G. M. Henkin

J. Leiterer

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von G. M. Henkin und J. Leiterer

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Dr. sc. Gennadi Markovič Henkin
Zentrales Ökonomisch-Mathematisches Institut
der Akademie der Wissenschaften der UdSSR, Moskau

Dr. sc. Jürgen Leiterer
Institut für Mathematik
der Akademie der Wissenschaften der DDR, Berlin

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Preface

Till the early fifties the theory of functions of several complex variables was mainly developed by constructive methods of analysis. We emphasize the work of A. WEIL in 1935 and of K. OKA in the period from 1936 till 1951. WEIL generalized the Cauchy integral formula to polynomial polyhedra in $\mathbb{C}^n$ and obtained an analogue of the Runge approximation theorem for such polyhedra. Equipped with the Weil formula OKA solved the so-called fundamental problems (Cousin problem, Levi problem, et al.).

In the fifties H. CARTAN, J. P. SERRE and H. GRAUERT discovered that by means of the theory of sheaves introduced in 1945 by J. LERAY the constructive methods of analysis in the theory of Oka can be reduced to a minimum and, moreover, that the theory of Oka admits far-reaching generalizations. In the sixties L. HÖRMANDER, J. J. KOHN and C. B. MORREY deduced the main results of OKA with the help of methods from the theory of partial differential equations and obtained, in addition, estimates in certain weighted L_2-metrics for solutions of the Cauchy-Riemann equations.

During the fifties and sixties it seemed that the method of integral representations, which works so successfully in the case of one variable, is not suitable to the case of several variables, because it is troublesome and gives only very special results.

However, in the seventies integral representations turned out to be the natural method for solving several problems related to Oka's theory, which are connected with the boundary behaviour of holomorphic functions. The basic tool is an integral representation formula for holomorphic functions discovered in 1955 by J. LERAY, which contains the Weil formula as a special case. Certain developments of this formula made it possible to solve several of such problems that are not easily obtained with other methods. Moreover, it turned out that by means of these formulas one can build up a large part of the theory of functions of several complex variables in a new and more constructive way.

It is the aim of this book to present such a new introduction to the theory of functions of several complex variables, where the main results will be obtained in a strengthened form — uniform estimates for solutions of the Cauchy-Riemann equations, uniform estimates for extensions of holomorphic functions from submanifolds, uniform approximation of holomorphic functions that are continuous on the boundary, et al.

It has been assumed that the reader has a certain knowledge of the theory of functions of one complex variable and the calculus of differential forms (Stokes' formula).

Chapter 1 starts with facts concerning holomorphic functions, plurisubharmonic functions, domains of holomorphy and pseudoconvex domains. Then we deduce from Stokes' formula the Martinelli-Bochner formula and the Leray formula as well as their generalizations to differential forms in $\mathbb{C}^n$ (Koppelman formula and Koppelman-Leray formula).

In Chapter 2 first the Cauchy-Riemann equations are solved by means of integral formulas in pseudoconvex open sets in $\mathbb{C}^n$. Then we prove this result on Stein manifolds, where an inductive procedure with respect to the levels of a strictly plurisubharmonic exhausting function will be used. For strictly pseudoconvex open sets with C^2-boundary, solutions of the Cauchy-Riemann equations with 1/2-Hölder estimates are obtained. The identity of domains of holomorphy (Stein manifolds) and pseudoconvex open sets in $\mathbb{C}^n$ (complex manifolds with strictly plurisubharmonic exhausting function) is proved, that is, the Levi problem is solved. Further, uniform approximation theorems are proved.

Chapter 3 is devoted to strictly pseudoconvex open sets in $\mathbb{C}^n$ with not necessarily smooth boundary. By means of integral formulas the Cauchy-Riemann equations are solved with uniform estimates in such sets. A uniform approximation theorem is proved for functions which are continuous on a strictly pseudoconvex compact set and holomorphic in the inner points. Further, for strictly pseudoconvex open sets D with not necessarily smooth boundary, an integral formula is constructed which gives bounded holomorphic extensions to D for bounded holomorphic functions defined on the intersection of D with a complex plane. In Chapter 4 this result will be generalized to the case of an intersection with an arbitrary closed complex submanifold in some neighbourhood of $\bar{D}$.

Chapters 1—3 are self-contained. Here we do not use without proof any result from the theory of functions of several complex variables. Only in Chapter 4 we use without proof some special results from the theory of coherent analytic sheaves, for the proof of which we can refer to several books devoted to this subject.

In Chapter 4 our principal aim is to extend the integral formulas introduced in the preceding sections to Stein manifolds. Moreover, in this chapter the Weil formula for analytic polyhedra as well as its generalization to differential forms and a more general class of polyhedra in Stein manifolds is proved. Some applications of these formulas are given. In the Notes at the end some further applications are outlined.

In our opinion Chapters 1 and 2 can be used as an elementary introduction to the theory of functions of several complex variables. Chapters 3 and 4 contain more special and more difficult results obtained only recently by means of complicated estimations, and references to the theory of coherent analytic sheaves. They can be used as an introduction to one of the actual fields of research in complex analysis.

