# Chaotic Behaviour of Deterministic Dissipative Systems

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CAMBRIDGE UNIVERSITY PRESS
*Cambridge*
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Published by
Academia, Publishing House of the Czechoslovak Academy of Sciences, Prague
in Albania, Bulgaria, China, Cuba, Czechoslovakia, Hungary, Mongolia, North Korea, Poland, Rumania, U.S.S.R., Vietnam and Yugoslavia

Published throughout the rest of the world by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
40 West 20th Street, New York, NY 10011, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia



First published 1991
Printed in Czechoslovakia

*British Library cataloguing in publication data*
Marek, Miloš
Chaotic behaviour of deterministic dissipative systems.
1. Nonlinear dynamical systems. Chaotic behaviour
I. Title II. Schreiber, Igor
515.3'5

*Library of Congress cataloging in publication data*
Marek, Miloš
Chaotic behaviour of deterministic dissipative systems/Miloš Marek and Igor Schreiber.
p. cm.
Includes bibliographical references.
ISBN 0 521 32167 0
1. Chaotic behavior in systems. 2. Differentiable dynamical systems. 3. Nonlinear theories.
I. Schreiber, Igor. II. Title. III. Title: Deterministic dissipative systems.
QA845.M37 1991
003—dc 19 88-2958 CIP

ISBN 0 521 32167 0

# PREFACE

Studies of nonlinear phenomena which occur in mathematical models and which are observed in experiments profit both from a general knowledge of the theory of dynamical systems and bifurcations, and from the experience accumulated in an interpretation of specific examples. The most interesting and important nonlinear phenomenon that has come to prominence recently is the chaotic behaviour of deterministic dissipative systems. The investigation of chaotic dynamics has undergone an explosive development over the past ten years but the results are still mostly scattered throughout the journal literature.

The number of interested students and research workers from diverse fields, ranging from mathematics and physics to engineering sciences and biology, increases continuously and many of them will find it useful to have an introductory text, that surveys both theoretical and experimental aspects of chaotic behaviour. We have attempted to provide this in the present book.

The introductory chapter discusses the significance of chaos as a model of many seemingly random processes in nature and a definition of the class of dissipative systems that we will study.

The second chapter considers basic notions of the theory of dynamical systems. The difference between linear and nonlinear systems is illustrated and asymptotic behaviour is discussed in more detail. Definitions of chaos and of strange attractors and a description of chaotic behaviour in the frame of ergodic theory are then surveyed.

The third chapter deals with qualitative changes of asymptotic behaviour as a chosen parameter is varied. These changes ('bifurcations') may lead to chaos in several well-defined routes. The role of bifurcation theory in understanding the onset of chaos is illustrated by a number of characteristic examples.

A review of the numerical methods used both in the treatment of mathematical models and in the interpretation of experimental data is provided in the fourth chapter. Methods for parametric dependences and for a characterization of chaotic behaviour are stressed.

The fifth chapter surveys some characteristic experimental observations of chaotic behaviour and includes data from mechanical systems, electronics, lasers, semiconductors, chemical and biological systems and hydrodynamics. It is stressed that most of these observations have many common features even though their physical nature is different.

The sixth chapter is based on our own experimental and numerical work and, using two detailed examples, it illustrates an interpretation of chaotic experimental data on the basis of one-dimensional mappings, and the role of numerical studies of bifurcations in the interpretation of complex periodic and chaotic behaviour in the system of two coupled cells.

In the final chapter we have tried to survey ways of approach to modelling of spatio-temporal chaotic behaviour in distributed systems. The results of analysis of cellular automata, coupled map lattices and partial differential equations are briefly reviewed and generally applicable methods are stressed.

The book contains two appendices. A brief survey of some normal forms of planar vector fields and the corresponding bifurcation diagrams can be found in the Appendix A. A program for continuation of stationary points and periodic orbits of dynamical systems and for the location and continuation of local bifurcation points is given in the Appendix B. Hence, an interested reader can with the help of these two appendices and the problems discussed in Chapters 5 and 6 obtain his own experience in the analysis of bifurcations and chaotic behaviour.

The individual chapters can be studied independently. References connected with the subjects discussed are given at the end of each chapter. In recent years the number of references has increased exponentially, and hence, we have had to limit ourselves to those references that, in our opinion, best illustrate the problems studied.

