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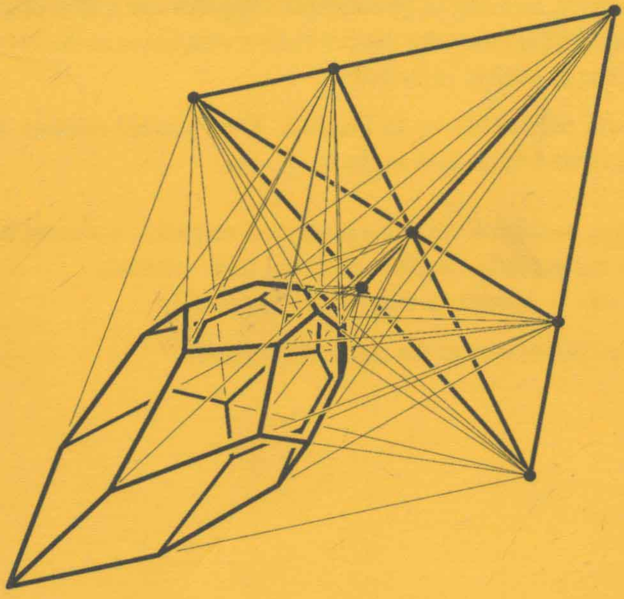
Jürgen Richter-Gebert

Realization Spaces of Polytopes



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For Ingrid and Angela-Sophia,
who helped me so much.

Preface

Steinitz's Theorem (proved in 1922) is one of the oldest and most prominent results in polytope theory. It gives a completely combinatorial characterization of the face lattices of 3-dimensional polytopes. Steinitz observed that the technique of proving his theorem also implies that for any 3-dimensional polytope the set of all its *realizations* is a trivial topological set. In other words: *realization spaces of 3-dimensional polytopes are contractible.* For a long time it was an open problem whether there exist similar results in spaces of dimension greater than three. It was proved by Mnëv in 1986 that the contrary is the case. As a consequence of his famous Universality Theorem for oriented matroids he showed that *realization spaces of polytopes with dimension-plus-four vertices can have arbitrary homotopy type.* The present research monograph studies the structure of realization spaces of polytopes in *fixed dimension.* The main result that is obtained is a Universality Theorem for 4-polytopes. It states that *for every primary basic semialgebraic set V there exists a 4-dimensional polytope whose realization space is stably equivalent to V.*

This research monograph has three goals. First of all it serves as a comprehensive source for all results that I have been able to obtain in connection to the Universality Theorem for 4-polytopes. It includes complete proofs of all these results including a proof of the Universality Theorem itself. Secondly, it is (as the title says) meant as an introduction to the beautiful theory of realization spaces of polytopes. For that purpose also a treatment of Steinitz's Theorem is included. Although the result is classical the proof presented here contains some new and fresh elements. In particular, we provide a new proof for Tutte's Theorem on equilibrium representations of planar graphs. We also give a complete proof of Mnëv's Universality Theorem for oriented matroids (and of its generalization: the Universal Partition Theorem). Last but not least, this monograph is written for the sake of enjoyment of geometric constructions. Most of the concepts and constructions that are needed here are elementary in nature. The final construction for the Universality Theorem is obtained by building larger and larger polytopal units of increasing geometric and algebraic complexity. We start from small incidence configurations, go to polytopes for addition and multiplication, and end up with polytopes that encode entire polynomial inequality systems. I hope that the reader can feel the fun that lies in these constructions.

There are many alternative ways of approaching the main results of this monograph. In particular, there are several different ways to build up the proof of the Universality Theorem for 4-polytopes. However, all the approaches kown to me rely on similar principles:

- first construct small and useful polytopes (using *Lawrence Extensions* or similar techniques) that have non-prescribable facets (or vertex figures),
- use *connected sums* to join these polytopes to larger units that are capable of encoding arithmetic operations,
- finally use *connected sums* to join these arithmetic units into even larger polytopes that encode entire polynomial inequality systems.

Here I have chosen an approach that is very modular. The basic building blocks are *very* simple polytopes, and the whole complexity is governed by the way of composing these blocks.

In order to obtain the strongest possible results it was necessary to set up a new concept of *stable equivalence* that compares realization spaces with other semialgebraic sets. The reader may excuse the fact that whenever stable equivalence between two spaces is proved the exposition becomes a bit technical. Everywhere else I used concrete geometric approaches rather than abstract settings. Whenever it is possible the constructions are carried out in an explicit manner.