There is also another way to develop the theory of functions of several complex variables by means of integral formulas. This way was outlined in 1961 by E. Bishop and is based on the concept of special analytic polyhedra. In distinction to the approach presented in this book, the way of Bishop is suitable not only for smooth complex manifolds but also for analytic spaces with singularities. However, this way seems to be more complicated and, above all, does not give uniform estimates, whereas in our opinion the latter is the main advantage of the method of integral formulas.

Finally, we point out that in our opinion there are also further interesting possibilities for applying the method of integral formulas, for example to the theory of CR-functions and to problems of complex analysis and integral geometry on projective manifolds connected with the theory of R. Penrose.

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Berlin and Moscow, July 1981 G. M. Henkin and J. Leiterer

Contents

1. Elementary properties of functions of several complex variables

Summary. In Section 1.1 we define holomorphic functions of several complex variables and prove simple properties of these functions. In Section 1.2, by means of a simple extension of the Cauchy-Green formula to several variables, we solve the inhomogeneous Cauchy-Riemann equations for some special cases. As a consequence we obtain a theorem of Hartogs which gives examples of open sets in $\mathbb{C}^n$ $(n \geqq 2)$ where all holomorphic functions can be continued holomorphically to a larger open set. Open sets for which this is not possible are called domains of holomorphy. These are investigated in Section 1.3. Section 1.4 is devoted to continuous plurisubharmonic and strictly plurisubharmonic C^2-functions. In Section 1.5, by means of these functions, pseudoconvex and strictly pseudoconvex open sets are introduced. We prove that every domain of holomorphy is pseudoconvex, but the converse (Levi's problem) is left to Section 2.7. Sections 1.6—1.12 are devoted to integral representation formulas for functions as well as for differential forms n $\mathbb{C}^n$. These formulas form the basic tool for the methods developed in the present book.

1.1. Holomorphic functions

We assume that the reader knows a certain amount of the theory of functions of one complex variable. Nevertheless we begin with a proof of the Cauchy-Green formula for one complex variable, because this proof is the model for the proofs of the integral representation formulas for several complex variables which form the basis of this book.

Notation. Let $\mathbb{C}^1$ be the complex plane. By x_1, x_2 we denote the real coordinates in $\mathbb{C}^1$ such that $\mathbb{C}^1 \ni z = x_1 + \mathrm{i}x_2$. For every complex-valued continuous function f in an open set in $\mathbb{C}^1$ we define (in the sense of distributions)

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x_1} + \frac{1}{\mathrm{i}}\frac{\partial f}{\partial x_2}\right), \qquad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x_1} - \frac{1}{\mathrm{i}}\frac{\partial f}{\partial x_2}\right),$$

$$\partial f := \frac{\partial f}{\partial z}\,\mathrm{d}z \quad \text{and} \quad \bar{\partial} f := \frac{\partial f}{\partial \bar{z}}\,\mathrm{d}\bar{z},$$

where $\bar{z} := x_1 - \mathrm{i}x_2$. Then the differential $\mathrm{d}f$ of f can be expressed as $\mathrm{d}f = \partial f + \bar{\partial} f$. We also write $\mathrm{d}_z f$, $\partial_z f$ and $\bar{\partial}_z f$ instead of $\mathrm{d}f$, ∂f and $\bar{\partial} f$ to make clear that the differentiation is with respect to z if f depends on other variables, too.

Recall that a C^1-function f is holomorphic if and only if $\bar{\partial} f = 0$ (the Cauchy-Riemann equations).

1.1.1. Theorem (Cauchy-Green formula). *Let $D \subset\subset \mathbb{C}^1$ be an open set with C^1-oundary ∂D, and let f be a complex-valued continuous function on $\bar{D}$ such that $\bar{\partial} f$ is* *also continuous on* $\bar{D}$[1]). *Then*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{D}\frac{\bar\partial f(\zeta)\wedge d\zeta}{\zeta - z}, \qquad z \in D. \tag{1.1.1}$$

Proof. Fix $z \in D$. Then, for $\zeta \in \bar{D} \setminus z$, $d_\zeta[(f(\zeta)\,d\zeta)/(\zeta - z)] = \bar\partial f(\zeta) \wedge d\zeta/(\zeta - z)$ and, therefore, by Stokes' formula, for sufficiently small $\varepsilon > 0$

$$\frac{1}{2\pi i}\int_{|\zeta - z| = \varepsilon}\frac{f(\zeta)\,d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{\zeta\in\partial D}\frac{f(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\zeta\in D,\,|\zeta - z| > \varepsilon}\frac{\bar\partial f(\zeta)\wedge d\zeta}{\zeta - z}.$$

It is clear that the right-hand side of this relation tends to the right-hand side of (1.1.1) when $\varepsilon \to 0$. Consider the left-hand side. Since $d\bar\zeta \wedge d\zeta/2i$ is the Lebesgue measure on $\mathbb{C}^1$, Stokes' formula gives

$$\int_{|\zeta - z| = \varepsilon}\frac{d\zeta}{\zeta - z} = \frac{1}{\varepsilon^2}\int_{|\zeta - z| = \varepsilon}(\bar\zeta - \bar z)\,d\zeta = \frac{1}{\varepsilon^2}\int_{|\zeta - z| < \varepsilon} d\bar\zeta \wedge d\zeta = 2\pi i.$$