The study of chaotic behaviour often requires a profound knowledge of different branches of mathematics. In attempting to provide an objective explanation of various aspects of chaos which is also accessible to readers who do not possess such knowledge we have made a number of simplifications in the explanations and used graphical representation of the features studied. We would be grateful to readers for any comments that may improve the text.

The text reflects the results obtained in our research group over the last 10 years. We would like to express our sincere thanks to our colleagues, Alois Klíč, Milan Kubíček, Martin Holodniok, Miloš Dolník and many others (including a number of students) for the discussions and the friendly atmosphere which helped to write the book.

We are thankful to Dr. S. Capelin, Ing. B. Kyselová, Ing. J. Malátková and to other members of the staff of Academia Publishers, Prague and Cambridge University Press, Cambridge for their persistent and invaluable help with the transformation of the manuscript into book.

Last but not least we are indebted to Dana Dopitová and Edita Padušáková for their assistance in producing the manuscript.

*M. Marek*
*I. Schreiber*

# CONTENTS

# 1

# Introduction

Observations of both natural and man-made systems evolving in time reveal an existence of various types of dynamics ranging from *steady time-independent* structures to very complicated *nonperiodic oscillations*. It is well known that the evolution of nonperiodic motions forms a basic problem in studies of hydrodynamical *turbulence*. However, both experimental and theoretical research in the last 20 years have clearly demonstrated that turbulent motion is in no case limited to fluids. It can exist in systems of different physical nature where oscillations occur, for example in mechanical vibrations, electronic circuits, chemical reactions, neurones, ecological systems, celestial mechanics, and so on.

An interpretation of experimental observations is closely coupled with the fast-developing theory of *nonlinear dynamical systems*. A typical mathematical model of an *evolution process* is in the form of a *differential equation*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = v(\mathbf{x}, \boldsymbol{\alpha}), \qquad \mathbf{x} \in \mathsf{R}^n \tag{1.1}$$

where the real variable $t$ denotes *time* and $\boldsymbol{\alpha}$ is a *parameter*. A *state* of the system (1.1) at a given time is determined by a point $\mathbf{x}$ in the *state space* $\mathsf{R}^n$. Evolution of the variable $\mathbf{x}$ in time is given by a solution of Eq. (1.1). A *discrete time evolution process* can be described by a *difference equation*

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \boldsymbol{\alpha}), \qquad \mathbf{x} \in \mathsf{R}^n \tag{1.2}$$

where the time, denoted by $k$, is discrete. The solution of Eq. (1.2) is given by repeated *iterations* of the mapping $f$.

Actual states of the above systems are described by the vector variable $\mathbf{x}$ consisting of $n$ independent components. However, the state variable is *spatially distributed* in the fluid flow as well as in a number of other systems. The state space then has an *infinite* dimension and the mathematical model is then formed by a system of *partial differential equations*. Evolution processes described by *integro-differential* equations and differential equations with a *time delay* also have infinite-dimensional state spaces.

If the evolution equations in the form of Eqs (1.1) and (1.2) are linear, then their solutions may be expressed in an explicit way and the evolution dynamics is relatively simple. It has become evident in the course of the last 20 years that even very simple but nonlinear equations can possess solutions, which from the statistical point of view look like random ones, although they are generated by a *deterministic system*. This behaviour is now called *deterministic chaos* and is generally believed to represent a valid model of low-dimensional turbulent behaviour in systems of various physical nature[1.56].

The onset of chaos can be studied by observing *asymptotic solutions* of (1.1) or (1.2) in dependence on the varying parameter $\boldsymbol{\alpha}$. An originally simple dynamic regime, represented, for example, by periodic oscillations, becomes more complex and finally leads to *chaotic behaviour*. In fact, a few degrees of freedom suffice to generate chaos and low-dimensional systems of the form of Eq. (1.1) and (1.2) are often used to study the onset of chaos. There exist several typical ways (routes) of *transition to chaos*[1.16]; the best known is the *period-doubling* route[1.18]. Further evolution of chaos may lead to an increased complexity of the chaotic motions as, for example, in hydrodynamical turbulence. On the other hand, *fully developed turbulence* in fluids[1.41] is a phenomenon too complex to be described by equations of the type (1.1) or (1.2) and its complete description is still an open problem. Thus the theory in its present state relates to the onset of turbulence and to the weak turbulence occurring in low-dimensional systems[1.62]. Nevertheless, the theory of dynamical systems in infinite-dimensional state spaces[1.63] as well as the theory of *cellular automata*[1.65] which are relevant to the problem of fully developed turbulence become more and more promising with a view of their applicability.

