Introductory Mathematical Analysis

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It is unworthy of a mathematician to see with
other people's eyes and to accept as true or
as proven that for which he himself has no proof.

Maestlin (Teacher of Johannes Kepler)

Preface

I have written this book in the belief that an introductory course of mathematical analysis should include some abstract structural mathematics and some discussion of numerical methods, as well as the more traditional classical analysis.

Roughly speaking, analysis is the study of limiting processes. This study involves, in particular, the convergence of infinite series, the continuity, differentiability and integrability of functions, and later the theory of function spaces and functional analysis. The applications of analysis in the physical and engineering sciences are legion, and the history of the subject spans over 2000 years, adorned with some of the greatest names in mathematics; to name but a few; Archimedes, Newton, Leibniz, Euler, Cauchy, Abel, Weierstrass, Cantor, Dedekind, Riesz, Hilbert and Banach.

Many courses of elementary analysis, which are traditionally given to first year students in universities, cover only a selection of the simpler aspects of pre-20th century classical analysis as it was handed down to us by Cauchy and Weierstrass. It seems to be an unfortunate fact that this purely 'classical' approach, often presented in an unnecessarily terse style, leads many students to regard analysis as calculus with tears. To help to avoid this situation I have tried to include, in addition to some basic classical analysis, enough numerical calculations, of a simple type, to enlighten and enliven the general theory. Of course the existence of mechanical calculating devices and computers has taken much of the labour out of numerical analysis, but there is a great danger for beginners in thinking that computers can solve any numerical problem. It is necessary to show, wherever possible, that the problem has a solution (this is where classical analysis is useful). Then one usually needs some procedure which gives a convergent sequence of approximations to the solution (this is where numerical analysis is required).

With regard to abstract structural mathematics, which is essentially a 20th century phenomenon, I have attempted to indicate something of the way in which its unifying spirit is valuable in analysis. The time has surely arrived for the introduction, even into elementary analysis, of such basic notions as groups, linear spaces, completeness, open sets and homeomorphisms.

Having met these ideas, the student will then be prepared for more advanced courses on functional analysis and topology with little or nothing to unlearn in the way of terminology.

The book is primarily addressed to students in universities and polytechnics. They will probably be in their first or second year and studying mathematics or the physical or engineering sciences. Large parts of the book could, however, be read by more advanced students in schools who have followed simple courses in calculus.

Every attempt has been made to give detailed proofs of all the results in the text. Many illustrative examples are incorporated, as well as a large number of exercises. Hints for the solution of many of the exercises are to be found at the end of the book. It is hoped that these will help the student who has to work without much supervision.

In order to make the book more useful I have given a fairly detailed treatment of the elementary functions and such topics as the fixed point principle, Newton's approximation for the roots of equations and Stirling's theorem on the asymptotic form of $n!$

A glance at the contents of the book will confirm that it has no claim to be encyclopaedic. Moreover, no course on a subject as vast as analysis can be an end in itself. When this work is put aside it is hoped only that the young analyst will have a firm base from which to explore the many aspects of what is one of the most fascinating creations of the human mind.

It is with pleasure that I thank Dr. P. L. Walker, a former colleague at the University of Lancaster. At very short notice he read a substantial part of the manuscript of the book, and has made a number of valuable comments and suggestions for improvement.

I. J. Maddox
The Queen's University of Belfast, 1975

Contents

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Logic, Sets, Algebraic Structures

1. Logic

Logical argument is as important in analysis as it is in any other branch of mathematics. However, we have neither the space nor the inclination to present a detailed formal account of all the logical ideas that we might need to use in this book. What we shall try to do is to point out some of the arguments commonly employed in the proofs of theorems in analysis, and to illustrate them with simple examples.

In order to give reasonably interesting examples we assume the reader has some rudimentary knowledge of the sets listed below. These sets pervade the whole of mathematics, and we shall be discussing them in greater detail in later chapters. The notations N, Z, Q, R, C will be reserved exclusively for the following sets:

$N=\{1, 2, 3, 4, \ldots\}$, the set of all natural numbers,
$Z=\{0, 1, -1, 2, -2, \ldots\}$, the set of all integers,
$Q=\{p/q \mid p \in Z \text{ and } q \in N\}$, the set of all rational numbers,
R, the set of all real numbers,
C, the set of all complex numbers.