Further

$$\left|\int_{|\zeta - z| = \varepsilon}\frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta\right| \leqq 2\pi \max_{|\zeta - z| = \varepsilon}|f(\zeta) - f(z)| \to 0 \quad \text{for} \quad \varepsilon \to 0.$$

Hence

$$\lim_{\varepsilon\to 0}\int_{|\zeta - z| = \varepsilon}\frac{f(\zeta)\,d\zeta}{\zeta - z} = 2\pi i f(z). \blacksquare$$

Since the first integral on the right-hand side of the Cauchy-Green formula (1.1.1) depends holomorphically on $z \in D$, and since $\bar\partial f = 0$ for holomorphic functions, we obtain:

1.1.2. Corollary. *Let f be a continuous complex-valued function in an open set $D \subseteqq \mathbb{C}^1$. Then f is holomorphic in D if and only if $\bar\partial f = 0$ in D.*

1.1.3. Theorem. *Let $D \subset\subset \mathbb{C}^1$ be a bounded open set, and let f be a bounded continuous complex-valued function in D. Then the continuous function*

$$u(z) := -\frac{1}{2\pi i}\int_{D}\frac{f(\zeta)\,d\bar\zeta\wedge d\zeta}{\zeta - z}, \qquad z \in D, \tag{1.1.1'}$$

is a solution of $\dfrac{\partial u}{\partial \bar z} = f$ *in D.*

Proof. First we consider the case that f is continuously differentiable in D. Fix $\xi \in D$. It is sufficient to prove that $\partial u/\partial\bar z = f$ in some neighbourhood of ξ. Choose a C^∞-function χ on $\mathbb{C}^1$ such that $\chi = 1$ in a neighbourhood $V_\xi \subset\subset D$ of ξ and $\chi = 0$ in some neighbourhood of $\mathbb{C}^1 \setminus D$. Then $u = u_1 + u_2$, where

$$u_1(z) := -\frac{1}{2\pi i}\int_{D}\frac{\chi(\zeta)f(\zeta)\,d\bar\zeta\wedge d\zeta}{\zeta - z}, \qquad u_2(z) := -\frac{1}{2\pi i}\int_{D}\frac{(1 - \chi(\zeta))f(\zeta)\,d\bar\zeta\wedge d\zeta}{\zeta - z}.$$

[1]) This means, $\partial f/\partial\bar z$ which is defined in the sense of distributions in D is continuous in D and admits a continuous continuation to $\bar{D}$.

Since $1 - \chi = 0$ in V_ξ, u_2 is holomorphic in V_ξ, that is, $\bar\partial u_2 = 0$ in V_ξ. Therefore it remains to prove that $\partial u_1/\partial\bar z = f$ in V_ξ. Since $\chi = 0$ in a neighbourhood of $\mathbb{C}^1 \setminus D$, the function χf can be continued by zero to $\mathbb{C}^1$, and we obtain

$$u_1(z) = -\frac{1}{2\pi i}\int_{\mathbb{C}^1}\frac{\chi(\zeta)f(\zeta)\,d\bar\zeta \wedge d\zeta}{\zeta - z} = -\frac{1}{2\pi i}\int_{\mathbb{C}^1}\frac{\chi(\zeta+z)f(\zeta+z)\,d\bar\zeta \wedge d\zeta}{\zeta}.$$

Taking into account that $\partial/\partial\bar z[\chi(\zeta+z)f(\zeta+z)]\,d\bar\zeta = \bar\partial_\zeta[\chi(\zeta+z)f(\zeta+z)]$, we conclude that

$$\frac{\partial u_1(z)}{\partial\bar z} = -\frac{1}{2\pi i}\int_{\mathbb{C}^1}\frac{\bar\partial_\zeta[\chi(\zeta+z)f(\zeta+z)] \wedge d\zeta}{\zeta} = -\frac{1}{2\pi i}\int_{\mathbb{C}^1}\frac{\bar\partial_\zeta[\chi(\zeta)f(\zeta)] \wedge \partial\zeta}{\zeta - z}.$$

Application of Theorem 1.1.1 with f replaced by χf and D equal to a disc containing the support of χ now gives that $\partial u_1/\partial\bar z = f$ in V_ξ.

Now we consider the case that f is an arbitrary continuous and bounded function on D. Choose a sequence f_k of continuously differentiable functions which converges uniformly on D to f. Then the sequence

$$u_k(z) := -\frac{1}{2\pi i}\int_D\frac{f_k(\zeta)\,d\bar\zeta \wedge d\zeta}{\zeta - z}$$

converges uniformly on D to u, and therefore we have in the sense of distributions $\partial u/\partial\bar z = \lim \partial u_k/\partial\bar z = \lim f_k = f$. ■

Now we pass to the case of several complex variables.