Here we shall study systems which *dissipate energy*; they are kept far from the thermodynamic equilibrium by an exchange of mass and/or energy with the environment. Mathematical models of *dissipative systems* possess the important property of the *contraction* of volumes in the state space.

Let $\mathsf{A}$ be a bounded set in the state space $\mathsf{R}^n$ and $V$ be its volume. The set $\mathsf{A}$ is generally deformed under the time evolution according to Eqs (1.1) or (1.2). The time evolution of the volume $V$ of set $\mathsf{A}$ is in the continuous time case given according to *Liouville's theorem*[1.43] as

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = \int_{\mathsf{A}(t)} \operatorname{div} \boldsymbol{v}(\mathbf{x}(t))\, \mathrm{d}\mathbf{x} \; ; \tag{1.3}$$

if the time is discrete we have similarly

$$V_{k+1} = \int_{\mathsf{A}_k} \left| \det \left( \frac{\partial \boldsymbol{f}(\mathbf{x}_k)}{\partial \mathbf{x}} \right) \right| \mathrm{d}\mathbf{x} \, . \tag{1.4}$$

The global contraction of subsets of the state space will be guaranteed if div $\boldsymbol{v}(\mathbf{x}) < 0$ or $|\det \partial f/\partial \mathbf{x}| < 1$, respectively, for all $\mathbf{x}$. Asymptotic motions will then occur on sets which have *zero volume*. If such asymptotic sets satisfy certain stability conditions, for example a stability against small bounded random perturbations, they are called *attractors*. Most of the points which are close to an attractor tend to it as time becomes large.

The structure of an attractor and of its dynamics can be simple, for example a point in the state space of continuous time systems corresponds to stationary behaviour and a closed curve to periodic oscillations. However, an overall contraction of volumes does not exclude complicated dynamics. The set A may be *expanding* in some directions in the state space even if its volume vanishes asymptotically. This may cause *folding* of different parts of A upon itself under time evolution and the asymptotic set may then have a very complicated geometric structure as well as complicated dynamics. In the case of a *chaotic attractor*, a locally *exponential expansion* of nearby points on the attractor is required.

In general dissipative systems need not contract all volumes in the state space; some sets may exist which expand their volumes with time, but such sets do not contain attractors. There is a special class of mathematical models characterized by the preservation of volumes under time evolution. Such systems are called *conservative* and are well known, for example from classical mechanics[1.2, 1.3]; they may also generate chaos but the phenomenon of attraction is lacking. Here we shall concentrate on *dissipative* systems.

One of the simplest discrete time systems which possesses a chaotic attractor is the following system of difference equations, studied first by M. Hénon[1.34]

$$x_{k+1} = f_1(x_k, y_k, a) = 1 - ax_k^2 + y_k ,$$
$$y_{k+1} = f_2(x_k, y_k, b) = bx_k , \qquad (1.5)$$

where $a$, $b$ are parameters.

The Jacobian of the mapping $f = (f_1, f_2)$ in (1.5) is $\det(\partial f/\partial \mathbf{x}) = -b$. Hence the system is dissipative if $|b| < 1$ and all areas of the state space $\mathsf{R}^2$ are contracted uniformly. Numerical computations show that Eqs (1.5) have for many values of the parameters $a$ and $b$ solutions which tend asymptotically to an attractor with complicated dynamics. This attractor also has a very complex geometric structure, as illustrated in Fig. 1.1 ($a$) and ($b$). The structure does not disappear upon magnification and is repeated on arbitrarily small scales. Such sets are called *fractals* and are characterized by a *noninteger dimension*[1.46]. However, the most important feature of the Hénon attractor is the existence of a chaotic dynamics; subsequent iterations of (1.5) fill up the attractor *randomly*. This randomness appears to be equivalent to that of tossing a coin [1.19] and is caused mainly by the exponential divergence of nearby points on the attractor.