With regard to Q, the vertical bar following p/q is read as 'such that' and $\in$ is read as 'belongs to' or 'is an element of'. N is also called the set of all positive integers. *Zahlen* is a German word for integers, whence the notation Z. Rational numbers are also called quotients, whence Q; R and C are self-explanatory. Most mathematicians now use these symbols, and those who do not ought to—but let us be charitable. Gauss himself said that Mathematics is about notions not notations.

One of the basic ideas of logic is that of a proposition. We define a **proposition** to be an assertion which is capable of being classified as true or false, but not both.

Here for example are two propositions:

It is raining (1)

The street is wet. (2)

From two such simple propositions we may form other propositions by inserting words such as 'and', 'not', 'or' etc. For example,

It is raining and the street is wet.

In general we shall denote propositions by p, q, r. Given p, q, the most important propositions that may be formed from them are:

NEGATION	not p, denoted by $\sim p$
CONJUNCTION	p and q, denoted by $p \wedge q$
DISJUNCTION	p or q, denoted by $p \vee q$
IMPLICATION	p implies q, denoted by $p \Rightarrow q$
EQUIVALENCE	p if and only if q, denoted by $p \Leftrightarrow q$.

By definition, $p \Leftrightarrow q$ means that $(p \Rightarrow q) \wedge (q \Rightarrow p)$, i.e. p if and only if q means that p implies q and q implies p. Two other ways of saying $p \Leftrightarrow q$ are:
(i) p is equivalent to q
(ii) p is a necessary and sufficient condition for q.
Other common ways of saying 'p implies q' are:
(iii) If p then q
(iv) p only if q
(v) p is a sufficient condition for q
(vi) q is a necessary condition for p.

Example 1. Let p denote (1) above, and q denote (2). Then $\sim q$; $p \wedge q$; $p \Leftrightarrow q$ are respectively: The street is not wet; It is raining and the street is wet; It is raining if and only if the street is wet.

It is stressed that we are saying nothing as to the truth or falsity of the propositions in Example 1; we are just illustrating the notations.

The reader will be glad to know (we hope) that mathematical analysis is not really much concerned with the aqueous state of the thoroughfare. The only reason for considering propositions like those of Example 1 is that they are perhaps the simplest which make clear the basic ideas. Moreover they make no mathematical demands—these come later.

By our definition of a proposition p it must be possible to classify it by exactly one of the words true or false. Hence we denote its **truth value** by T or F.

The following truth table is to be taken as the *definition* of the truth values of the indicated propositions:

p	q	$p \wedge q$	$p \vee q$	$\sim p$	$p \Rightarrow q$
T	T	T	T	F	T
T	F	F	T	F	F
F	T	F	T	T	T
F	F	F	F	T	T

(3)

Example 2. Using (3), the truth table for $(\sim p) \vee q$ is

p	$\sim p$	q	$(\sim p) \vee q$
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T

(4)

Note that the truth tables of $(\sim p) \vee q$ and $p \Rightarrow q$ are the same, in the sense that the last columns are the same. In such a case we say that the propositions $(\sim p) \vee q$ and $p \Rightarrow q$ are logically equivalent.

It is of course possible to motivate the assignment of truth values in (3), but the motivation for implication is never very convincing and the table is best taken as a definition which is almost universally accepted.

As a point of interest, we remark that a proposition which has only T for its truth value is called a **tautology**. A proposition which has only F is called a **contradiction**.

Example 3. By (3) the table for $p \vee (\sim p)$ is:

p	$\sim p$	$p \vee (\sim p)$
T	F	T
F	T	T

whence $p \vee (\sim p)$ is a tautology. Similarly we see that $p \wedge (\sim p)$ is a contradiction.

It is less trivial, though still quite easy, to construct the truth table to show that

$$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r) \tag{5}$$

is a tautology. The formal name for (5) is the **law of syllogism**, and it is used all the time in mathematics. It is so obvious if put in words that most people are not aware that they are using it.

Methods of proof

There are three main methods of proof which one finds in analysis:

P1 Direct proof
P2 Contrapositive proof
P3 Proof by contradiction, or *reductio ad absurdum.*

By **P1** we mean that, to prove $p \Rightarrow q$ we start with p, make various deductions depending on our knowledge of the situation, and finish with q.

