Number 363

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## ABSTRACT

In this paper we study the trace ring of $m$ generic 2 by 2 matrices $\mathbb{T}_{m,2}$.It is shown that it is a polynomial ring over the generic Clifford algebra for $m$-ary quadratic forms of rank $\leq 3$. We prove that it is a Cohen-Macaulay module,i.e. it is a free module of finite rank over a polynomial subring of the center.This explains the existence of a functional equation for its Poincaré series.



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## CONTENTS

# INTRODUCTION

Throughout these notes, $F$ will be a field of characteristic zero, algebraically closed if necessary. $\mathbb{F}_m = F < x_1, \cdots, x_m >$ will be the free $F$-algebra in $m$ variables, i.e. $\mathbb{F}_m$ is the tensor algebra of a vector space of dimension $m$ over $F$.

If $I_{m,n}$ is the ideal of $\mathbb{F}_m$ of all identities satisfied by $n$ by $n$ matrices in $m$ variables, then

$$\mathbb{G}_{m,n} = F < x_1, \cdots, x_m > / I_{m,n}$$

is the ring of $m$ generic $n$ by $n$ matrices.

A more convenient description of this ring is obtained as follows. Let $\mathcal{P}_{m,n}$ be the commutative polynomial ring

$$\mathcal{P}_{m,n} = F[x_{ij}(l) \mid 1 \leq i,j \leq n; 1 \leq l \leq m]$$

and consider the so called generic matrices

$$X_l = (x_{ij}(l))_{i,j} \in M_n(\mathcal{P}_{m,n})$$

then $\mathbb{G}_{m,n}$ is the $F$-subalgebra of $M_n(\mathcal{P}_{m,n})$ generated by the elements $\{X_1, ..., X_m\}$.

It is well known, see for example [P1,Ch.4], that $\mathbb{G}_{m,n}$ is a domain and if we localize it at the multiplicative set of all nonzero central elements we obtain a division ring $\Delta_{m,n}$ which is of dimension $n^2$ over its center $\mathcal{K}_{m,n}$. Let $Tr : \Delta_{m,n} \rightarrow \mathcal{K}_{m,n}$ be the usual (reduced) trace morphism, then the trace ring of $m$ generic $n$ by $n$ matrices, $\mathbb{T}_{m,n}$, is the $F$-subalgebra of $\Delta_{m,n}$ generated by $\mathbb{G}_{m,n}$ and $Tr(\mathbb{G}_{m,n})$. There are two main motivations to study them :

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**a. Representation theory.**

We assume that $F$ is algebraically closed,for arbitrary fields similar results hold,see for example [AS]. An $n$-dimensional representation of the free algebra $\mathbb{F}_m$ is an algebra morphism

$$\phi : \mathbb{F}_m \rightarrow M_n(F)$$

Note that this is equivalent to giving $m$ elements $\phi(x_1), \cdots, \phi(x_m)$ of $M_n(F)$. Therefore, the set of all $n$-dimensional representations of $\mathbb{F}_m$ can be identified to the affine variety associated to $\mathcal{P}_{m,n}$ i.e. $A^{mn^2}$.

Two $n$-dimensional representations $\phi_1$ and $\phi_2$ are said to be equivalent if they differ only up to an $F$-automorphism of $M_n(F)$

$$\begin{array}{ccc} \mathbb{F}_m & \longrightarrow & M_n(F) \\ \downarrow & & \downarrow \\ \mathbb{F}_m & \longrightarrow & M_n(F) \end{array}$$

Therefore, $Aut_F(M_n(F)) = PGL_n(F)$ acts on $REP_n(\mathbb{F}_m) = A^{mn^2}$ and the orbits under this action are the equivalence classes of representations. To classify representations up to equivalence is thus the study of the orbit space of $A^{mn^2}$ under the action of $PGL_n(F)$.

