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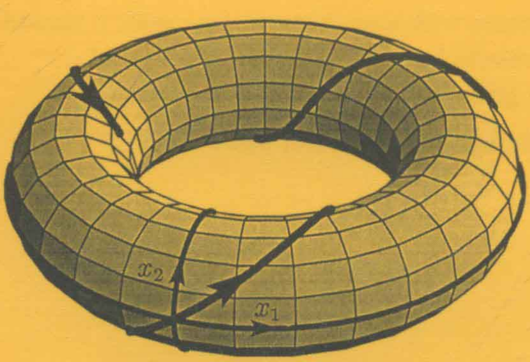
Quasi-Periodic Motions in Families of Dynamical Systems



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Order amidst Chaos

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Preface

This book is devoted to the phenomenon of quasi-periodic motion in dynamical systems. Such a motion in the phase space densely fills up an invariant torus. This phenomenon is most familiar from Hamiltonian dynamics. Hamiltonian systems are well known for their use in modelling the dynamics related to frictionless mechanics, including the planetary and lunar motions. In this context the general picture appears to be as follows. On the one hand, Hamiltonian systems occur that are in complete order: these are the integrable systems where all motion is confined to invariant tori. On the other hand, systems exist that are entirely chaotic on each energy level. In between we know systems that, being sufficiently small perturbations of integrable ones, exhibit coexistence of order (invariant tori carrying quasi-periodic dynamics) and chaos (the so called stochastic layers). The Kolmogorov–Arnol'd–Moser (KAM) theory on quasi-periodic motions tells us that the occurrence of such motions is *open* within the class of all Hamiltonian systems: in other words, it is a phenomenon persistent under small Hamiltonian perturbations. Moreover, generally, for any such system the union of quasi-periodic tori in the phase space is a nowhere dense set of positive Lebesgue measure, a so called Cantor family. This fact implies that open classes of Hamiltonian systems exist that are not ergodic.

The main aim of the book is to study the changes in this picture when other classes of systems – or contexts – are considered. Examples of such contexts are the class of reversible systems, of volume preserving systems or the class of all systems, often referred to as "dissipative". In all these cases, we are interested in the occurrence of quasi-periodic motions, or tori, persistent under small perturbations within the class in question. By an application of the KAM theory it turns out that in certain cases, in order to have this persistence, the systems are required to depend on *external* parameters. An example of such a situation is the dissipative class, where quasi-periodic attractors are found. These attracting quasi-periodic tori are isolated in the phase space and they are only persistent when at least one parameter is present. In that case, generally, the set of parameters for which such an attractor occurs has positive Lebesgue measure. Quasi-periodic attractors are well known to be a transient stage in a bifurcation route from order to chaos.

The KAM theory is a powerful instrument for the investigation of this problem in a broad sense, describing the organization of invariant tori as Cantor families of positive Lebesgue (Hausdorff) measure. It yields a unifying approach for all cases, leading to a formulation with a minimal number of parameters. In this book, we discuss various aspects of the KAM theory. However, there are still many problems of the theory outside the scope of the present text. Some of these will be briefly indicated.

We proceed in giving an outline of the text. In introductory Chapter 1 we present a more precise formulation of our main problem illustrating this with many examples. Here we also define the contexts to which we apply our approach throughout. These include two different reversible contexts and, in the Hamiltonian setting next to the isotropic case, also the coisotropic one. The Chapters 2 and 3 form the "kernel" of the book. In Chapter 2 first we formulate the conjugacy or stability theory. Depending on the context, we introduce a suitable number of unfolding parameters which stabilize the systems within their context for small perturbations. This stability only holds on Cantor families of invariant tori with Diophantine frequencies (or KAM tori), the corresponding conjugacies being smooth in the sense of Whitney. This approach first leaves us with families that depend on a great many parameters, but the discussion continues by systematically reducing the number of parameters to a minimum, where still sufficiently many tori are left, in the sense of Hausdorff measure. Main tool here is the Diophantine approximation lemma in the form close to that of V.I. Bakhtin. Chapter 3 subsequently discusses the theory on the continuation of analytic tori due to A.D. Bruno.

Next, in Chapter 4, we discuss the organization of the Cantor families of tori as they occur in our various contexts, including estimates of the appropriate Hausdorff measure. We also present some considerations regarding the dynamics in the "resonance zones", i.e., in the complement of the KAM tori (including Nekhoroshev estimates on solutions near those tori).