Example 4. Let us give a direct proof that (for natural numbers n), n even implies n^2 even:

$$\begin{aligned} n \text{ even} &\Rightarrow n = 2a, \text{ for some natural number } a, \\ &\Rightarrow n^2 = 4a^2 = 2(2a^2) \\ &\Rightarrow n^2 \text{ even.} \end{aligned}$$

Nothing could be simpler than this, and in general direct proofs are the most natural and satisfying. Unfortunately, many of the most important theorems of analysis cannot be proved directly, and we have to resort to the indirect methods of **P2** and **P3**. We now make the definitions:

Contrapositive proof. *A contrapositive proof is one in which, instead of proving $p \Rightarrow q$ (which is what we really want) we actually prove that $\sim q \Rightarrow \sim p$. We call the proposition $\sim q \Rightarrow \sim p$ the contrapositive of the proposition $p \Rightarrow q$.*

Proof by contradiction. *In such a proof, instead of proving $p \Rightarrow q$ (which is what we really want) we actually prove that*:

$$p \wedge (\sim q) \Rightarrow \textit{any false proposition } r.$$

Both of these proof methods will probably look rather peculiar to the beginner, but nearly all books on analysis use them extensively, and indeed usually without saying so.

The next theorem justifies the use of **P2** and **P3** in that it shows that they are logically equivalent to the direct proof $p \Rightarrow q$.

Theorem 1. (*i*) *$\sim q \Rightarrow \sim p$ is logically equivalent to $p \Rightarrow q$.*
(*ii*) *Let r be any false proposition. Then $p \wedge (\sim q) \Rightarrow r$ is logically equivalent to $p \Rightarrow q$.*

Proof. We have to show that the truth tables for the propositions $\sim q \Rightarrow \sim p$, and $p \wedge (\sim q) \Rightarrow r$ are the same as the table for $p \Rightarrow q$. Now by the definition (3) we find that

p	q	$\sim q$	$\sim p$	$\sim q \Rightarrow \sim p$	$p \wedge (\sim q)$	r	$p \wedge (\sim q) \Rightarrow r$
T	T	F	F	T	F	F	T
T	F	T	F	F	T	F	F
F	T	F	T	T	F	F	T
F	F	T	T	T	F	F	T

Columns five and eight of this table are identical with the last column of (3), whence the theorem is proved.

Example 5. For natural numbers n let us use the contrapositive method to prove that n^2 even implies n even (say $p \Rightarrow q$). Note that we proved $q \Rightarrow p$ in Example 4. Of course $q \Rightarrow p$ is quite a different proposition from $p \Rightarrow q$, and the student is strongly warned never to say that: if $q \Rightarrow p$ then $p \Rightarrow q$.

Now $\sim q$ means that n is not even, i.e. n is odd, i.e. $n = 2a - 1$ for some natural number a. Hence $n^2 = 2a(2a-2)+1$, so that n^2 is odd, whence n^2 is not even, which is $\sim p$. Thus we have proved that $\sim q \Rightarrow \sim p$, so by Theorem 1(i) we see that $p \Rightarrow q$.

Example 6. In the important Theorem 7 on p. 147, we use the proof by contradiction. In that theorem we have to prove that, on a certain set, continuity (p say) implies boundedness (q say). It is not relevant at this stage that you know the meaning of the words continuity and boundedness since we are concerned only with the structure of the proof.

In fact in Theorem 7 we prove that $p \wedge (\sim q) \Rightarrow r$, where r is the false proposition $1 \leqslant 0$. Informally the argument goes as follows: if we have continuity and not boundedness then we are able to deduce that $1 \leqslant 0$, and this contradiction shows that continuity implies boundedness.

Universal and existential quantifiers

These are rather fearsome logical names for two simple and useful things. The symbol $\forall$ is read as 'for all' and is called the universal quantifier. The symbol $\exists$ is read as 'there exists' and is called the existential quantifier. Apart from this section we shall not usually employ these symbols, since we prefer to use the relevant phrases. However, the student is likely to meet the symbols in works on logic and in informal teaching in mathematics.

Example 7. We may assert as true that (i) $x^2 \geqslant 0$ for all real numbers x, and (ii) there exists a real number y such that $y^4 = 16$. Symbolically: (i) $\forall x \in R$, $x^2 \geqslant 0$, and (ii) $\exists y \in R | y^4 = 16$.

In (i) we are expressing a general theorem about all real numbers, and in (ii) we may note that $y = 2$ satisfies the condition, as does $y = -2$. When we say 'there exists' we mean, more precisely, that 'there exists at least one'.