If $\phi : \mathbb{F}_m \rightarrow M_n(F)$ is a representation then $F^n = F \oplus \cdots \oplus F$ ($n$ times) becomes an $\mathbb{F}_m$-module via $\phi$.If $F^n$ is completely reducible as a $\mathbb{F}_m$-module, then $\phi$ is said to be semi-simple.In general we can find a decomposition series

$$0 = V_t \subset \cdots \subset V_1 \subset V_0 = F^n$$

for $F^n$ as a $\mathbb{F}_m$-module. Then $W = \oplus(V_i/V_{i+1})$ is completely reducible and $dim_F(W) = n$. Choose a basis for $W$ as follows : the first $dim_F(V_{t-1})$-vectors form a basis for $V_{t-1}$, the next $dim_F(V_{t-2}/V_{t-1})$-vectors form a basis for $V_{t-2}/V_{t-1}$ and so on.With respect to this basis, $\phi$ will have the matrix-form

$$\phi = \begin{pmatrix} \phi_1 & & N \\ & \ddots & \\ 0 & & \phi_t \end{pmatrix}$$

where $\phi_j : \mathbb{F}_m \rightarrow M_{n_j}(F)$, for $n_j = dim_F(V_{j-1}/V_j)$, are the irreducible components of $\phi$. In particular $\phi_j$ is epimorphic, $n = n_1 + \cdots + n_t$ and $N$ is the nilradical of $\phi$. With $\phi$ we can therefore associate a semi-simple representation

$$\phi^{ss} = \phi_1 \oplus \cdots \oplus \phi_t$$

Artin has proved in [Ar] that $\phi^{ss}$ lies in the closure of the orbit of $\phi$ under action of $PGL_n(F)$, i.e. we are not able in any type of quotient variety of $A^{mn^2}$ by $PGL_n(F)$ to distinguish between $\phi$ and $\phi^{ss}$.

This motivates us to construct an affine variety whose $F$-points are in one-to-one correspondence with the equivalence classes of semi-simple $n$-dimensional representations of $\mathbb{F}_m$.Let $V$ be the variety corresponding to the center of $\mathbb{T}_{m,n}$,$\mathcal{R}_{m,n}$, which is an affine algebra by a result of Procesi's [P3,Th.3.3].Then the natural inclusion $\mathcal{R}_{m,n} \subset \mathcal{P}_{m,n}$ induces a map between the varieties

$$p : A^{mn^2} \to V$$

which is onto and $p(\phi) = p(\psi)$ if and only if $\phi^{ss} = \psi^{ss}$. Therefore

**THEOREM 0.1** (Artin-Procesi)

The points of $V = AFF(\mathcal{R}_{m,n})$ are in one-to-one correspondence with the equivalence classes of semi-simple $n$-dimensional representations of $\mathbb{F}_m$.

The trace ring $\mathbb{T}_{m,n}$ itself is also affine, so we can associate to it the set $Y = AFF(\mathbb{T}_{m,n})$ consisting of all maximal twosided ideals equipped with the usual induced Zariski topology. Since $\mathbb{T}_{m,n}$ is integral over $\mathcal{R}_{m,n}$,we have a surjection

$$\delta : Y \to V$$

It is easy to describe the fibers of this map. Let $\phi$ be an $F$-point in $V$,i.e. $\phi$ corresponds to an equivalence class of a semi-simple representation

$$\phi = \phi_1 \oplus \cdots \oplus \phi_t$$

then $\delta^{-1}(\phi)$ consists of maximal ideals of $\mathbb{T}_{m,n}$ corresponding to distinct irreducible components of $\phi$.

**THEOREM 0.2** (Artin-Schelter,[AS,Th.3.20])

The points of $Y = AFF(\mathbb{T}_{m,n})$ corresponds to couples $(\phi, \phi_j)$ where $\phi$ is a representant of an equivalence class of a semi-simple $n$-dimensional representation of $\mathbb{F}_m$ and $\phi_j$ is an irreducible component of $\phi$.

Further, one can verify that the dimension of $V$ , which is equal to the Krull dimension of the affine $F$-algebra $\mathcal{R}_{m,n}$ is $(m-1).n^2+1$.

**b. Invariant theory of $n$ by $n$ matrices.**

In the foregoing paragraph we have seen that there is an action of $GL_n(F)$, actually of $PGL_n(F)$ on $\mathcal{P}_{m,n}$.This action is defined in the following way : if $P \in GL_n(F)$ and if $X_l$ is the $l$-th generic matrix,then

$$P.X_l.P^{-1} = (\psi_{ij}(l))_{i,j}$$

where the $\psi_{ij}(l)$ are $F$-linear combinations of the $x_{ij}(l)$. Then,sending $x_{ij}(l)$ to $\psi_{ij}(l)$ induces an $F$-automorphism on $\mathcal{P}_{m,n}$ which we denote by $\alpha_P$.

