SÉMINAIRE DE MATHÉMATIQUES SUPÉRIEURES
SÉMINAIRE SCIENTIFIQUE OTAN (NATO ADVANCED STUDY INSTITUTE)

CONVOLUTIONS IN GEOMETRIC FUNCTION THEORY

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To my Friends and Colleagues
in Afghanistan

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INTRODUCTION

For two functions f analytic in $|z| < R_1$, g analytic in $|z| < R_2$ and represented by their power series expansions

$$(0.1) \qquad f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad g(z) = \sum_{k=0}^{\infty} b_k z^k ,$$

let $f * g$ denote the function

$$(0.2) \qquad (f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k .$$

A simple calculation shows that $f * g$ is analytic in $|z| < R_1 R_2$. It is called the *Hadamard product* of f and g in honor of J. Hadamard's famous theorem concerning the location of singularities of $f * g$ in terms of the singularities of the "factors". He used the alternative representation as a convolution integral:

$$(0.3) \qquad (f * g)(z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} f(z/\zeta) g(\zeta) d\zeta/\zeta, \quad |z|/R_1 < \rho < R_2 .$$

For this reason $f * g$ is also called the *convolution* of f and g.

The constant theme in this book is to study properties of operators

$$f * \; : \; \mathcal{B} \to \mathcal{A} ,$$

where f is an element of the set $\mathcal{A}$ of functions analytic in the unit disc $U = \{|z| < 1\}$ and $\mathcal{B} \subset \mathcal{A}$. In particular, we try to characterize operators which send a certain given $\mathcal{B} \subset \mathcal{A}$ into itself. An important classical example of such a result is the following (unit disc version of a) theorem due to Szegö [85] (a corollary to the famous theorem of Grace [23]).

THEOREM 0.1: *For* $n \in \mathbf{N}$ *let*

$$f(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k , \quad g(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$$

be nonvanishing in U. *Then*

$$h(z) = \sum_{k=0}^{n} \binom{n}{k} a_k b_k z^k$$

has the same property.

Note that it is possible to state Theorem 0.1 as a convolution theorem characterizing functions which preserve the class of nonvanishing polynomials of fixed degree. Other operators preserve the range, the univalence, geometric properties of the image domains, certain norms, etc. A stimulating result in this direction was Robertson's convolution theorem for typically real functions (i.e., functions $f \in \mathcal{A}$ with $f(0) = 0$, $f'(0) = 1$, $\operatorname{Im} f(z) \cdot \operatorname{Im} z \geq 0$ in U).

THEOREM 0.2 [45]: *If*

$$f(z) = \sum_{k=1}^{\infty} a_k z^k , \quad g(z) = \sum_{k=1}^{\infty} b_k z^k$$

are typically real, then

$$h(z) = \sum_{k=1}^{\infty} \frac{a_k b_k}{k} z^k$$

has the same property.

Since this theorem has many applications, among them a simple solution of the coefficient problem for these functions, it was hoped that the class S of normalized univalent functions in A or certain subclasses of S share this invariance property (Mandelbrojt-Schiffer, Pólya-Schoenberg). The attempts to solve these problems produced a number of important general insights into properties of Hadamard products which are discussed in the following chapters.

The following notion turned out to be useful: let A_0 consist of the functions $f \in A$ with $f(0) = 1$. Then for $V \subset A_0$ define the *dual set*

(0.4) $$V^* = \{g \in A_0 \mid \forall f \in V: \ (f * g)(z) \neq 0 \ \text{ in } U\} ,$$

and $V^{**} = (V^*)^*$, the second dual. For instance, Theorem 0.1 has the equivalent formulation

(0.5) $$\{(1 + z)^n\}^{**} = \{P \in A_0 \mid P \text{ polynomial of degree } \leq n, P(z) \neq 0 \ \text{ in } U\} .$$

The "duality principle" states that under fairly weak conditions on V, many linear (and other) extremal problems in V^{**} are solved in V. This is a useful information since in many cases of interest V^{**} is much larger than V (compare (0.5)), and various classical theorems from different fields can be obtained by a unified approach.

