibit the singular behavior,

$$\sigma \to \infty \quad \text{as} \quad \det \underset{\sim}{F} \to 0 \tag{4}$$

This condition, of course, constrains the domain of the operators of elastostatics. Moreover, the set of admissible motions K where

$$K = \{\underset{\sim}{x}: \underset{\sim}{F} = \nabla \underset{\sim}{x} ; \det \underset{\sim}{F} \geq 0 \tag{5}$$

is not convex, an unfortunate fact which lifts the elastostatics problem, subject to the constraint (3), outside the realm of modern developments in the theory of variational inequalities.

In very recent times, some progress has been made toward the development of a general existence theory for finite elasticity. Two entirely different approaches have been proposed, each having some attractive features and each having some undesirable features which include serious questions yet to be satisfactorily resolved. An elaborate existence theory for hyperelastic materials has recently been proposed by Ball [20]. Ball's theory is based on the idea of polyconvex functions, and he addresses the question of existence by showing that motions exist which make the total potential energy of a hyperelastic body assume a stationary value. Ball's theory extends that of Morrey [21] on quasiconvex functions. While Ball's theory is restricted to hyperelastic materials, it does accomodate one-, two-, and three-dimensional problems, a wide variety of boundary conditions, unilateral constraints such as $\det \nabla \underset{\sim}{x} > 0$ (local invertibility) or $\det \nabla \underset{\sim}{x} = 1$ (incompressibility) and singular behavior of the type imposed by the natural condition, $\sigma \to \infty$ as $\det \underset{\sim}{F} \to 0$, $\underset{\sim}{F}$ being the deformation gradient tensor.

Unfortunately, Ball is unable to show that any vector which renders the total energy stationary is even a weak solution of the equilibrium equations of elasticity. This critical step is missing in his theory because of difficulties in determining the differentiability of certain functionals on convex (or quasi convex) sets. In addition, the (constitutive) assumption of quasiconvexity of a twice differentiable strain energy function σ in turn implies that σ satisfies the Legendre-Hadamard condition (1). In other words, Ball effectively assumes strong ellipticity as a constitutive requirement. Assumptions of this type for two- and three- dimensional theories have come under serious criticism lately. Indeed, Knowles and Sternberg [22,23] have recently shown that the equations of nonlinear elastostatics for hyperelastic materials may suffer a loss of strong ellipticity for solutions which exhibit "sufficiently severe local deformations." Also Ericksen [24] has shown that the violation of the strong ellipticity condition at a point may lead to physically reasonable instabilities.

An alternate existence theory for a class of nonlinear boundary-value problems has recently been developed by Oden [25]. In [25] a general existence theorem for a class of nonlinear operators on reflexive Banach spaces is developed which generalizes the theory of "operators of the type of the calculus of variations" developed by Lions [14]. These operators fall into the category of pseudomonotone operators introduced by Brezis [26], but they do not lead to operators which are necessarily monotone in the highest derivatives. Oden shows that a useful element in a theory of this type is the availability of a nonlinear Gårding-type inequality. While a necessary and sufficient condition for the existence of a Gårding-inequality for certain linear second-order elliptic operators with well-behaved coefficients defined on sufficiently smooth functions is that the operator be strongly elliptic (see, e.g., Agmon [27]), this is apparently not the case in nonlinear problems. The theory presented in [25], therefore, does not rely on strong ellipticity - or, at the very least, the role of strong ellipticity in implementation of this theory is, at present, very obscure.

In applications of the theory in [25] to specific problems in nonlinear elastostatics, difficulties in developing general Gårding-type inequalities have not been completely overcome. The accomodation of general boundary conditions and unilateral constraints also impose problems which are yet to be resolved.