Chapter 5 presents conclusive remarks on the subject. Correspondences and differences between the cases of vector fields and diffeomorphisms are discussed. Also we show that, generally, the KAM tori accumulate very much on each other, which can be concisely formulated in terms of Lebesgue density points.

Chapter 6 consists of appendices. In the first of these, the stability theorem is proven in one of its simplest forms. Other appendices fully describe the Bruno theory and the Diophantine approximation lemma.

The style of the book makes it suitable for both experts and beginners regarding the KAM theory. On the one hand, it presents an up to date and therefore quite advanced overview of the theory. On the other hand, it contains an elementary introduction to Whitney differentiability and a complete proof of the simplest stability theorem in this respect. By this and the other details of the appendices, the text is largely self-contained. Also it contains an extended bibliography (which does not, of course, claim to be complete).

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H.W.B., G.B.H. & M.B.S.
Groningen, September 1996

Notation

Although, as a rule, we explain each notation where it first appears in the book, some notations used frequently in the sequel are collected here. Some basic sets are denoted by the "open font" (or "blackboard bold") characters. The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ require no comments, $\mathbb{R}_+$ will denote the set of *non-negative* real numbers, $\mathbb{Z}_+$ is the set of *non-negative* integers, and $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$ is the set of positive integers. The symbol $\mathbb{S}^n$ denotes the unit n-dimensional sphere in $\mathbb{R}^{n+1}$, and $\mathbb{T}^n = (\mathbb{S}^1)^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ is the standard n-torus. Also, $\mathbb{RP}^n$ is the n-dimensional real projective space, and $\Pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ denotes the natural projection. Note that each of the spaces $\mathbb{T}^0$ and $\mathbb{RP}^0$ is a point, while $\mathbb{S}^0$ is the disjoint union of two points. Note also that $\mathbb{RP}^1 \simeq \mathbb{S}^1 = \mathbb{T}^1$.

The symbols $\mathcal{O}_n(a)$ will denote a neighborhood of a point $a \in \mathbb{R}^n$. In particular, $\mathcal{O}_n(0)$ is a neighborhood of the origin in $\mathbb{R}^n$. We write $\mathcal{O}(a)$ instead of $\mathcal{O}_1(a)$ for $a \in \mathbb{R}$.

By the angle brackets, we will denote the standard inner product of two vectors, so that

$$\langle a, b \rangle = \textstyle\sum_{i=1}^n a_i b_i \qquad \text{for} \qquad a \in \mathbb{R}^n, b \in \mathbb{R}^n.$$

The symbols $|a|$ and $\|a\|$ will denote the l_1-norm and l_2-norm (Euclidean norm), respectively, of vector a (unless when stated otherwise):

$$|a| = \textstyle\sum_{i=1}^n |a_i| \quad \text{and} \quad \|a\|^2 = \sum_{i=1}^n |a_i|^2 \qquad \text{for} \qquad a \in \mathbb{C}^n.$$

For the Landau symbols, we write $O_m(u)$ instead of $O(|u|^m)$ [and $O(u)$ instead of $O_1(u) = O(|u|)$] for $m \in \mathbb{N}$ and any (scalar or vector) independent variable u. By $D_u F$ we will sometimes denote the Jacobi matrix $\partial F/\partial u$. The relation $a := b$ will mean that equality $a = b$ is the definition of quantity a. By $\log u$ we denote $\log_e u$ (which is often designated by $\ln u$ elsewhere). The boundary of manifold or set M is denoted by ∂M. The interior of set M is denoted by $\mathrm{int}(M)$ and the closure of set M by $\mathrm{cl}(M)$ or $\overline{M}$. The symbols $diag(a_1, a_2, \ldots, a_n)$ mean the diagonal $n \times n$ matrix with diagonal entries $a_1, a_2, \ldots, a_n$. The matrix transposed to A is denoted by A^t. The dot means differentiation with respect to time: $\dot{x} := dx/dt$ and $\ddot{x} := d^2x/dt^2$. The average of a function over $\mathbb{T}^n$ will be sometimes denoted by the square brackets $[\cdot]$. Mark $\square$ means the end of the proof.