Most of the results in this book are no more than ten years old (a considerable number of them have not even been published before) and many parts of the theory are still developing and have not yet found a final form. Although it was impossible to include every result in the field, I have tried to give a fairly complete survey of the available material.

These notes are an enlarged version of a series of lectures delivered at the Séminaire de mathématiques supérieures, Université de Montréal, August 1981. I should like to thank the organizers of this conference, Prof. Q.I. Rahman and Prof. G. Sabidussi, for the opportunity to present this part of convolution theory.

Würzburg, January 1982 St. Ruscheweyh

Chapter 1

DUALITY

1.1. The duality principle

We are using dual sets as defined in 0.4. With the topology of compact convergence in U the space A is a locally convex separated topological vector space. The space Λ of continuous linear functionals on A is described in the following basic theorem of Toeplitz [88].

THEOREM A: *$\lambda \in \Lambda$ if and only if there is a function g analytic in $|z| \leq 1$ such that for $f \in A$*

$$\lambda(f) = (g * f)(1) . \tag{1.1}$$

The correspondence (1.1) is denoted by $\lambda \doteqdot g$. A subset $V \subset A_0$ is said to be *complete* if it has the following property:

$$f \in V \Longrightarrow \forall\, |x| \leq 1:\ f_x \in V . \tag{1.2}$$

Here we used the notation $f_x(z) = f(xz)$, $z \in U$. Note that any dual set is complete (and closed).

THEOREM 1.1 (Duality principle, [50]): *Let $V \subset A_0$ be compact and complete. Then*

(1.3) $$\forall\ \lambda \in \Lambda:\ \ \lambda(V) = \lambda(V^{**})\ ,$$

(1.4) $$\overline{co}(V) = \overline{co}(V^{**})\ .$$

($\overline{co}$ stands for the closed convex hull of a set.)

PROOF: Since $V \subset V^{**}$ we have $\lambda(V) \subset \lambda(V^{**})$, $\lambda \in \Lambda$. To prove the inverse inclusion we need to show that $a \notin \lambda(V)$ implies $a \notin \lambda(V^{**})$ for $a \in \mathbf{C}$, $\lambda \in \Lambda$, and clearly we can restrict ourselves to the case $a = 0$. Thus consider $\lambda \in \Lambda$, $\lambda \doteqdot g$, with $0 \notin \lambda(V)$. g is analytic in $|z| < R$ for a certain $R > 1$. The compactness of V then shows that

$$U = \{f * g \mid f \in V\}$$

is a compact set of analytic functions in $|z| < R$. Our assumption on λ and (1.1) gives $u(1) \neq 0$ for $u \in U$. By compactness we conclude that the same is true in a certain neighborhood of 1, in particular in a point x_0 with $1 < x_0 < R$. Let $\tilde{g}(z) = g(x_0 z)$. Then $\tilde{g}$ is analytic in $|z| \leq 1$ and for $f \in V$ we obtain $(\tilde{g} * f)(1) = (g * f)(x_0) \neq 0$. Since V is complete we have for $|x| < 1$ and $f \in V$: $(\tilde{g} * f)(x) = (\tilde{g} * f_x)(1) \neq 0$. Thus $\tilde{g} \in V^*$ and for arbitrary $z \in U$, $f \in V^{**}$, we get $(\tilde{g} * f)(z) \neq 0$. The choice $z = 1/x_0$ then gives

$$\lambda(f) = (g * f)(1) = (\tilde{g} * f)(1/x_0) \neq 0$$

for $f \in V^{**}$. This proves (1.3). Now assume $\overline{co}(V) \neq \overline{co}(V^{**})$. Then by a separation theorem in locally convex separated topological vector spaces (compare [30, section 20, 7.(1); section 16, 3.(1)]) there exists $\lambda \in \Lambda$ which separates elements of V^{**} from V. This is impossible by (1.3).

Although (1.4) indicates a certain connection of duality and convexity it turns out that the second dual V^{**} is in general more closely related to a set