2. Theoretical Developments

The following existence theorem generalizes the theory of monotone operators

Theorem 1. Let $\mathcal{U}$ and $\mathcal{V}$ be real reflexive Banach spaces, with $\mathcal{U}$ compactly imbedded in $\mathcal{V}$, and let A be bounded, hemicontinuous, coercive operator from $\mathcal{U}$ into the topological dual $\mathcal{U}'$ of $\mathcal{U}$. Moreover, for every pair of vectors u , v in $\mathcal{U}$ such that

$$u,v \in B_\mu(0) = \{w \in \mathcal{U} : \|w\|_{\mathcal{U}} < \mu\} \tag{6}$$

let there exist a continuous nonnegative function $H: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, with the property

$$\lim_{\theta\to 0} \frac{1}{\theta} H(x, \theta y) = 0 \quad \forall \quad x \in \mathbb{R}^+ \tag{7}$$

such that

$$\langle A(u) - A(v), u - v\rangle \geq -H(\mu, \|u - v\|_{\mathcal{V}}) \tag{8}$$

where $\langle\cdot,\cdot\rangle$ denotes duality pairing from $\mathcal{U}' \times \mathcal{U}$ into $\mathbb{R}$. Then for every $f \in \mathcal{U}$, there exists at least one $u \in \mathcal{U}$ such that

$$A(u) = f \tag{9}$$

For a proof of this and related theorems, see Oden [25].

In applications to elasticity problems, (8) may manifest itself in the form of a nonlinear Gårding inequality. For example, the Dirichlet problem in elastostatics of hyperelastic materials involves seeking a displacement field $\underset{\sim}{v}$ such that

$$\int_\Omega \left(\frac{\partial\sigma(u)}{\partial u^i_{,\alpha}} v_{i,\alpha} - \rho_o f^i v_i\right) dv = 0 \quad \forall \quad \underset{\sim}{v} \in \mathcal{U}(\Omega) \tag{10}$$

where Ω is an open subset off particles) in $\mathbb{R}^n$, n = 1,2,3 , and $\mathcal{U}(\Omega)$ is the space of admissible displacements, $\rho_o f^i$ the components of body force, etc. Then

$$\langle A(\underset{\sim}{u}), \underset{\sim}{v}\rangle \equiv \int_\Omega \frac{\partial\sigma(\underset{\sim}{u})}{\partial u^i_{,\alpha}} v_{i,\alpha}\, dv \tag{11}$$

For example, suppose σ is a polynomial on the principal invariants of the deformation tensor, which, when expressed in terms of $\nabla \underset{\sim}{w}$, is a polynomial of degree $p > 2$ in the components $w_{i,\alpha}$. Then we may usually encounter the Sobolev spaces

$$\left(W_o^{1,p}(\Omega)\right)^n = \mathcal{U}(\Omega) \quad ; \quad \left(L^p(\Omega)\right)^n = \mathcal{V}(\Omega) \tag{12}$$

To apply Theorem 1, we seek a Gårding-type inequality of the form

$$\left\langle A(\underset{\sim}{u}) - A(\underset{\sim}{v}),\ \underset{\sim}{u} - \underset{\sim}{v} \right\rangle \geq C_o \|\underset{\sim}{u} - \underset{\sim}{v}\|_{1,p}^{p} - \gamma(\mu)\, \|\underset{\sim}{u} - \underset{\sim}{v}\|_{o,p}^{p'}$$

$$\forall\ \underset{\sim}{u},\ \underset{\sim}{v} \in B_\mu(0) \subset \underset{\sim}{W}_o^{1,p}(\Omega) \tag{13}$$

where C_o, $\gamma(\mu)$ are positive constants, γ depending continuously on μ, $p' = p/(p-1)$, and

$$\|\underset{\sim}{u} - \underset{\sim}{v}\|_{1,p}^{p} = \int_\Omega \sum_{i,\alpha=1}^{n} \left|u_{i,\alpha} - v_{i,\alpha}\right|^p dv$$

$$\|\underset{\sim}{u} - \underset{\sim}{v}\|_{o,p}^{p} = \int_\Omega \left|(\underset{\sim}{u} - \underset{\sim}{v}) \cdot (\underset{\sim}{u} - \underset{\sim}{v})\right|^{p/2} dv \tag{14}$$

Clearly, for small enough $\gamma(\mu)$, the right hand side of (13) may be positive, in which case A is strictly monotone. As $\gamma(\mu)\, \|\underset{\sim}{u} - \underset{\sim}{v}\|_{o,p}^{p'}$ increases relative to $C_o \|\underset{\sim}{u} - \underset{\sim}{v}\|_{1,p}^{p}$, a point is reached where the right hand side passes from positive to negative. This corresponds to a point of primary bifurcation, and beyond it A is not monotone.