Notation. Let $\mathbb{C}^n$ $(n = 1, 2, \ldots)$ be the space of all n-tuples $z = (z_1, \ldots, z_n)$ of complex numbers z_j. The components $z_1, \ldots, z_n$ of $z \in \mathbb{C}^n$ will be called the *canonical* (*complex*) *coordinates* of z. By $x_1, \ldots, x_{2n}$ we denote the *real coordinates in* $\mathbb{C}^n$ such that $z_j = x_j + i x_{j+n}$. For $Y \subseteq \mathbb{C}^n$, let $C^0(Y)$ be the space of continuous complex-valued functions on Y. If $D \subseteq \mathbb{C}^n$ is open, then $C^k(D)$ $(k = 1, 2, \ldots, \infty)$ is the space of k-times continuously differentiable (with respect to the real coordinates x_j) complex-valued functions in D, and $C_0^k(D)$ is the subspace of all $f \in C^k(D)$ vanishing outside a compact subset of D. We introduce the differential operators

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \frac{1}{i}\frac{\partial}{\partial x_{j+n}}\right) \quad\text{and}\quad \frac{\partial}{\partial\bar z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \frac{1}{i}\frac{\partial}{\partial x_{j+n}}\right).$$

These operators $\partial/\partial z_j$ and $\partial/\partial\bar z_j$ will also be denoted by ∂_j and $\bar\partial_j$, respectively. A *multi-order* is an n-tuple $k = (k_1, \ldots, k_n)$ of non-negative integers. For every multi-order k we write $z^k := z_1^{k_1} \ldots z_n^{k_n}$ if $z \in \mathbb{C}^n$, $k! := k_1! \ldots k_n!$, $\partial^k := \partial_1^{k_1} \circ \ldots \circ \partial_n^{k_n}$ and $\bar\partial^k := \bar\partial_1^{k_1} \circ \ldots \circ \bar\partial_n^{k_n}$. If $D \subseteq \mathbb{C}^n$ is open and $f \in C^0(D)$, then we define

$$\partial f := \sum_{j=1}^n \frac{\partial f}{\partial z_j}\,dz_j \quad\text{and}\quad \bar\partial f := \sum_{j=1}^n \frac{\partial f}{\partial\bar z_j}\,d\bar z_j \text{ in } D.$$

Then $df = \partial f + \bar\partial f$ in D.

(Remark. In general, ∂f and $\bar\partial f$ are differential forms whose coefficients are distributions. However, in this book we need ∂f $(\bar\partial f)$ only if ∂f $(\bar\partial f)$ is continuous.)

A set $P \subseteq \mathbb{C}^n$ is said to be an *open* (*closed*) *polydisc* if there are open (closed) discs $P_1, \ldots, P_n$ in $\mathbb{C}^1$ such that $P = P_1 \times \ldots \times P_n$. If ξ_j is the center of P_j, then the point

$(\xi_1, \dots, \xi_n)$ is called the *center* of P. If r_j is the radius of P_j, then $(r_1, \dots, r_n)$ is called the *multi-radius* of P.

1.1.4. Lemma. *Let a_k be complex numbers defined for all multi-orders k. Suppose that for some $\xi \in \mathbb{C}^n$ ($\xi_j \neq 0$ for all j) the power series $\sum_k a_k \xi^k$ converges. Then the series $\sum_k a_k z^k$ converges normally in the polydisc $|z_j| < |\xi_j|$, $j = 1, \dots, n$, that is, for all $r < 1$*

$$\sum_k \sup_{|z_j| \leqq r|\xi_j|} |a_k z^k| < \infty .$$

Proof. By hypothesis there is $C < \infty$ such that $|a_k \xi^k| \leqq C$ for all k. Consequently,

$$\sum_k \sup_{|z_j| \leqq r|\xi_j|} |a_k z^k| \leqq C \sum_k r^{k_1 + \dots + k_n} < \infty . \blacksquare$$

1.1.5. Theorem. *Let $D \subseteqq \mathbb{C}^n$ be an open set and f a complex-valued function in D. Then the following conditions are equivalent:*

(i) *$f \in C^0(D)$ and $\bar{\partial} f = 0$ in D.*

(ii) *$f \in C^0(D)$ and f is holomorphic in each variable z_j when the other variables are kept fixed.*

(iii) *$f \in C^0(D)$ and, for every polydisc $P = P_1 \times \dots \times P_n \subset\subset D$,*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial P_1 \times \dots \times \partial P_n} \frac{f(\zeta)\, d\zeta_1 \wedge \dots \wedge d\zeta_n}{(\zeta_1 - z_1) \cdot \dots \cdot (\zeta_n - z_n)}, \qquad z \in P . \tag{1.1.2}$$

(iv) *f admits local power series expansions, that is, for every point $\xi \in D$ there are complex numbers a_k defined for every multi-order k such that, for all z in some neighbourhood of ξ,*

$$f(z) = \sum_k a_k (z - \xi)^k . \tag{1.1.3}$$

If these equivalent conditions are fulfilled, then, moreover:

(v) *$f \in C^\infty(D)$.*

(vi) *For every polydisc $P = P_1 \times \dots \times P_n \subset\subset D$ and every multi-order k,*

$$\partial^k f(z) = \frac{k!}{(2\pi i)^n} \int_{\partial P_1 \times \dots \times \partial P_n} \frac{f(\zeta)\, d\zeta_1 \wedge \dots \wedge d\zeta_n}{(\zeta_1 - z_1)^{k_1+1} \dots (\zeta_n - z_n)^{k_n+1}}, \qquad z \in P . \tag{1.1.4}$$

(vii) *The coefficients in the power series expansion* (1.1.3) *are uniquely determined, where*

$$a_k = \frac{\partial^k f(\xi)}{k!} . \tag{1.1.5}$$

(viii) *The power series expansion* (1.1.3) *converges uniformly in each polydisc $P \subset\subset D$ centered at ξ.*

Proof. (i) $\Leftrightarrow$ (ii) according to Corollary 1.1.2. By repeated use of (1.1.1), we obtain the implication (i) $\Rightarrow$ (iii).

We prove that (iii) $\Rightarrow$ (iv). Let $\xi \in D$ and $P = P_1 \times \dots \times P_n \subset\subset D$ be a polydisc centered at ξ. Since for $z \in P$ and $\zeta \in \partial P_1 \times \dots \times \partial P_n$

$$\frac{1}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)} = \sum_{k_1, \dots, k_n = 0}^{\infty} \frac{(z_1 - \xi_1)^{k_1} \dots (z_n - \xi_n)^{k_n}}{(\zeta_1 - \xi_1)^{k_1+1} \dots (\zeta_n - \xi_n)^{k_n+1}}$$

with uniform convergence in ζ, it follows from (1.1.2) that

$$f(z) = \sum_k \left[\frac{1}{(2\pi i)^n} \int\limits_{\partial P_1 \times \dots \times \partial P_n} \frac{f(\zeta)\, d\zeta_1 \wedge \dots \wedge d\zeta_n}{(\zeta_1 - \xi_1)^{k_1+1} \dots (\zeta_n - \xi_n)^{k_n+1}}\right] (z - \xi)^k, \qquad z \in P. \tag{1.1.5'}$$

Now we prove that (iv) $\Rightarrow$ (ii). It follows from (1.1.3) and Lemma 1.1.4 that f is locally the uniform limit of polynomials in $z_1, \dots, z_n$. Since such polynomials fulfil (ii), and since the uniform limit of continuous functions is continuous and the uniform limit of holomorphic functions of one complex variable is holomorphic, we conclude that f satisfies condition (ii).

Now we suppose that the equivalent conditions (i)—(iv) are fulfilled. Then (v) follows from (iii) and (vi) can be obtained by differentiation under the sign of integration in (1.1.2). Property (vii) follows, having chosen a sufficiently small polydisc $P = P_1 \times \dots \times P_n$ centered at ξ, by the following computation, which is permissible in view of (1.1.4) and Lemma 1.1.4:

$$\partial^k f(\xi) = \frac{k!}{(2\pi i)^n} \sum_l a_l \int\limits_{\partial P_1 \times \dots \times \partial P_n} \frac{(\zeta - \xi)^l\, d\zeta_1 \wedge \dots \wedge d\zeta_n}{(\zeta_1 - \xi_1)^{k_1+1} \dots (\zeta_n - \xi_n)^{k_n+1}}$$

$$= \frac{k!}{(2\pi i)^n} \sum_l a_l \frac{(2\pi i)^n}{l!} \partial_\zeta^k (\zeta - \xi)^l \big|_{\zeta=\xi} = k!\, a_k .$$

(viii) follows from (vii), (1.1.5') and Lemma 1.1.4. ■

1.1.6. Definition. Let $D \subseteq \mathbb{C}^n$ be an open set. A complex-valued function f in D is said to be *holomorphic* (or *analytic*) if the equivalent conditions in Theorem 1.1.5 are fulfilled. The set of all holomorphic functions in D will be denoted by $\mathcal{O}(D)$.

1.1.7. Corollary. *For every open set $D \subseteq \mathbb{C}^n$, $\mathcal{O}(D)$ is a ring, that is, if $f, g \in \mathcal{O}(D)$, then $f + g \in \mathcal{O}(D)$ and $fg \in \mathcal{O}(D)$. Further, if $f \in \mathcal{O}(D)$ and $f(z) \neq 0$ for all $z \in D$, then $1/f \in \mathcal{O}(D)$.*

Proof. This follows from condition (ii) in Theorem 1.1.5 and the corresponding properties of holomorphic functions of one complex variable.