Part I to Part III are based on my Habilitationsschrift at the Technical University Berlin, 1995. The typesetting of this monograph relies on LaTeX. Most of the drawings are done with *Cinderella*.

There are many people who have made the writing of this monograph possible. First of all I want to thank Günter M. Ziegler for offering me a position where I could concentrate mainly on this work. I am extremely grateful to him for his careful reading of every page and for the uncountably many valuable suggestions, discussions, comments and protests that encouraged me to go always one step further than I had already done.

Also I am very grateful to Anders Björner, Marie-Françoise Coste-Roy, Henry Crapo, Eva-Maria Feichtner, Eli Goodman, Martin Henk, Peter Kleinschmidt, Ulli H. Kortenkamp, Peter McMullen, Ricky Pollack, Jörg Rambau, and Bernd Sturmfels for many inspiring discussions and valuable comments on my manuscript in its various stages.

I especially, want to thank my wife Ingrid and my little daughter Angela-Sophia, who was born on the day of the "breakthrough" for the main theorem. Angela-Sophia's inspiring presence definitely helped me to keep my thoughts as simple as possible. Without Ingrid I would have never been able to write all this. She always had an open ear for me that helped me to clarify my ideas, and she accompanied me through all the "dead ends" that are unavoidable in such a kind of work.

Berlin, October 1996 JÜRGEN RICHTER-GEBERT

Contents

INTRODUCTION

1 Polytopes and their Realizations

Polytopes have a long tradition as objects of mathematical study. Their historical roots reach back to the ancient Greek mathematicians, having a first highlight in their enumeration of the famous Platonic Solids. Already at this point strong impetus came from the fact that polytopes intimately connect topics from geometry and from combinatorics (the Platonic Solids solve a first enumerative question in polytopal geometry, to find all polytopes with a *flag transitive* symmetry group — a combinatorial concept). The work presented in this research monograph is also motivated from questions that are on the borderline of geometry, algebra and combinatorics. We investigate the structure of the *realization spaces* of polytopes with fixed combinatorial types. Our aim is to exhibit a radical contrast between the behavior of realization spaces for polytopes in dimensions three and four.

For three-dimensional polytopes the structure of the realization spaces turns out to be rather simple (a consequence of the classical *Steinitz's Theorem* that was already known in 1922). However, realization spaces of four-dimensional polytopes can behave as complicated as one can think of (as a consequence of the *Universality Theorem* first presented in this monograph). We will give complete proofs of these two theorems and explore their far reaching consequences.

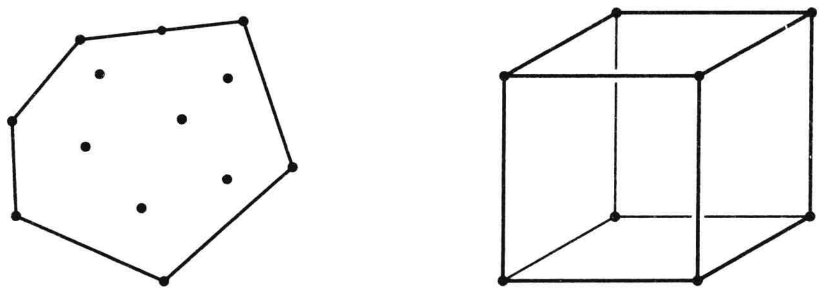
1.1 Polytopes

Formally, polytopes are the convex hulls of finite point sets in $\mathbb{R}^d$:

DEFINITION 1. Let $\boldsymbol{P} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) \in \mathbb{R}^{d \cdot n}$ be a finite collection of points that affinely span $\mathbb{R}^d$. The set

$$P = \mathbf{conv}(\boldsymbol{P}) := \Big\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{p}_i \mid \sum_{i=1}^{n} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for } i = 1, \ldots, n \Big\},$$

the *convex hull* of the point set $\boldsymbol{P}$, is called a *d-dimensional polytope* (a "*d-polytope*" for short). The *faces* of P are P itself and the intersections $P \cap A$, such that A is an affine hyperplane that does not meet the interior of P. The *face lattice* of P is the set of all faces of P, partially ordered by inclusion.



(a) (b)

Figure 1: A convex polygon and a cube. Two simple examples of polytopes.