Another classical example of a chaotic attractor is the Lorenz attractor, arising in a continuous time system of three nonlinear ordinary differential equations (ODEs), studied first by E. N. Lorenz[1.44]. The severe truncation of the set of partial differential equations describing thermal convection in the atmosphere leads to the following set of ODEs

$$\frac{dx}{dt} = -\sigma x + \sigma y \, ,$$

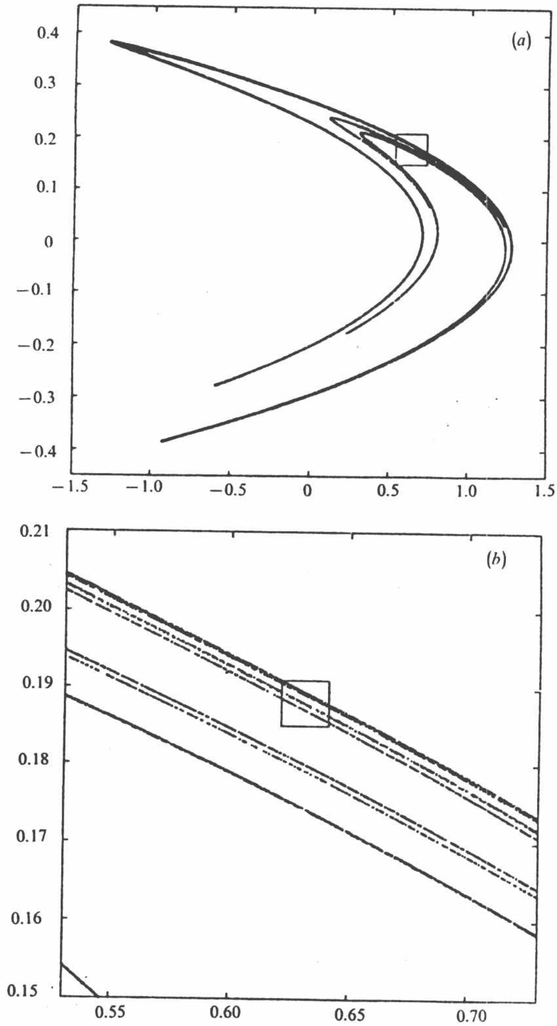


*Fig.* 1.1. *Several thousands of iterates of Eqs* (1.5) *for* $a = 1.4$, $b = 0.3$ plotted after ignoring several hundreds of initial iterations. (*a*) Entire attractor is covered by a single solution. (*b*) The magnified part of the attractor reveals a complex internal structure.

$$\frac{dy}{dt} = -xz + rx - y \,, \tag{1.6}$$

$$\frac{dz}{dt} = \quad xy - bz \,,$$

where $\sigma$, $r$, $b$ are positive real-valued parameters. The state space of Eqs (1.6) is three-dimensional. The divergence of the vector field on the right hand sides of Eqs (1.6) is $\operatorname{div} \boldsymbol{v}(\mathbf{x}) = -\sigma - 1 - b$ and thus, as in the Hénon system, the flow of Eqs (1.6) uniformly contracts the volumes of the state space.