1.1.8. Corollary (Maximum principle). *Let $D \subseteq \mathbb{C}^n$ be an open set and $f \in \mathcal{O}(D)$. Suppose that there exists a point $\xi \in D$ such that $|f(z)| \leqq |f(\xi)|$ for all $z \in D$. Then f is constant in D if D is connected.*

Proof. This follows from condition (ii) in Theorem 1.1.5 and the maximum principle for holomorphic functions of one complex variable. ■

1.1.9. Corollary. *Let $D \subseteq \mathbb{C}^n$ be an open set and $f_j \in \mathcal{O}(D)$. If $f_j \to f$ when $j \to \infty$, uniformly on every compact subset of D, then $f \in \mathcal{O}(D)$.*

Proof. This follows from condition (ii) in Theorem 1.1.5 and the corresponding property of holomorphic functions of one complex variable. Clearly, it can also be obtained from condition (iii) in Theorem 1.1.5. ■

1.1.10. Corollary (the uniqueness of holomorphic continuation). *Let $D \subseteq \mathbb{C}^n$ be an open set and $f \in \mathcal{O}(D)$. If there is a point $\xi \in D$ where $\partial^k f(\xi) = 0$ for all multi-orders k, then $f = 0$ in D if D is connected.*

Proof. This follows from condition (iv) and relation (1.1.5) in Theorem 1.1.5. ■

1.1.11. Corollary. *For every open set $D \subseteqq \mathbb{C}^n$ and all multi-orders k, we have $\partial^k \mathcal{O}(D) \subseteqq \mathcal{O}(D)$.*

Proof. This follows from property (vi) in Theorem 1.1.5. ■

1.1.12. Corollary (Cauchy's inequality). *If f is holomorphic in the polydisc $P := \{z \in \mathbb{C}^n : |z_j| < r_j, j = 1, \dots, n\}$, where $0 < r_j < \infty$, $j = 1, \dots, n$, then for all multi-orders k*

$$|\partial^k f(0)| \leqq k!\, r^{-k} \sup_{z \in P} |f(z)| \,.$$

Proof. This follows from property (vi) in Theorem 1.1.5. ■

1.1.13. Theorem. *Let $D \subseteqq \mathbb{C}^n$ be an open set. Then, for every compact set $K \subset\subset D$ and every multi-order k, there exists a constant $C = C(K, k)$ such that*

$$\max_{z \in K} |\partial^k f(z)| \leqq C \int_D |f| \, d\sigma_{2n} \quad \textit{for all} \quad f \in \mathcal{O}(D) \,,$$

where $d\sigma_{2n}$ is the Lebesgue measure in $\mathbb{C}^n$.

Proof. First let $n = 1$. Choose $\chi \in C_0^\infty(D)$ such that $\chi = 1$ in a neighbourhood U_K of K. Then, by (1.1.1), for every $f \in \mathcal{O}(D)$

$$\chi(z) f(z) = \frac{-1}{2\pi i} \int_{D \setminus U_K} \frac{f(\zeta)\, \bar{\partial}\chi(\zeta) \wedge d\zeta}{\zeta - z} \,,$$

and differentiation leads to the required estimate. Repeated use of this gives the theorem for polydiscs D of arbitrary dimension. The general case follows by use of a covering of K by a finite number of polydiscs $\subset\subset D$. ■

1.1.14. Corollary (Stieltjes-Vitali). *Let $D \subseteqq \mathbb{C}^n$ be an open set and let f_k be a sequence of holomorphic functions in D which is uniformly bounded on every compact subset of D. Then there is a subsequence f_{k_j} converging uniformly on every compact subset of D to a limit in $\mathcal{O}(D)$.*

Proof. Since $\bar{\partial} f_k = 0$, it follows from Theorem 1.1.13 that all first-order derivatives (with respect to the real coordinates) of f_k are uniformly bounded on any compact subset of D. Hence the corollary follows from Ascoli's theorem and Corollary 1.1.9. ■

1.1.15. Definition. Let $D \subseteqq \mathbb{C}^n$ be an open set. A map $f = (f_1, \dots, f_m): D \to \mathbb{C}^m$ is said to be *holomorphic* in D if $f_1, \dots, f_m \in \mathcal{O}(D)$. The set of all holomorphic maps $f: D \to \mathbb{C}^m$ will be denoted by $\mathcal{O}^m(D)$. If $f \in \mathcal{O}^m(D)$ and $\xi \in D$, then the matrix

$$J_f(\xi) := \left(\frac{\partial f_j(\xi)}{\partial \xi_k} \right)_{k=1,\dots,n}^{j=1,\dots,m} \quad (j \text{ is the row index})$$

is called the (*complex*) *Jacobi matrix* of f at ξ. f is called *regular* at $\xi \in D$ if rank $J_f(\xi) = \min\{n, m\}$. Let $D, G \subseteqq \mathbb{C}^n$ be open sets. A *biholomorphic map from D onto G* is by definition a bijective map f from D onto G such that both $f \in \mathcal{O}^n(D)$ and $f^{-1} \in \mathcal{O}^n(G)$. Then we shall also say that f is *biholomorphic in D*.