The proposition (iii) $\forall x \in R$, $x^2 > 0$ is false, since $x = 0$ is such that x^2 is not greater than 0.

It is of crucial importance that the student realizes the difference between the quantifiers 'for all' and 'there exists' and never casually interchanges them in definitions or proofs.

Sometimes it is necessary to negate a proposition involving $\forall$ and $\exists$. As a general rule, in negating such a proposition we change $\forall$ into $\exists$ and change $\exists$ into $\forall$, and finally negate any proposition following the quantifiers.

For example, the negation of $\exists a \exists b \forall c$ such that $p(a, b, c)$ is $\forall a \forall b \exists c$ such that $\sim p(a, b, c)$.

Example 8. The following is the definition that a sequence (x_n) converges to zero (see Chapter 2):

$$\forall \varepsilon > 0, \exists n_0 \quad \text{such that } |x_n| < \varepsilon, \forall n > n_0. \tag{6}$$

In this definition the Greek letter ε (epsilon) denotes a positive real number and must not be confused with $\in$ which means 'belongs to'.

The negation of (6), i.e. (x_n) does not converge to zero, is:

$$\exists \varepsilon > 0, \forall n_0, \exists n > n_0 \quad \text{such that } |x_n| \geqslant \varepsilon. \tag{7}$$

In words (7) says that: there exists a positive number ε such that for all n_0, there exists $n > n_0$ such that $|x_n| \geqslant \varepsilon$.

Exercises 1.1

(Hints for some of these exercises are on p. 302)

1. Write down the truth table for $p \Leftrightarrow q$.
2. Classify as tautology, contradiction, or neither: (i) $p \Rightarrow (p \vee q)$, (ii) $p \Rightarrow \sim p$, (iii) $[p \wedge (p \Rightarrow q)] \Rightarrow q$, (iv) $(\sim p \Rightarrow p) \wedge (p \Rightarrow \sim p)$.
3. Use a truth table to show that the law of syllogism is a tautology.
4. Use truth tables to show the logical equivalence (E) of each of:
   (i) $\sim(p \wedge q)$ E $(\sim p) \vee (\sim q)$
   (ii) $\sim(p \vee q)$ E $(\sim p) \wedge (\sim q)$
   (iii) $\sim(p \Leftrightarrow q)$ E $p \Leftrightarrow \sim q$
   (iv) $p \wedge (q \vee r)$ E $(p \wedge q) \vee (p \wedge r)$.
5. If Stoke City avoid injuries then they will win the championship. They avoid injuries or the referee is biased. If the referee is biased then the crowd is unhappy. But the crowd is happy.
   Given the truth of these propositions, will Stoke win the championship?
6. Let $x \in R$; p denote $x = 1$; q denote $x^2 = 1$. Classify as true or false: (a) $p \Rightarrow q$, (b) $q \Rightarrow p$.
7. Write each of the following propositions as $p \Rightarrow q$, or as $p \Leftrightarrow q$. In each case classify as true or false:
   (i) $3x^2 + 4 = 7$ if $x = 1$, where $x \in R$.
   (ii) A necessary and sufficient condition for $3x^2 + 4 = 7$ is $x = 1$, where $x \in R$.

(iii) An integer greater than 2 is prime only if it is odd.
(iv) A sufficient condition that an integer is even is that it is a multiple of 4.
(v) $z \in C$ and $z = 1$, if and only if, $z \in C$ and $z^3 = 1$.

8. For $n \in N$, prove that n^3 even implies n even.
9. Negate the following: If the lecturer is lazy then some students do not finish their tutorial work.

2. Sets and functions

Simple set theory, and the concept of a function (also called a mapping or a map) now appear in even the most elementary school textbooks of mathematics. Consequently we give only a brief account (but adequate for the purposes of basic analysis) which explains our notation and presents definitions and some useful results.

A **set** is any collection of definite, distinguishable objects of our thought, to be conceived as a whole. So said the great German mathematician G. Cantor (1845–1918), the creator of set theory. His definition will do for us, though it is regarded as naive by mathematical logicians.

The objects of his definition are called the elements of the set, or the members of the set. Usually sets are denoted by capital letters and elements by lower case letters. If X is a set then $x \in X$ means that x is an element of X, and we also say that x belongs to X. If an object y is not an element of a set X, then we write $y \notin X$, and also say that y does not belong to X.