A polynomial $f(x_{ij}(l)) \in \mathcal{P}_{m,n}$ is said to be an invariant of $m$ copies of $n$ by $n$ matrices iff $\alpha_P(f) = f$ for all $P \in GL_n(F)$. The set of all invariants is called the ring of invariants of $m$ copies of $n$ by $n$ matrices.

Artin [Ar] conjectured that any invariant is an element of $\mathcal{R}_{m,n}$. For $n = 2$, this fact was proved as far back as 1903 by J.H. Grace and A. Young [GY]. For arbitrary $n$, Artins conjecture was proved independently by Gurevich [Gu],Siberskii [Si] and Procesi [P3,Th.1.3].The proof of this result relies heavily on the so called "first fundamental theorem" on vector invariants [Gu,Th.16.2] which gives a generating set for the invariants of $m$ vectors and $m$ covectors, i.e. invariants of $GL_n(F)$ acting on the symmetric algebra of

$$(V^{\otimes m}) \otimes (V^*)^{\otimes m}$$

where $V$ is an $n$-dimensional $F$-vector space and $V^*$ is its dual. This theorem is quite old but the first complete proof seems to be that of Gurevich.The solution of Artins conjecture

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is a translation of the first fundamental theorem,using the dictionary

$$V \otimes V^* \simeq M_n(F)$$

$$\text{vector invariant} = \text{trace}$$

for details see [P3,pp 310-313].

**THEOREM 0.3** (Gurevich-Siberskii-Procesi)

The ring of invariants of $m$ copies of $n$ by $n$ matrices under action of $GL_n(F)$ is equal to $\mathcal{R}_{m,n}$.

We will now define an action of $GL_n(F)$ on $M_n(\mathcal{P}_{m,n})$. Let $P \in GL_n(F)$ and $(a_{ij})_{i,j} \in M_n(\mathcal{R}_{m,n})$ then there is an action by conjugation

$$(a_{ij})_{i,j} \to P.(a_{ij})_{i,j}.P^{-1}$$

and an action extending $\alpha_P$

$$(a_{ij})_{i,j} \to (\alpha_P(a_{ij}))_{i,j}$$

If we regard $M_n(\mathcal{P}_{m,n})$ as $M_n(F) \otimes \mathcal{P}_{m,n}$ then the first action is on the first factor,fixing the second, whereas the second action is vice-versa,thus the two actions commute.Note that the two actions agree on the generic matrices.Define

$$\beta_P : M_n(\mathcal{P}_{m,n}) \to M_n(\mathcal{P}_{m,n})$$

sending a matrix $(a_{ij})_{i,j}$ to $P^{-1}.(\alpha_P(a_{ij}))_{i,j}.P$. This defines an action of $GL_n(F)$ on $M_n(\mathcal{P}_{m,n})$. The ring of matrix-concomitants is by definition the fixed ring under this action.

**THEOREM 0.4** (Procesi)

The ring of matrix-concomitants is equal to the trace ring of $m$ generic $n$ by $n$ matrices,$\mathbb{T}_{m,n}$.

Apart from these general results, allmost nothing is known on the ringtheoretical structure of $\mathcal{R}_{m,n}$ and $\mathbb{T}_{m,n}$. In fact, the only trace ring for which an explicit description exists in the literature is $\mathbb{T}_{2,2}$,see [He] or [FH].We will recall their result.

Working with 2 by 2 matrices, one basicly uses only two identities

$$A^2 - Tr(A).A + D(A) = 0 \tag{1}$$

$$A.B + B.A = Tr(A.B) - Tr(A).Tr(B) + Tr(A).B + Tr(B).A \tag{2}$$

Consider the $F$-subalgebra $\mathcal{R}$ of $\Delta_{2,2}$ generated by the elements

$$\{Tr(X_1), Tr(X_2), D(X_1), D(X_2), Tr(X_1.X_2)\}$$

Using the identities given above, one can verify that the $F$-subalgebra $\mathcal{R}\{X_1, X_2\}$ of $\Delta_{2,2}$ is a finite module over $\mathcal{R}$ generated by the elements