1.1.16. Proposition. *Let $D \subseteqq \mathbb{C}^n$ be an open set and $f = (f_1, \dots, f_m) \in \mathcal{O}^m(D)$. Then*
(i) *For every $\xi \in D$,*

$$f(\xi + z) = f(\xi) + J_f(\xi)\, z + O(|z|^2) \quad \textit{when} \quad \mathbb{C}^n \ni z \to 0 \,. \tag{1.1.6}$$

(ii) *If g is a continuous complex-valued function in some neighbourhood of $f(D)$, then (in the sense of distributions)*

$$\frac{\partial(g \circ f)}{\partial z_j} = \sum_{k=1}^{m} \frac{\partial g}{\partial f_k} \frac{\partial f_k}{\partial z_j} \quad \text{for} \quad j = 1, \dots, n, \tag{1.1.7}$$

and

$$\frac{\partial(g \circ f)}{\partial \bar{z}_j} = \sum_{k=1}^{m} \frac{\partial g}{\partial \bar{f}_k} \frac{\partial \bar{f}_k}{\partial \bar{z}_j} \quad \text{for} \quad j = 1, \dots, n. \tag{1.1.8}$$

Proof. Since $\bar{\partial} f_j = 0$ for $j = 1, \dots, n$, and $\mathrm{d} = \partial + \bar{\partial}$, (1.1.6) follows from the Taylor formula. Since $\mathrm{d} = \partial + \bar{\partial}$ and $\bar{\partial} f_k = \partial \bar{f}_k = 0$, we obtain

$$\sum_{j=1}^{n} \left(\frac{\partial(g \circ f)}{\partial z_j} \mathrm{d}z_j + \frac{\partial(g \circ f)}{\partial \bar{z}_j} \mathrm{d}\bar{z}_j \right) = \mathrm{d}(g \circ f)$$

$$= \sum_{k=1}^{m} \left(\frac{\partial g}{\partial f_k} \mathrm{d}f_k + \frac{\partial g}{\partial \bar{f}_k} \mathrm{d}\bar{f}_k \right) = \sum_{j=1}^{n} \sum_{k=1}^{m} \left(\frac{\partial g}{\partial f_k} \frac{\partial f_k}{\partial z_j} \mathrm{d}z_j + \frac{\partial g}{\partial \bar{f}_k} \frac{\partial \bar{f}_k}{\partial \bar{z}_j} \mathrm{d}\bar{z}_j \right),$$

which implies (1.1.7) amd (1.1.8) by comparison of the coefficients of $\mathrm{d}z_1, \dots, \mathrm{d}z_n$, $\mathrm{d}\bar{z}_1, \dots, \mathrm{d}\bar{z}_n$. ■

1.1.17. Corollary. *Let $D \subseteq \mathbb{C}^n$, $G \subseteq \mathbb{C}^m$ be open sets, and let $f \in \mathcal{O}^m(D)$, $g \in \mathcal{O}^k(G)$ such that $f(D) \subseteq G$. Then $g \circ f \in \mathcal{O}^k(D)$ and*

$$J_{g \circ f}(\xi) = J_g(f(\xi))\, J_f(\xi) \quad \text{for all} \quad \xi \in D. \tag{1.1.9}$$

Proof. That $g \circ f \in \mathcal{O}^k(D)$ follows from (1.1.8). (1.1.9) follows from (1.1.7). ■

1.1.18. Theorem (Implicit function theorem). *Let U be a neighbourhood of $\xi \in \mathbb{C}^n$ and $f \in \mathcal{O}^n(U)$. Then f is biholomorphic in some neighbourhood of ξ if and only if*

$$\det J_f(\xi) \neq 0. \tag{1.1.10}$$

Proof. It follows from (1.1.9) that (1.1.10) is necessary. We prove that condition (1.1.10) is sufficient. Without loss of generality we can assume that $\xi = 0$, $f(\xi) = 0$, and $J_f(\xi)$ is the unit matrix. Denote by id the identity map of $\mathbb{C}^n$ and define

$$\tilde{f} := \mathrm{id} - f \quad \text{on} \quad U.$$

Choose a neighbourhood V of $0 \in \mathbb{C}^n$ such that $V \subset\subset U$. Then, by (1.1.6), for some $C < \infty$

$$|\tilde{f}(z)| \leqq C\,|z|^2 \quad \text{for} \quad z \in V.$$

We use the notation $E_0(\delta) := \{z \in \mathbb{C}^n : |z| < \delta\}$, $\delta > 0$, and choose $\varepsilon > 0$ so small that $\varepsilon < 1/2C$ and $E_0(2\varepsilon) \subseteq V$. Then $\tilde{f}(E_0(\varepsilon/2^k)) \subseteq E_0(\varepsilon/2^{k+1})$ for $k = 0, 1, 2, \dots$. Consequently, the series

$$h := \sum_{k=1}^{\infty} \tilde{f}^k, \quad \text{where} \quad \tilde{f}^k := \underbrace{\tilde{f} \circ \dots \circ \tilde{f}}_{k \text{ times}},$$

converges uniformly on $E_0(\varepsilon)$ and $h(E_0(\varepsilon)) \subseteq E_0(2\varepsilon) \subseteq U$. As the uniform limit of holomorphic maps h is holomorphic. Further, $(\mathrm{id} + h) \circ f = f \circ (\mathrm{id} + h) = \mathrm{id}$. ■