While a polytope is a geometric object, its face lattice is purely combinatorial in nature. Figure 1(a) illustrates a 2-polytope as the convex hull of a finite number of points in the plane. We see that those points that are not in an extreme position make no contribution to the polytope itself. The points in extreme position (i.e., the 0-dimensional faces) are the *vertices* of a polytope. Figure 1(b) shows a cube as an example of a 3-dimensional polytope. The face lattice of the cube consists of the cube itself, 6 facets, 12 edges, 8 vertices and the empty set.

The need to structure the set of all polytopes of a fixed dimension leads to two main lines of study:

- to list all possible combinatorial types of polytopes (in other words, to determine which finite lattices correspond to face lattices of polytopes, and which do not),
- to describe the set of all realizations of a given combinatorial type.

The "set of all realizations" of a combinatorial type is formalized below by the concept of the *realization space* of a polytope. Besides their intrinsic importance for questions of real discrete geometry, such spaces appear in subjects as diverse as algebraic geometry (moduli spaces), differential topology (see Cairns' smoothing theory [21]), and nonlinear optimization (see Günzel et al. [33]).

Assume that in Definition 1 each point $\boldsymbol{p}_i$ for $i = 1, \ldots, n$ is a vertex of P. A *realization* of a polytope P is a polytope $Q = \mathbf{conv}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$ such that the face lattices of P and Q are isomorphic under the correspondence $\boldsymbol{p}_i \to \boldsymbol{q}_i$. The sequence of vertices $B = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{d+1})$ is a *basis* of $\boldsymbol{P}$ if these points are affinely independent in any realization of P.

DEFINITION 2. Let $P = \mathbf{conv}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) \subset \mathbb{R}^d$ be a d-polytope with n vertices and with a basis $B = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{d+1})$. The *realization space* $\mathcal{R}(P, B)$ is the set of all matrices $\boldsymbol{Q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \in \mathbb{R}^{d \cdot n}$ for which $\mathbf{conv}(\boldsymbol{Q})$ is a realization of P and $\boldsymbol{q}_i = \boldsymbol{p}_i$ for $i = 1, \ldots, d+1$.

By the choice of a certain basis for which the points have to stay fixed, we factor out components in the set of all realizations of a polytope that come from rotations and translations. It turns out that the realization space $\mathcal{R}(P, B)$ is essentially (up to "stable equivalence," see below) independent of the choice of an admissible basis. Hence it makes sense to speak of *the* realization space $\mathcal{R}(P)$ of a polytope.

Every realization space is a *primary basic semialgebraic set*: it is the set of solutions of a finite system of polynomial equations $f_i(x) = 0$ and strict inequalities $g_j(x) > 0$, where the f_i and g_j are polynomials with integer coefficients on $\mathbb{R}^{d \cdot n}$. To see this, one checks that the realization space is the set of all matrices $\boldsymbol{Q} \in \mathbb{R}^{d \cdot n}$ for which some entries are fixed, and the determinants of certain $d \times d$ minors have to be positive, negative, or zero.

Our main aim here is a Universality Theorem for 4-polytopes, stating that for every primary semialgebraic V set there exists a 4-polytope whose realization space is "stably equivalent" to V. The concept of *stable equivalence* will be clarified in Section 2. It can be considered as a strengthened version of homotopy equivalence that preserves also information on the underlying algebraic structure. In particular, if two semialgebraic sets V and V' are stably equivalent and V contains non-rational points, then V' contains non-rational points as well.

1.2 History I: Steinitz's Theorem

What does the realization space of a polytope look like? Which algebraic numbers are needed to coordinatize the vertex set of a given d-dimensional polytope? How can one tell whether a finite lattice is the face lattice of a polytope or not?

For 3-dimensional polytopes, Steinitz's work [56, 55] answered the basic questions about realization spaces more then seventy years ago. In particular, Steinitz's "Fundamentalsatz der konvexen Typen" (today known as *Steinitz's Theorem*) and its modern relatives (see [31] and [65]) provide complete answers to the above questions for this special case.