$$\{1, X_1, X_2, X_1.X_2\} \tag{*}$$

Because

$$\mathbb{G}_{2,2} \subset \mathcal{R}\{X_1, X_2\} \subset \Delta_{2,2}$$

we get that $Kdim(\mathcal{R}) = trdeg_F(\mathcal{K}_{2,2}) = 5$.Therefore, the generating elements of $\mathcal{R}$ are algebraically independent,i.e. $\mathcal{R}$ is the polynomial ring

$$F[Tr(X_1), Tr(X_2), D(X_1), D(X_2), Tr(X_1.X_2)]$$

Further, since $\mathbb{G}_{2,2} \subset \mathcal{R}\{X_1, X_2\} \subset \mathbb{T}_{2,2}$ and $Tr(\mathcal{R}\{X_1, X_2\}) \subset \mathcal{R}\{X_1, X_2\}$ we get that $\mathbb{T}_{2,2} = \mathcal{R}\{X_1, X_2\}$. Let $\mathcal{K}$ be the field of fractions of $\mathcal{R}$,then the $\mathcal{K}$-dimension of $\Delta_{2,2}$ is smaller or equal to 4 since $\mathcal{K}\{X_1, X_2\}$ has $(*)$ as a generating set. On the other hand

$$dim_{\mathcal{K}}(\Delta_{2,2}) = dim_{\mathcal{K}_{2,2}}(\Delta_{2,2}).dim_{\mathcal{K}}(\mathcal{K}_{2,2}) = 4.dim_{\mathcal{K}}(\mathcal{K}_{2,2})$$

so $\mathcal{K} = \mathcal{K}_{2,2}$ and the set $(*)$ is linearly independent. This finishes the proof of

**THEOREM 0.5** (Procesi,Formanek-Halpin-Li)

(a) : $\mathcal{R}_{2,2} = F[Tr(X_1), Tr(X_2), D(X_1), D(X_2), Tr(X_1.X_2)]$

(b) : $\mathbb{T}_{2,2} = \mathcal{R}_{2,2}.1 \oplus \mathcal{R}_{2,2}.X_1 \oplus \mathcal{R}_{2,2}.X_2 \oplus \mathcal{R}_{2,2}.X_1.X_2$

Surprisingly,a similar result for $\mathbb{T}_{3,2}$ does not exists in the literature, apart from a description of $\mathcal{R}_{3,2}$ in [Fo,Th.22]. Nevertheless, in [LV] it is shown that one can describe $\mathcal{R}_{3,2}$ and $\mathbb{T}_{3,2}$ using the same methods as above.

Consider the $F$-subalgebra $\mathcal{R}$ of $\Delta_{3,2}$, the generic division algebra of 3 generic 2 by 2 matrices, generated by the elements

$$\{Tr(X_1), Tr(X_2), Tr(X_3), D(X_1), D(X_2), D(X_3), Tr(X_1X_2), Tr(X_2X_3), Tr(X_1X_3)\}$$

Using the identities (1) and (2) above, one verifies that the $F$-subalgebra of $\Delta_{3,2}$, $\mathcal{R}\{X_1, X_2, X_3\}$ is a finite module over $\mathcal{R}$ generated by the elements

$$(*) = \{1, X_1, X_2, X_3, X_1X_2, X_2X_3, X_1X_3, X_1X_2X_3\}$$

Since $\mathbb{G}_{3,2} \subset \mathcal{R}\{X_1, X_2, X_3\} \subset \Delta_{3,2}$, we get that $Kdim(\mathcal{R}) = trdeg_F(Z(\Delta_{3,2})) = 9$. Therefore, the generating elements of $\mathcal{R}$ are algebraically independent,i.e. $\mathcal{R}$ is the polynomial ring

$$F[Tr(X_1), Tr(X_2), Tr(X_3), D(X_1), D(X_2), D(X_3), Tr(X_1X_2), Tr(X_2X_3), Tr(X_1X_3)]$$