We have also developed inequalities of the type

$$\left\langle A(\underset{\sim}{w}) - A(\bar{\underset{\sim}{w}}),\ \underset{\sim}{z} \right\rangle \leq \|\underset{\sim}{z}\|_{1,p}\, \|\underset{\sim}{w} - \bar{\underset{\sim}{w}}\|_{1,p}\, G(\underset{\sim}{w}, \bar{\underset{\sim}{w}})$$

$$G(\underset{\sim}{w}, \bar{\underset{\sim}{w}}) = C_1\left(1 + \|\underset{\sim}{w}\|_{1,p}^{p-2} + \|\bar{\underset{\sim}{w}}\|_{1,p}^{p-2}\right) \tag{15}$$

Clearly, if A: $\underset{\sim}{W}_o^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is hemicontinuous, coercive i.e., if

$$\lim_{\|w\|_{1,p} \to \infty} \frac{\left\langle A(\underset{\sim}{w}),\ \underset{\sim}{w} \right\rangle}{\|\underset{\sim}{w}\|_{1,p}} = +\infty \tag{16}$$

and if A satisfies (13) and (15), then (10) has at least one solution.

3. Error Estimates

In approximating (10), we introduce the usual finite-element spaces of $S_h(\Omega)$ of piecewise polynomials of degree k. If h is the mesh parameter, then these spaces have the following well-known interpolation property: if $\underset{\sim}{w} \in \underset{\sim}{W}^{\ell,p}(\Omega)$, there exists a $\underset{\sim}{\tilde{w}}_h \in (S_h(\Omega))^n$ such that

$$\|\underset{\sim}{w} - \underset{\sim}{\tilde{w}}_h\|_{1,p} \leq C_3\, h^{\mu}\, \|w\|_{\ell,p} \tag{17}$$

$$\mu = \min(k, \ell-1)$$

Here

$$\|w\|^p_{\ell,p} = \int_\Omega \sum_{i=1}^{n} \sum_{\alpha\leq\ell} |D^\alpha w_i|^p \, d\upsilon \tag{18}$$

If $\underset{\sim}{w}$ is the solution of (10), its finite element approximation $\underset{\sim}{w}_h \in \underset{\sim}{S}_h(\Omega) = (\mathring{S}_h(\Omega))^3 \cap \underset{\sim}{W}_o^{1,p}(\Omega)$ satisfies

$$\langle A(\underset{\sim}{w}_h), \underset{\sim}{v}_h\rangle = \int_\Omega \rho_o f \cdot \underset{\sim}{v}_h \, d\upsilon \quad \forall\ \underset{\sim}{v}_h \in \mathring{\underset{\sim}{S}}_h(\Omega) \tag{19}$$

and the orthogonality condition

$$\langle A(\underset{\sim}{w}) - A(\underset{\sim}{w}_h), \underset{\sim}{v}_h\rangle = 0 \quad \forall\ \underset{\sim}{v}_h \in \mathring{\underset{\sim}{S}}_h(\Omega) \tag{20}$$

The error $\underset{\sim}{e}_h \equiv \underset{\sim}{w} - \underset{\sim}{w}_h$ satisfies the inequality

$$\begin{aligned}\|\underset{\sim}{e}_h\|_{1,p} &\leq \|\underset{\sim}{w} - \underset{\sim}{\tilde{w}}_h\|_{1,p} + \|\underset{\sim}{w}_h - \underset{\sim}{\tilde{w}}_h\|_{1,p} \\ &\leq C_3 h^{\mu}\, \|\underset{\sim}{w}\|_{\ell,p} + \|\underset{\sim}{w}_h - \underset{\sim}{\tilde{w}}_h\|_{1,p}\end{aligned} \tag{21}$$