STEINITZ'S THEOREM (1922): *A graph G is the edge graph of a 3-polytope if and only if G is simple, planar and 3-connected.*

The classical proof by Steinitz is done by a clever combinatorial reduction technique that allows one to generate larger 3-polytopes from smaller ones. Alternatively, Steinitz's Theorem can be proved using the *Koebe-Andreev-Thurston Circle Packing Theorem* (see [65]), or by arguments using the concept of *self-stresses of planar graphs* (see [23, 36, 47], and Part IV). The statements in the following list can be derived from a careful inspection of the known proofs of Steinitz's Theorem.

- For every 3-polytope $P \subseteq \mathbb{R}^3$ the realization space $\mathcal{R}(P)$ is a smooth open ball. (This ball has dimension $e-6$, if P has e edges.)
- For every 3-polytope P the space $\mathcal{R}(P)$ contains rational points, that is, every 3-polytope can be realized with integral vertex coordinates.
- Every combinatorial 2-sphere is polytopal.
- BARNETTE, GRÜNBAUM, 1970, [12]: The shape of one 2-face in the boundary of a 3-polytope P can be arbitrarily prescribed, that is, the canonical map $\mathcal{R}(P) \to \mathcal{R}(F)$ is surjective for every facet $F \subseteq P$.
- ONN, STURMFELS, 1992, [51]: If a 3-polytope has n vertices then it can be realized with integral coefficients smaller than n^{169n^3}.

In Part IV of this monograph we will present a proof of Steinitz's Theorem that is based on the self stress approach. This approach also proves that the realization space of any 3-polytope is contractible, and that it contains rational points. In particular, our treatment will improve the bound given by Sturmfels and Onn.

- Any 3-polytope P with n vertices can be realized with integral coordinates smaller than 2^{18n^2}.
- If P furthermore contains a triangle, then it can be realized with integral coordinates smaller than 43^n.

One can prove statements similar to the above corollaries for d-polytopes that have at most $d+3$ vertices. Under (affine) Gale duality these polytopes are encoded by certain point arrangements on a line. This fact leads to a classification method that allows one to analyze these polytopes. Most of this analysis has been done by Mani [44] and Kleinschmidt [41].

- Every combinatorial $(d-1)$-sphere with $d+3$ vertices is polytopal.
- Every d-polytope with $d+3$ vertices can be realized with integral coefficients.
- The realization space of every d-polytope with $d+3$ vertices is contractible.

1.3 History II: Polytopes in Dimension Higher than 3

Over the years, it became clear that no similar positive answer could be expected for high-dimensional polytopes. The situation becomes much more complicated if either the dimension or the codimension exceeds three. We first discuss the case of fixed dimension. There are several d-polytopes (with $d \geq 3$) known that behave differently from 3-polytopes with respect to realizability. The following list summarizes chronologically the counterexamples that are found to contrast with the 3-dimensional case.

- PERLES, 1967, [31]:
  Non-rational 8-polytope (12 vertices, 28 facets).

- BARNETTE, 1971, [8]:
  Non-polytopal combinatorial 3-sphere (8 vertices, 19 facets).
- KLEINSCHMIDT, 1976, [40]:
  4-polytope with non-prescribable 3-face (10 vertices, 15 facets).
- BARNETTE, 1980, [11]:
  4-polytope with non-prescribable 3-face (12 vertices, 7 facets).
- BOKOWSKI, EWALD, KLEINSCHMIDT, 1984, [15, 16]:
  4-polytope with disconnected realization space (10 vertices, 28 facets).
- ZIEGLER, 1992, [65]:
  5-polytope with non-prescribable 2-face (12 vertices, 10 facets).

Besides these "sporadic examples," no general construction technique was known to produce polytopes with a "controllably bad" behavior for any fixed dimension. The σ-construction presented in [57] for that purpose turned out to be incorrect [65].

If we investigate the case of codimension four much more is known and general tools are applicable. In 1986 N.E. Mnëv proved a *Universality Theorem for oriented matroids* of rank 3 (see [7, 33, 52, 48, 49]). This result leads, via Gale diagram techniques, to a universality theorem for d-polytopes with $d+4$ vertices: in general for such polytopes the realization spaces can be arbitrarily complicated. In technical terms the Universality Theorem can be stated as:

MNËV'S UNIVERSALITY THEOREM (1986):

(i) *For every primary basic semi-algebraic set V defined over $\mathbb{Z}$ there is a rank 3 oriented matroid whose realization space is stably equivalent to V.*

(ii) *For every primary basic semi-algebraic set V defined over $\mathbb{Z}$ there is an integer $d > 1$ and a d-polytope P with $d+4$ vertices whose realization space is stably equivalent to V.*

Stable equivalence is a strong concept of topological equivalence, that in particular preserves homotopy type and the algebraic complexity of test points. So Mnëv's construction implies:

- The realizability problem for d-polytopes with $d+4$ vertices is (polynomial time) equivalent to the "Existential Theory of the Reals."
- The realizability problem for d-polytopes with $d+4$ vertices is NP-hard.
- All algebraic numbers are needed to coordinatize all d-polytopes with $d+4$ vertices.
- For every finite simplicial complex Δ there is a d-polytope with $d+4$ vertices whose realization space is homotopy equivalent to Δ.