Further, $\mathbb{G}_{3,2} \subset \mathcal{R}\{X_1, X_2, X_3\} \subset \mathbb{T}_{3,2}$ and $Tr(\mathcal{R}\{X_1, X_2, X_3\}) \subset \mathcal{R}\{X_1, X_2, X_3\}$. This entails that $\mathbb{T}_{3,2} = \mathcal{R}\{X_1, X_2, X_3\}$. Now, let $\mathcal{K}$ be the field of fractions of $\mathcal{R}$, then

$$dim_K(\Delta_{3,2}) = dim_K(K\{X_1, X_2, X_3\}) \leq 8$$

because $K\{X_1, X_2, X_3\}$ has generating set (*).Further, $dim_K(Z(\Delta_{3,2})) \geq 2$ because $Tr(X_1X_2X_3) \notin \mathcal{K}$. For otherwise, because $Tr(X_1X_2X_3)$ is linear in each of the generic matrices this would entail that $Tr(X_1X_2X_3)$ can be written as

$$\alpha Tr(X_1)Tr(X_2)Tr(X_3) + \beta(Tr(X_1)Tr(X_2X_3) + Tr(X_2)Tr(X_1X_3) + Tr(X_3)Tr(X_1X_2))$$

and by specializing

$$X_1 \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad X_2 \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad X_3 \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

one obtains a contradiction.Combining all this,we get that

$$8 \leq dim_K(\mathcal{K}_{3,2}).dim_{\mathcal{K}_{3,2}}(\Delta_{3,2}) = dim_K(\Delta_{3,2}) \leq 8$$

i.e. the set (*) is linearly independent over $\mathcal{K}$ or $\mathcal{R}$. Taking traces in the identity

$$(X_1X_2X_3)^2 - Tr(X_1X_2X_3)X_1X_2X_3 + D(X_1)D(X_2)D(X_3) = 0$$

and simplifying the first term we get that $Tr(X_1X_2X_3)$ satisfies the quadratic equation

$$X^2 - A.X + B = 0 \qquad (**)$$

where

$$A = Tr(X_1)Tr(X_2X_3) + Tr(X_2)Tr(X_1X_3) + Tr(X_3)Tr(X_1X_2) - Tr(X_1)Tr(X_2)Tr(X_3)$$

$$\begin{aligned} B = {} & D(X_1)Tr(X_2X_3)^2 + D(X_2)Tr(X_1X_3)^2 + D(X_3)Tr(X_1X_2)^2 \\ & - Tr(X_1)Tr(X_2)Tr(X_1X_2)D(X_3) - Tr(X_1)Tr(X_3)Tr(X_1X_3)D(X_2) \\ & - Tr(X_2)Tr(X_3)Tr(X_2X_3)D(X_1) \\ & + Tr(X_1)^2D(X_2)D(X_3) + Tr(X_2)^2D(X_1)D(X_3) + Tr(X_3)^2D(X_1)D(X_2) \\ & - 4.D(X_1)D(X_2)D(X_3) + Tr(X_1X_2)Tr(X_1X_3)Tr(X_2X_3) \end{aligned}$$

**THEOREM 0.6 :** If $\mathcal{R}$ is the polynomial ring

$$F[Tr(X_1), Tr(X_2), Tr(X_3), D(X_1), D(X_2), D(X_3), Tr(X_1X_2), Tr(X_2X_3), Tr(X_3X_1)]$$

(1) : $\mathcal{R}_{3,2}$ is the free $\mathcal{R}$-module of rank 2 generated by $\{1, Tr(X_1X_2X_3)\}$ and $Tr(X_1X_2X_3)$ satisfies the quadratic equation (**) over $\mathcal{R}$.

(2) : $\mathbb{T}_{3,2}$ is the free $\mathcal{R}$-module of rank 8 generated by $\{1, X_1, X_2, X_3, X_1X_2, X_1X_3, X_2X_3, X_1X_2X_3\}$

Being free over a polynomial subring of the center, $\mathbb{T}_{3,2}$ is a reflexive module over $\mathcal{R}_{3,2}$.Further, we claim that the localization of $\mathbb{T}_{3,2}$ at a central height one prime ideal is an Azumaya algebra For, such a prime $p$ cannot contain simultaneously the elements

$(X_1X_2 - X_2X_1)^2$,$(X_1X_3 - X_3X_1)^2$ and $(X_2X_3 - X_3X_2)^2$ belonging to the Formanek center, whence $(\mathbb{T}_{3,2})_p$ is a localization of an Azumaya algebra.This proves that $\mathbb{T}_{3,2}$ is a reflexive Azumaya algebra and entails that $\mathbb{T}_{3,2}$ cannot be a free module over $\mathcal{R}_{3,2}$ since this would imply that $\mathbb{T}_{3,2}$ is an Azumaya algebra ,but dividing out the commutator ideal one finds a commutative epimorphic image.