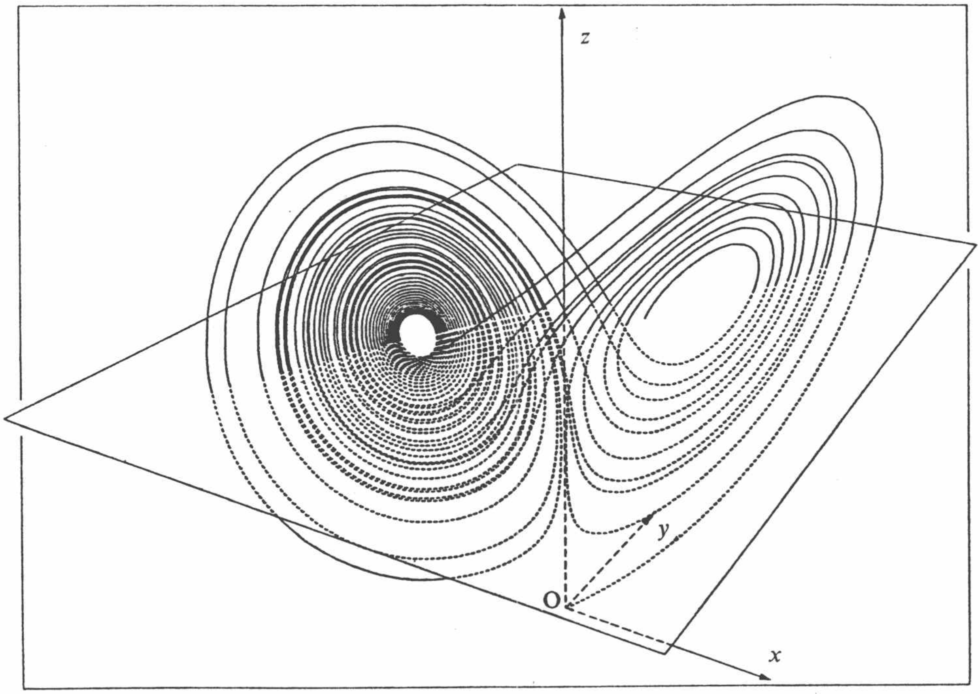


Fig. 1.2. The Lorenz attractor. The solution starting at $x = y = z = 0$ forms irregularly alternating loops to the left and to the right and quickly approaches the chaotic Lorenz attractor.

Despite its simplicity, the model yields chaotic oscillations at many parameter values, for example at $\sigma = 10$, $b = 8/3$, $r = 28$, see Fig. 1.2. Again this attractor has complex geometric structure as well as chaotic dynamics.

These two examples indicate the lowest possible dimension of the state space of a chaotic system. In the *discrete time* case given by Eq. (1.2), with $\boldsymbol{f}$ an *invertible* mapping, the dimension must be at least two, while the *continuous time* system (1.1) must have at least three-dimensional state space. If we consider a *noninvertible* mapping $f$ in Eq. (1.2) then even a one-dimensional state space will admit the existence of chaos.

An example of an infinite-dimensional system may be provided by the Mackey–Glass mathematical model of haematologic disorders[1.45, 1.17] formed by a differential equation with a time delay

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \lambda(\mathbf{x}(t - \tau)) - \gamma\, \mathbf{x}(t)\,, \tag{1.7}$$

where

$$\lambda(\mathbf{x}(t - \tau)) = \frac{\alpha \mathbf{x}(t - \tau)}{1 + (\mathbf{x}(t - \tau))^{\beta}}\,.$$