Example 9. If X is the set whose elements are the alphabetical letters a, b and c then we write $X = \{a, b, c\}$. It is usual to employ curly brackets to enclose the elements of a set. To illustrate our notation we may say that $b \in X$, but $d \notin X$, and $2 \notin X$.

Example 10. A set may have 'mixed elements', e.g., $X = \{a, b, 3, \text{my cat}\}$, but such sets do not occur much in analysis.

Example 11. It is incorrect to say that $\{3, 1, 6, 1, 5\}$ is a set, since by Cantor's definition the elements have to be *distinguishable*. Thus elements of a set must appear once and only once in the set.

Example 12. A set remains the same if the elements are written in a different order, e.g. $\{2, 4, 3, 1\}$ is the same set as $\{1, 2, 3, 4\}$. Of course it is usually more satisfying if the elements can be arranged in some 'natural' or systematic way.

In general it is impossible to write down all the elements of a set, e.g., we cannot exhibit all the elements of the set of natural numbers N, and this is one of the simplest sets in mathematics. In this case we write $N = \{1, 2, 3, \ldots\}$, using curly brackets to enclose the elements and three dots ... to imply that the law of formation of other elements is 'known'. More will be said about N in Chapter 2.

Notation such as $\{x \in N | x > 2\}$ is read as 'the set of all x belonging to N such that $x > 2$'. The vertical bar following N is read as 'such that'. Another way of writing $\{x \in N | x > 2\}$ is $\{3, 4, 5, \ldots\}$.

As another example, we have $\{x \in R | x^2 = 1\} = \{1, -1\}$.

We now define the basic notions of subsets, unions, intersections, and complements for sets, together with related ideas.

Subset. *If A and B are sets, then A is defined to be a subset of B if and only if every element of A is also an element of B. If A is a subset of B then we write $A \subset B$, or equivalently $B \supset A$. Also, if $A \subset B$ we say that A is included in B, that B includes A, or that B is a superset of A.*

Equal sets. *We define $A = B$ if and only if $A \subset B$ and $B \subset A$, which means that A and B have the same elements.*

Proper subset. *We define A to be a proper subset of B, and we write $A \subset B$ strictly, if and only if $A \subset B$ but $A \neq B$.*

The empty set. *If A is a set, then $\varnothing = \{x \in A \mid x \neq x\}$ is a subset of A, and we call $\varnothing$ the empty set. The set $\varnothing$ has no elements.*

There is a very important property of the empty set $\varnothing$, namely that it is the *unique* set which is a subset of *every* set.

To see this, take any set B. Suppose, if possible, that $\varnothing$ was not a subset of B. Then there exists $x \in \varnothing \backslash B$, contrary to the fact that $\varnothing$ has no elements. Hence $\varnothing \subset B$ for every set B. By $x \in \varnothing \backslash B$ we mean $x \in \varnothing$ but $x \notin B$.

Now we show that $\varnothing$ is unique. Suppose that ψ is also a set which is a subset of every set. Then $\psi \subset \varnothing$. But we know already that $\varnothing \subset \psi$, so by the definition of equality of sets we must have $\psi = \varnothing$, whence $\varnothing$ is unique.

Example 13. Two proper subsets of N are $\{1, 3, 8\}$, and $\{1, 3, 5, 7, \ldots\}$ the set of odd numbers. Hence we may write $\{1, 3, 8\} \subset N$ strictly.

Union of sets. *The union of sets A and B is:*

$$A \cup B = \{x | x \text{ belongs to at least one of } A \text{ and } B\}.$$

If S is a class of sets A then we define

$$\bigcup\{A | A \in S\} = \{x | x \in A \text{ for at least one } A \in S\}.$$

Intersection of sets. *The intersection of sets A and B is:*

$$A \cap B = \{x | x \in A \quad \text{and} \quad x \in B\}.$$

If S is a class of sets A then we define

$$\bigcap\{A | A \in S\} = \{x | x \in A \text{ for all } A \in S\}.$$

Disjoint sets. *Sets A and B are called disjoint if and only if $A \cap B = \varnothing$, i.e. A and B have no elements in common.*

Example 14. $\{a, b, c\} \cup \{a, c, d\} = \{a, b, c, d\}$; $\{b, c\} \cap \{a, b, c\} = \{b, c\}$; $\{a, b\} \cap \{c, d\} = \varnothing$.