For $m$ generic 2 by 2 matrices one would similarly like to consider the $F$-subalgebra $\mathcal{R}$ of $\Delta_{m,2}$ generated by the elements $Tr(X_i), D(X_i), Tr(X_iX_j)$. But, for $m \geq 4$, $\mathcal{R}$ can never be a polynomial ring since the number of generators is $2m + \binom{m}{2}$ whereas the $Kdim(\mathcal{R}) = 4m - 3$. Therefore, a similar approach fails for $m \geq 4$. Nevertheless, one can ask whether a similar structural result holds

**QUESTION** : Is the trace ring of $m$ generic 2 by 2 matrices a free module of finite rank over a polynomial subring of its center ?

Because $\mathbb{T}_{m,2}$ is the fixed $F$-algebra under action of the reductive linear algebraic group $GL_2(F)$ on an $F$-algebra of finite global dimension $M_2(\mathcal{P}_{m,2})$, this question could be answered affirmatively, provided one has a noncommutative version of the famous Hochster-Roberts theorem,cfr. III.2. At present, it is not clear how one might prove such a result since the main tool of the commutative proof, i.e. reduction to finite characteristic and investigation of the Frobenius morphism on local cohomology, clearly does not generalize (directly) to the noncommutative setting.

Therefore, we had to find another approach. In the first chapter we recall some basic results on the invariant theory of the (special) orthogonal groups. We give an explicit proof of an observation due to C.Procesi that the center $\mathcal{R}_{m,2}$ of the trace ring is a polynomial ring over the ring of invariants $\mathcal{R}^o_m$ of the special orthogonal group $SO_3(F)$. This enables us to give another proof of the fact that $\mathcal{R}_{m,2}$ is the ring of invariants of $m$ copies of 2 by 2 matrices under componentswise action of $GL_2(F)$. Similarly, the trace ring $\mathbb{T}_{m,2}$ is the polyncmial ring

$$\mathbb{T}_{m,2} = \mathbb{T}^o_m[Tr(X_1), \cdots, Tr(X_m)]$$

where $\mathbb{T}_m^o$ is the $F$-subalgebra of $\mathbb{T}_{m,2}$ generated by the generic trace zero matrices

$$X_i^o = X_i - \frac{1}{2}Tr(X_i)$$

Further, we recall Procesi's description of $\mathcal{R}_m^o$ and $\mathbb{T}_m^o$ using standard Young tableaux. Using these results, we have reduced our question to the corresponding question for $\mathbb{T}_m^o$.

From the identity (2) above, we get that the generators of $\mathbb{T}_m^o$ satisfy the relation

$$X_i^o X_j^o + X_j^o X_i^o = Tr(X_i^o X_j^o)$$

This motivates us to construct and study the iterated Öre extension

$$\Lambda_m = F[a_{ij} : 1 \leq i < j \leq m][a_1][a_2, \sigma_2, \delta_2]...[a_m, \sigma_m, \delta_m]$$

where $\sigma_j(a_i) = -a_i$ and $\delta_j(a_i) = 2a_{ij}$ for all $i < j$ and trivial action on the other variables. Then sending $a_i$ to $X_i^o$ and $a_{ij}$ to $\frac{1}{2}Tr(X_i^o X_j^o)$ we get an epimorphism

$$\phi_m : \Lambda_m \to \mathbb{T}_m^o$$

It is possible to describe the kernel of this morphism in the following way. $\Lambda_m$ turns out to be the Clifford algebra over the commutative polynomial ring

$$S_m = F[a_{ij} : 1 \leq i \leq j \leq m]$$

where $a_{ii} = a_i^2$ , i.e. the coordinate ring of the variety of all symmetric $m$ by $m$ matrices over $F$, associated to the quadratic form

$$\sum_{i,j} a_{ij} X_i X_j$$

For this reason, we christen $\Lambda_m$ the generic Clifford algebra $Cl_m$. This observation combined with the classical structure theory of Clifford algebras makes it possible to describe the prime ideal structure of $Cl_m$. In particular we get that there is only one prime ideal of $Cl_m$ lying over the ideal of $S_m$ generated by all $k$ by $k$ minors of the (generic) symmetric $m$ by $m$ matrix

$$\mathcal{A} = (a_{ij})_{i,j}$$

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