Let $\underset{\sim}{E}_h \equiv \underset{\sim}{w}_h - \underset{\sim}{\tilde{w}}_h$. Then from (13),

$$\begin{aligned}\|\underset{\sim}{E}_h\|^p_{1,p} &\leq C_o^{-1} \langle A(\underset{\sim}{w}_h) - A(\underset{\sim}{\tilde{w}}_h), \underset{\sim}{w}_h - \underset{\sim}{\tilde{w}}_h\rangle + C_o^{-1}\, \gamma(\mu)\, \|\underset{\sim}{E}_h\|^{p'}_{o,p} \\ &= C_o^{-1} \langle A(\underset{\sim}{w}) - A(\underset{\sim}{\tilde{w}}_h), \underset{\sim}{w}_h - \underset{\sim}{\tilde{w}}_h\rangle + C_o^{-1}\, \gamma(\mu)\, \|\underset{\sim}{E}_h\|^{p'}_{o,p} \quad \text{(by (20))} \\ &\leq C_o^{-1} \|\underset{\sim}{E}_h\|_{1,p}\, \|\underset{\sim}{w} - \underset{\sim}{\tilde{w}}_h\|_{1,p}\, G(\underset{\sim}{w}, \underset{\sim}{\tilde{w}}_h) + C_o^{-1}\, \gamma(\mu)\, \|\underset{\sim}{E}_h\|^{p'}_{o,p} \\ &\leq C_o^{-1} \|\underset{\sim}{E}_h\|_{1,p}\, C_3\, h^{\mu}\, \|\underset{\sim}{w}\|_{\ell,p}\, G(\underset{\sim}{w}, \underset{\sim}{\tilde{w}}_h) + C_o^{-1}\, \gamma(\mu)\, \|\underset{\sim}{E}_h\|^{p'}_{o,p}\end{aligned} \tag{22}$$

Next we observe that as $h \to 0$

$$G(\underset{\sim}{w}, \underset{\sim h}{w}) = G_o(\underset{\sim}{w}) + O(h^\mu)$$
$$G_o(\underset{\sim}{w}) = C_1\left(1 + 2\,\|\underset{\sim}{w}\|_{1,p}^{p-2}\right) \tag{23}$$

Also, by the Poincaré inequality,

$$\|\underset{\sim h}{E}\|_{o,p} \le C_4\,\|\underset{\sim h}{E}\|_{1,p} \qquad C_4 > 0 \tag{24}$$

Hence, as $h \to 0$, (22), (23) and (24) combine to give

$$\|\underset{\sim h}{E}\|_{1,p}^{p-1} \le C_o^{-1}\, C_3\, h^\mu\, \|\underset{\sim}{w}\|_{\ell,p}\, G_o(\underset{\sim}{w}) + C_o^{-1}\,\gamma(\mu)\, C_4\, \|\underset{\sim h}{E}\|_{1,p}^{p'/p} \tag{25}$$

We reach, at this point, a problem not yet resolved. We must determine a number σ such that

$$C_5 x^\sigma \le x^{p-1} - C_o^{-1}\,\gamma(\mu)\, C_4 x^{p'/p} \quad , \quad x \ge 0 \tag{26}$$

where C_5 is a constant greater than 0. If σ is known, we have

$$\|\underset{\sim h}{E}\|_{1,p} \le C_6\, h^{\mu/\sigma}\, H(\underset{\sim}{w}) \tag{27}$$

where

$$C_6 = (C_5 C_o)^{-1/\sigma} \quad , \quad H(\underset{\sim}{w}) = \left(\|\underset{\sim}{w}\|_{\ell,p}\, G_o(\underset{\sim}{w})\right)^{1/\sigma} \tag{28}$$

The final error estimate is then obtained from (27) and (21):

$$\|\underset{\sim h}{e}\|_{1,p} \le C_7\left(h^\mu\, \|\underset{\sim}{w}\|_{\ell,p} + h^{\mu/\sigma}\, H(\underset{\sim}{w})\right) \tag{29}$$

This estimated rate of convergence is generally not optimal.

4. Closing Comments

Much additional work remains to be done before meaningful results can be obtained in the theory of finite elements in nonlinear elasticity. While some recent advances have been made in establishing a constructive existence theory for nonlinear elastostatics, several significant open questions stand in the way of developments of approximation theories.

Inequalities of the type (25) must be "solved" in the sense of (26) and techniques for obtaining optimal $W^{1,p}$- error estimates, when possible, must be developed. The development of techniques for constructing $\underset{\sim}{L}^p$- and $\underset{\sim}{L}^\infty$ estimates seems to be a remote prospect for the near future. Techniques for the assessment of bifurcation effects and regularity of solutions are still unknown, as are the effects of quadrature errors, singularities, boundary errors and the use of approximate data and material coefficients. All of these subjects provide a formidable challenge for researchers in computational mechanics and numerical analysis for many years to come.

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