The notations $[a, b]$, $[a, b[$, $]a, b]$, and $]a, b[$ for $-\infty \le a < b \le +\infty$ mean respectively the intervals $\{x \ : \ a \le x \le b\}$, $\{x \ : \ a \le x < b\}$, $\{x \ : \ a < x \le b\}$, and $\{x \ : \ a < x < b\}$. For example, $\mathbb{R}_+ = [0, +\infty[$. Given $x \in \mathbb{R}$, the integral part of x is denoted by $\mathrm{Entier}(x) := \max\{m \in \mathbb{Z} \ : \ m \le x\} = \max(\mathbb{Z} \cap] - \infty, x])$. If $\mathrm{Entier}(x) = \ell$ then $\ell \le x < \ell + 1$.

The n-dimensional Hausdorff measure in $\mathbb{R}^N$ for $N \ge n$ (see Federer [114] or Morgan [242]) will be denoted by $meas_n$ (elsewhere usually denoted by $\mathcal{H}^n$). For $N = n$ the measure $meas_n$ coincides with the standard Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$.

The term "analytic" will always refer to mappings between *real* manifolds (equipped with an analytic structure), whereas holomorphic functions $f : D \to (\mathbb{C}/2\pi\mathbb{Z})^{N_1} \times \mathbb{C}^{N_2}$, $D \subset (\mathbb{C}/2\pi\mathbb{Z})^{n_1} \times \mathbb{C}^{n_2}$, that are real-valued for real arguments will be called "real analytic".

... there are so many ways to deal with formulas.

Donald E. Knuth. *The TEXbook.* Addison-Wesley, 1986.

There are some formulas that can't be handled easily ...

Leslie Lamport. LATEX. *A Document Preparation System.* Addison-Wesley, 1986.

Contents

Chapter 1

Introduction and examples

1.1 A preliminary setting of the problem

This book investigates the occurrence of quasi-periodic motions in dynamical systems with special emphasis on the persistence of these motions under small perturbations of the system. Quasi-periodic motions densely fill up invariant tori, therefore this study can be regarded as a part of a more general theory of invariant manifolds. The existence, persistence and other properties of invariant manifolds play a fundamental rôle in the analysis of nonlinear dynamical systems [67, 115, 158, 356]. In this book we confine ourselves with finite dimensional systems. For the theory of quasi-periodic motions in infinite dimensional dynamical systems, the reader is recommended to consult, e.g., [185, 186, 279–281] and references therein.

The perturbations we will consider, although small in an appropriate sense, will be arbitrary. However, it is important to specify whether the whole perturbation problem, for example, sits in the Hamiltonian context, or has to respect a certain symmetry, or is subject to no restriction whatsoever. Such a class of vector fields to be examined is often referred to as a "*context*" of the problem [60, 62, 162]. In many of these cases one needs parameters to achieve persistent occurrence of quasi-periodicity, where the specific rôle of the parameters depends on the context at hand.

One reason for the need of parameters is the following. In the perturbation analysis of quasi-periodic tori we follow a specific torus through the perturbation. In this "continuation process" the frequencies of the torus are kept constant. This means that these frequencies somehow have to be treated as parameters of the system. In the classical Hamiltonian context with Lagrangian tori, these frequency-parameters can be accounted for by the action variables, granted some nondegeneracy. However, in the context of general (or "dissipative") systems this is not possible and parameters have to be explicitly present in the setting. Therefore in the title of this book we speak of "families of dynamical systems".

The main problem will be what is the *minimal* number of parameters needed in order to have persistence of quasi-periodicity. We will discuss the *organization* of the tori in families parametrized over Cantor sets of positive Lebesgue measure in a Whitney-smooth manner. A related problem is how to apply the theory to examples where a number of parameters is available.

Several types of examples of dynamical systems with quasi-periodicity will appear in the sequel. Among these are oscillators with weak forcing, either periodic or quasi-periodic, or with weak couplings between them. Another class of applications is given by Bifurcation Theory: subordinate to some degenerate bifurcations quasi-periodic motion shows up in a persistent way.

1.1.1 Definitions

A torus with parallel dynamics. Consider a smooth vector field X on a manifold M with an invariant n-torus T. We say that X on T induces *parallel* (or *conditionally periodic*, or *Kronecker*, or *linear*) motion, evolution, dynamics, or flow, if there exists a diffeomorphism $T \to \mathrm{T}^n$ transforming the restriction $X|_T$ to a constant vector field $\sum_{i=1}^{n} \omega_i \partial/\partial x_i$ on the standard n-torus $\mathrm{T}^n := (\mathrm{S}^1)^n = (\mathrm{R}/2\pi\mathrm{Z})^n$ with angular coordinates $x_1, x_2, \ldots, x_n$ modulo 2π. In a more familiar notation, this vector field determines the system $\dot{x}_i = \omega_i$, $1 \le i \le n$, of differential equations. The numbers $\omega_1, \omega_2, \ldots, \omega_n$ are called *(internal) frequencies* of the motion (evolution, dynamics, or flow) on T, but also of the invariant torus T itself.