1.1.19. Corollary. *If $X \subseteq \mathbb{C}^n$ and $k \in \{1, 2, \dots, n-1\}$, then the following conditions are equivalent:*

(i) *For every point $\xi \in X$ there exists a biholomorphic map $f = (f_1, \dots, f_n)$ in some neighbourhood U of ξ such that $X \cap U = \{z \in U : f_{k+1}(z) = \dots = f_n(z) = 0\}$.*

(ii) *For every point* $\xi \in X$ *there exist a neighbourhood* V *of* ξ *and a regular holomorphic map* $g\colon V \to \mathbb{C}^{n-k}$ *such that* $X \cap V = \{z \in V : g(z) = 0\}$.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. We prove that (ii) $\Rightarrow$ (i). Let $\xi \in X$ and let V, g as in condition (ii). Denote by G the linear map from $\mathbb{C}^n$ into $\mathbb{C}^{n-k}$ defined by the Jacobi matrix $J_g(\xi)$. Since rank $J_g(\xi) = n - k$, G is onto and we can find a linear map $A\colon \mathbb{C}^n \to \mathbb{C}^k$ such that the map $A \oplus G\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by $(A \oplus G)(z) := \big(A(z), G(z)\big)$ is invertible. Define $f(z) := \big(A(z), g(z)\big)$ for $z \in V$. Then $\det J_f(\xi) = \det A \oplus G \neq 0$ and, by Theorem 1.1.18, f is biholomorphic in some neighbourhood $U \subseteq V$ of ξ. Since $X \cap U = \{z \in U : g(z) = 0\}$, and since $(f_{k+1}, \ldots, f_n) = g$, it follows that $X \cap U = \{z \in U : f_{k+1}(z) = \ldots = f_n(z) = 0\}$. ■

1.1.20. Definition. Let $D \subseteq \mathbb{C}^n$ be an open set. A subset $X \subseteq D$ is said to be a *complex submanifold of* $\mathbb{C}^n$ if the equivalent conditions (i) and (ii) in Corollary 1.1.19 are fulfilled. If, moreover, X is a closed subset of D, then X is called a *closed complex submanifold of* D.

It follows from condition (i) in Corollary 1.1.19 that every complex submanifold of $\mathbb{C}^n$ in a natural way becomes a complex manifold in the following abstract sense:

1.1.21. Definition. A *complex manifold of complex dimension* n is a real $2n$-dimensional C^∞-manifold X together with a family $\{(U_j, \varphi_j)\}_{j \in I}$ satisfying the following conditions:

(i) For every $j \in I$, U_j is an open subset of X, and $\bigcup_{j \in I} U_j = X$.

(ii) For all $j \in I$, φ_j is a C^∞-diffeomorphism from U_j onto some open set in $\mathbb{C}^n$.

(iii) For all $i, j \in I$, $\varphi_j \circ \varphi_i^{-1}$ is a biholomorphic map from $\varphi_i(U_i \cap U_j)$ onto $\varphi_j(U_i \cap U_j)$.

Then a couple (V, ψ) is called a *system of holomorphic coordinates in* V, if the family $\{(U_j, \varphi_j)\}_{j \in I} \cup \{(V, \psi)\}$ fulfils conditions (i)–(iii). A family $(V_\alpha, \psi_\alpha)_{\alpha \in A}$ is called a *holomorphic atlas of* X if every (V_α, ψ_α) is a system of holomorphic coordinates and $X = \bigcup_{\alpha \in A} V_\alpha$. (In particular, the system $\{(U_j, \varphi_j)\}_{j \in I}$ above is a holomorphic atlas of X.) Every open subset of a complex manifold becomes in an obvious sense a complex manifold.

A function f defined on a complex manifold X is said to be *holomorphic* on X if, for every system (V, ψ) of holomorphic coordinates in V, the function $f \circ \psi^{-1}$ is holomorphic on $\psi(V)$. If X, Y are two complex manifolds, then a map $f\colon X \to Y$ is called *holomorphic* if, for every system (U, φ) of holomorphic coordinates in X and every system (V, ψ) of holomorphic coordinates in Y, the map $\psi \circ f \circ \varphi^{-1}$ is holomorphic in $\varphi\big(U \cap f^{-1}(V)\big)$. If, moreover, f is a bijective map from X onto Y and f^{-1} is also holomorphic, then f is said to be *biholomorphic from* X *onto* Y. If there exists a biholomorphic map from X onto Y, then X and Y are called *biholomorphically equivalent*.

A subset Z of a complex manifold X is said to be a *complex submanifold of* X if, for every system (U, φ) of holomorphic coordinates in X, $\varphi(U \cap Z)$ is a complex submanifold of $\varphi(U)$. If, moreover, Z is a closed subset of X, then Z is called a *closed complex submanifold of* X. Every complex submanifold of a complex manifold becomes in an obvious sense a complex manifold.

If U is an open set in a complex manifold, then we denote by $\mathcal{O}(U)$ the set of all holomorphic functions in U, and by $\mathcal{O}^k(U)$ we denote the set of all holomorphic maps from U into $\mathbb{C}^k$.