It will be the main purpose of this monograph to establish similar results for the case of polytopes in fixed dimension $d = 4$.

1.4 New Results on 4-Polytopes

We will constructively prove that the realization spaces of 4-polytopes can be "arbitrarily ugly," in a well defined sense.

UNIVERSALITY THEOREM FOR 4-POLYTOPES:
For every primary basic semi-algebraic set V defined over $\mathbb{Z}$, there is a 4-polytope P whose realization space is stably equivalent to V. The face lattice of P can be generated from defining equations of V in polynomial time.

The following new results are corollaries of the Universality Theorem or consequences of the construction we provide for it.

(i) There is a non-rational 4-polytope with 33 vertices.

(ii) All algebraic numbers are needed to coordinatize all 4-polytopes.

(iii) The realizability problem for 4-polytopes is NP-hard.

(iv) The realizability problem for 4-polytopes is (polynomial time) equivalent to the "Existential Theory of the Reals" (see [53]).

(v) For every finite simplicial complex Δ, there is a 4-polytope whose realization space is homotopy equivalent to Δ.

(vi) There is a 4-polytope for which the shape of some 2-face cannot be arbitrarily prescribed.

(vii) Polytopality of 3-spheres cannot be characterized by excluding a finite set of "forbidden minors".

(viii) In order to realize all combinatorial types of integral 4-polytopes with n vertices in the integer grid $\{1, 2, \ldots, f(n)\}^4$, the "coordinate size" function $f(n)$ has to be at least doubly exponential in n.

In particular these consequences solve all the problems that were recently emphasized in Ziegler's article *"Three problems about 4-polytopes"* [64].

The proof of the Universality Theorem is constructive. We will describe 4-polytopes that model *addition* and *multiplication* by the non-prescribability of a 2-dimensional face. The addition- and multiplication-polytopes will be joined into larger units that model systems of polynomial equations and inequalities.

Our approach is in some sense analogous to Mnëv's original proof of his Universality Theorem for oriented matroids. He uses the classical *von Staudt constructions* (which model addition and multiplication for points on a line in the projective plane) to compose large planar incidence structures that model arbitrary polynomial computations. The main difficulty in Mnëv's proof is to organize the construction in a way such that different basic calculations do not interfere and such that the underlying oriented matroid stays invariant for all instances of a geometric computation. *Our* main difficulty will be the construction of polytopes for addition and multiplication.

1.5 Polytopal Tools

Lawrence extensions and connected sums are elementary geometric operations on polytopes that form the basis for the constructions we need in order to prove the Universality Theorem. They are very simple and innocent looking operations, but they are very powerful.

For *Lawrence extensions* the basic operation is the following: take a point $\boldsymbol{p}$ in a d-dimensional point configuration, and replace it by two new points $\overline{\boldsymbol{p}}$ and $\underline{\boldsymbol{p}}$ that lie on a ray that starts at the original point and leaves the d-dimensional space spanned by the point configuration in a "new" direction of $(d+1)$-dimensional space (see Figure 2).

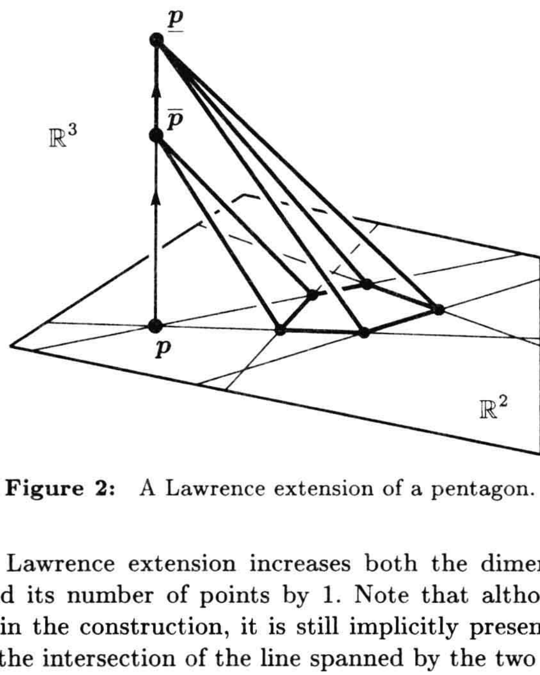


Figure 2: A Lawrence extension of a pentagon.