Remark 1. The *frequency vector* $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathrm{R}^n$ is determined uniquely up to changes of the form $\omega \mapsto A\omega$, where $A \in GL(n, \mathrm{Z})$, i.e., A is an $n \times n$ matrix with integer entries and determinant ± 1.

Remark 2. Invariant tori with parallel dynamics are of great importance in the theory of dynamical systems which stems, in the long run, from the fact that *any finite dimensional connected and compact abelian Lie group is a torus* [2, 50, 236]. A close more geometric statement is that the factor group R^N/Γ of R^N by a discrete subgroup Γ is T^N provided that R^N/Γ is compact [13].

More generally, any finite dimensional connected abelian Lie group is the product $\mathrm{T}^n \times \mathrm{R}^m$ and the factor group of R^N by any discrete subgroup is $\mathrm{T}^n \times \mathrm{R}^{N-n}$ for some $0 \le n \le N$. The latter statement is the key one in the proof of the Liouville–Arnol'd theorem on completely integrable Hamiltonian systems (see Theorem 1.2 in § 1.3.2 below).

The dynamical properties of an invariant torus with linear flow are very sensitive to the number-theoretical properties of its frequency vector.

A quasi-periodic torus. A parallel motion on an invariant n-torus T with frequency vector ω is called *quasi-periodic* or *nonresonant* if the frequencies $\omega_1, \omega_2, \ldots, \omega_n$ are rationally independent, i.e., if for all $k \in \mathrm{Z}^n \setminus \{0\}$ one has $\langle \omega, k \rangle := \sum_{i=1}^{n} \omega_i k_i \neq 0$. In this case the torus T itself also is said to be quasi-periodic. Otherwise an invariant torus T with parallel dynamics is called *resonant*. For example, the 2-torus of Figure 1.1 with parallel dynamics is quasi-periodic if and only if the ratio of the corresponding frequencies ω_1 and ω_2 is irrational.

Quasi-periodic tori are densely filled up by each of the orbits (or solution curves) contained therein. The whole motion then is ergodic [17]. However, quasi-periodic dynamics is not chaotic, since by the parallelity there is no sensitive dependence on the initial conditions.

The resonant tori are foliated by invariant subtori of smaller dimension. In all the contexts to be met below, a Kupka–Smale theorem holds (cf. [260, 265]), generically for-

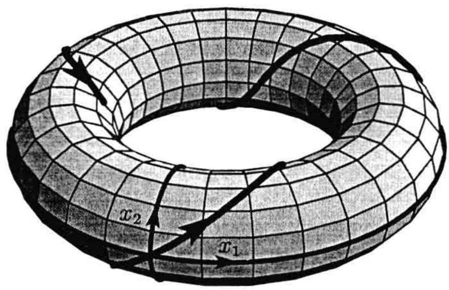

Figure 1.1: Evolution or solution curve of a constant vector field $\omega_1 \partial/\partial x_1 + \omega_2 \partial/\partial x_2$ on the two-dimensional torus T^2.

bidding the existence of resonant tori. For a more detailed discussion, see one of the examples in the next section. As a consequence, all the invariant tori with parallel flow of a *generic* dynamical system are quasi-periodic. Moreover, it turns out that most of the invariant tori with parallel dynamics in the phase space of a generic vector field satisfy stronger nonresonance conditions, as we shall introduce now.