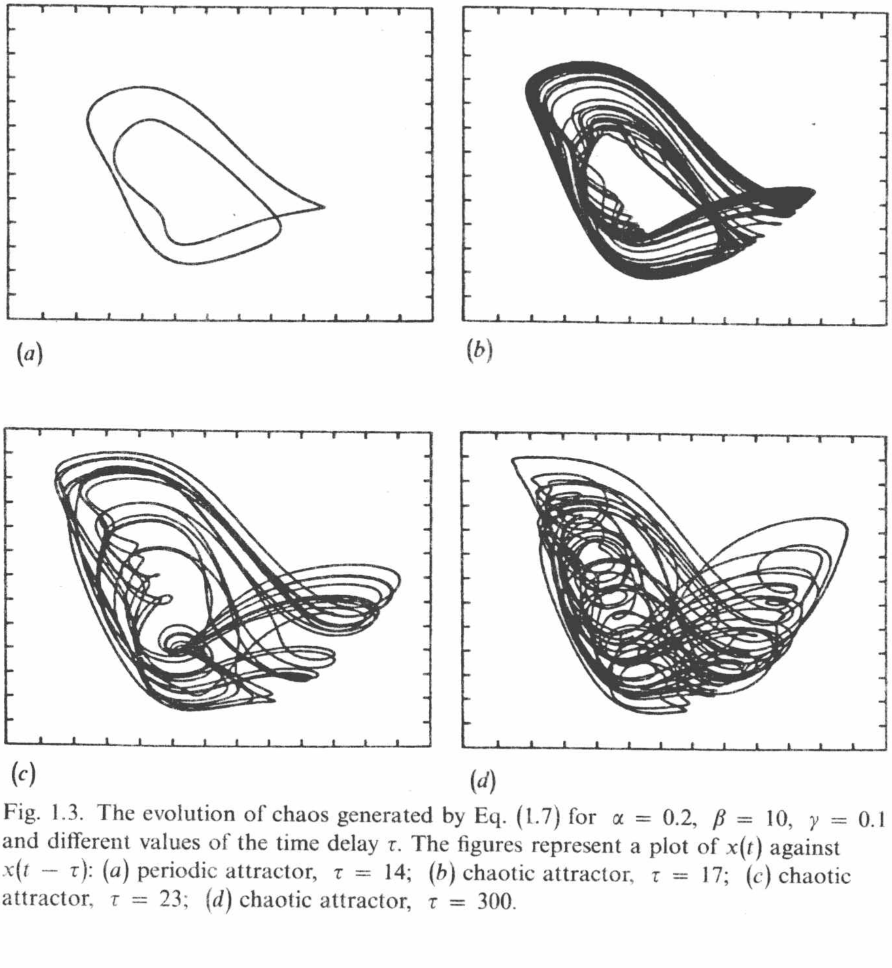

Fig. 1.3. The evolution of chaos generated by Eq. (1.7) for $\alpha = 0.2$, $\beta = 10$, $\gamma = 0.1$ and different values of the time delay $\tau$. The figures represent a plot of $x(t)$ against $x(t - \tau)$: (*a*) periodic attractor, $\tau = 14$; (*b*) chaotic attractor, $\tau = 17$; (*c*) chaotic attractor, $\tau = 23$; (*d*) chaotic attractor, $\tau = 300$.

Here $\mathbf{x}(t)$ is the concentration of blood cells at time $t$. Cells are lost from the circulation at a rate $\gamma$ and the flux $\lambda$ of cells into the circulation from the stem cell compartment depends on $\mathbf{x}$ at a delayed time $(t - \tau)$. Hence, the time derivative of $\mathbf{x}$ at time $t$ is dependent both on the actual state $\mathbf{x}(t)$ and on the delayed state $\mathbf{x}(t - \tau)$. To generate a solution of Eq. (1.7) we have to know all states in the time interval $[t - \tau, t]$. The evolution of asymptotic regimes from periodic to chaotic attractors is shown in Fig. 1.3 $(a)$–$(d)$. The onset of chaos is here accompanied by a sequence of periodic oscillations with a successively doubled period.

The existence of complex solutions of a system of ODEs was already known to H. Poincaré at the end of the last century[1.53]. Similar systems were later studied by Birkhoff[1.5] and in the 1940s by Littlewood and Cartwright[1.11]. The theory of chaos was developed during the 1960s for conservative systems (Kolmogorov–Arnold–Moser theorem[1.3]) and for special types of dissipative systems (Smale[1.58]). First demonstrations of chaos in numerical solutions of the systems of the type (1.1) and (1.2) appeared at the same time[1.35, 1.44]. At the beginning of the 1970s Ruelle and Takens[1.56] expressed an idea about the possibility of the description of turbulence by chaotic solutions (strange attractors). The term 'chaos' was first used by Li and Yorke[1.42].

Then there was a large increase in the number of studies on chaos. A number of review articles, conference proceedings, reprint collections and several monographs on the subject of chaos have appeared in the last several years. A list of some of them is provided in the literature references to this chapter.

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# 2

# Differential equations, maps and asymptotic behaviour

## 2.1 Time evolution and dynamical systems

Physical, chemical, biological or social phenomena can be seen as systems characterized by a time evolution of their properties. Such *evolution systems* are ubiquitous in nature. Often we are able to express the rate of change of the properties of a considered evolution system in the form of equations, applying and combining the relevant laws of nature. Solutions of the constructed *mathematical model* then mimic the time evolution of the real system.

Our aim is to predict this evolution using a proper mathematical model. An instantaneous *state* of the model system can be given by a finite set of numbers or by a finite set of functions. A set of all states of the system will be called a *state space* (in physics literature it is also called a *phase space*). A system will be considered as *deterministic* if its future and past are fully determined by its current state. In a *semideterministic* system only the future is uniquely determined, while in a *stochastic* system neither the past nor the future is unique (this type of system will not be treated here).

A system of bodies moving according to laws of classical mechanics, electronic circuits or interacting populations in a closed ecological system may be considered as deterministic systems. An isothermal chemical reaction in an ideally stirred (homogeneous) environment is another example of a deterministic system, while the consideration of molecular diffusion makes the system semideterministic.

A substantial difference between the last two examples of evolution systems is in the *dimension* of the corresponding state space. Let $\mathsf{X}$ denote the state space and $\mathbf{x}$ its elements (states of the system). The state of the system is, in the first case, described at a given time by $n$ values of concentrations $x_1, \ldots, x_n$ of components taking part in $n$ independent reaction steps. Hence $\mathbf{x}$ is an $n$-vector with the state space defined on a subset of $n$-dimensional *Euclidean space* $\mathsf{R}^n$. In the second case, the concentration of each of the $n$ reaction components is dependent on the spatial coordinates and also has to satisfy conditions imposed on the boundaries of the system. The corresponding state space consists of elements formed by $n$-tuples of concentration profiles satisfying the boundary conditions. As each element can be generally described by an infinite number of