Every such Lawrence extension increases both the dimension of a point configuration and its number of points by 1. Note that although the original point is deleted in the construction, it is still implicitly present: it can be "reconstructed" as the intersection of the line spanned by the two new points with the d-hyperplane spanned by the original point configuration.

The "classical" use of Lawrence extensions [13, 6, 49] starts with a 2-dimensional configuration of n points, and performs Lawrence extensions on all these points, one by one. The resulting configuration of $2n$ points is the vertex set of an $(n+2)$-dimensional polytope, the *Lawrence polytope* of the point configuration. Every realization of the Lawrence polytope determines a realization of the original point configuration, including all collinearities and all orientations of triples. In fact, the realization spaces of the Lawrence polytope and the planar configuration are stably equivalent. This can be used to lift Mnëv's Universality Theorem from planar point configurations (oriented matroids) to d-polytopes.

If one wants to stay within the realm of 4-polytopes, then it is not permissible to use more than two Lawrence extensions. However, careful use of just one or two Lawrence extensions on some points outside a 2- or 3-polytope leads to extremely interesting and useful polytopes — such as the basic building blocks for the Universality Theorem (see Section 5).

Connected sums are the operations that compose these basic building blocks into larger units. They are performed as follows: Assume that one is given two d-polytopes P_1 and P_2 that have projectively equivalent facets F_1 resp. F_2. We use F to denote the combinatorial type of $F_1 \cong F_2$. Then, using a projective transformation, one can "merge" P_1 and P_2 into a more complicated polytope, the *connected sum* $Q := P_1 \#_F P_2$. The polytope Q has all the facets of P_1 and P_2, except for F_1 and F_2. However, the boundary complex ∂F, consisting of all the proper faces of F, is still present in Q (Figure 3).

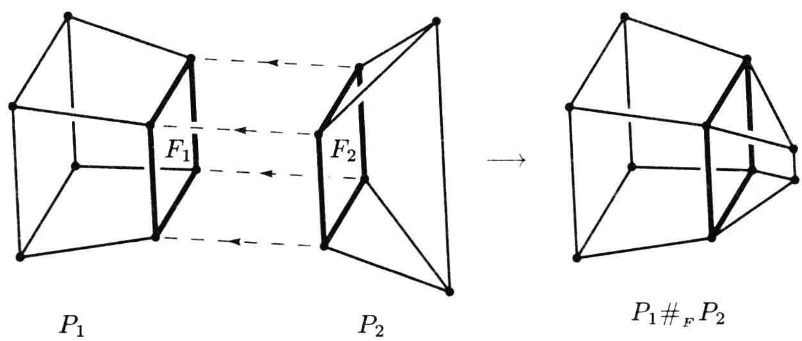


Figure 3: The connected sum of a cube and a triangular prism.

Now, if one takes an arbitrary realization of Q, then it is not in general true that this realization arises as a connected sum of realizations of P_1 and of P_2: in a "bad" realization of Q the boundary complex ∂F may not be flat. In fact, in dimension $d = 3$ one can see that the complex ∂F in Q is necessarily flat if and only if F is a triangular facet. In dimension 4, there are much more different types of facets that are "necessarily flat," among them pyramids, prisms, and "tents." Only such necessarily flat facets are used in connected sum operations for the proof of the Universality Theorem.

1.6 Sketch of the Proof of the Universality Theorem

Our proof starts from the defining equations of a primary basic semialgebraic set, and uses them explicitly to construct the face lattice of a 4-polytope. A result of Shor [53] is used, which states that every primary semialgebraic set V is stably equivalent to a semialgebraic set $V' \in \mathbb{R}^n$ whose variables

$$1 = x_1 < x_2 < x_3 < \ldots < x_n$$