A Diophantine torus. Our study is a part of the Kolmogorov–Arnol'd–Moser (KAM) theory (named after its founders A.N. Kolmogorov [179, 180], V.I. Arnol'd [4, 5, 8], and J. Moser [243, 245]) which generally establishes the existence and persistence of quasi-periodic tori in dynamical systems. In this theory the frequencies of quasi-periodic tori are not only rationally independent, but have to meet the following, stronger nonresonance condition. We say that an invariant n-torus T with parallel dynamics is *Diophantine* if for some constants $\tau > 0$ and $\gamma > 0$ the corresponding frequency vector ω satisfies the following infinite system of inequalities:

$$|\langle \omega, k \rangle| \geq \gamma |k|^{-\tau} \tag{1.1}$$

for all $k \in \mathrm{Z}^n \setminus \{0\}$, where $|k| := \sum_{i=1}^{n} |k_i|$. Clearly Diophantine tori are quasi-periodic, but not *vice versa*. For $\tau > n-1$ the set of all frequency vectors $\omega \in \mathrm{R}^n$ that are Diophantine in the above sense has positive Lebesgue measure [337].

Example 1.1 Each equilibrium point of a vector field is a Diophantine invariant 0-torus. Each S-periodic trajectory of a vector field is a Diophantine invariant 1-torus with frequency $2\pi/S$.

A Floquet torus. An invariant n-torus T with parallel dynamics of a vector field X on an $(n+m)$-dimensional manifold is called *Floquet* if near T, one can introduce coordinates $(x \in \mathrm{T}^n, y \in \mathrm{R}^m)$ in which the torus T itself gets the equation $\{y = 0\}$ while the field X determines the system of differential equations of the so called Floquet form

$$\begin{aligned} \dot{x} &= \omega + O(y) \\ \dot{y} &= \Omega y + O_2(y) \end{aligned} \tag{1.2}$$

with $\Omega \in gl(m, \mathrm{R})$ independent of $x \in \mathrm{T}^n$, compare [60, 62, 162]. If this is the case, matrix Ω is called the *Floquet matrix* of torus T. Of course, near *any* invariant n-torus T with parallel dynamics, one can introduce coordinates $(x \in \mathrm{T}^n, y \in \mathrm{R}^m)$ in which the torus T itself gets the equation $\{y = 0\}$ while the field X determines the system of differential equations

$$\begin{aligned} \dot{x} &= \omega + O(y) \\ \dot{y} &= \Omega(x)y + O_2(y) \end{aligned} \tag{1.3}$$

with $\Omega = \Omega(x) \in gl(m, \mathrm{R})$ depending smoothly on $x \in \mathrm{T}^n$ (provided that a certain neighborhood of T is diffeomorphic to $\mathrm{T}^n \times \mathrm{R}^m$, i.e., the normal bundle of T in the phase space is trivial[1] [157]). Note that $\dot{y} = \Omega(x)y$ is *the variational equation* along T. The torus T is Floquet if matrix Ω can be made independent of the point on T (*reduced* to a constant) by an appropriate choice of local coordinates.

To any Floquet invariant torus with parallel dynamics, one associates the so called *normal frequencies*, i.e., the *positive imaginary parts* of the eigenvalues of its Floquet matrix. The Diophantine tori that the KAM theory can deal with in general also have to be Floquet [12, § 26], for some exceptions see, e.g., [137, 138, 364] (these papers concern the so called hyperbolic lower-dimensional tori in Hamiltonian systems, cf. the remark at the end of § 2.3.4) and Chapter 3 below as well. Moreover, the internal $\omega_1, \omega_2, \ldots, \omega_n$ and normal $\omega_1^N, \omega_2^N, \ldots, \omega_r^N$ frequencies of these tori $(0 \le r \le m)$ should satisfy further Diophantine conditions

$$|\langle \omega, k \rangle + \langle \omega^N, \ell \rangle| \ge \gamma |k|^{-\tau} \tag{1.4}$$

for all $k \in \mathrm{Z}^n \setminus \{0\}$, $\ell \in \mathrm{Z}^r$, $|\ell| \le 2$ with some constants $\tau > 0$ and $\gamma > 0$, as we will see below in Section 2.1. Note that for $\ell = 0$, inequalities (1.4) are just the standard Diophantine inequalities (1.1).

Remark 1. The Floquet matrix is determined up to similarity, i.e., up to changes of the form $\Omega \mapsto A^{-1}\Omega A$, where $A \in GL(m, \mathrm{R})$.

Remark 2. Recall that the *Floquet multipliers* of a periodic trajectory are the eigenvalues of its monodromy operator [12, § 34] (sometimes these are also called *characteristic multipliers* [1, Sect. 7.1]). If in (1.2) $n = 1$, then the period and monodromy operator of closed trajectory $\{y = 0\}$ are equal respectively to $2\pi/\omega$ and $e^{2\pi\Omega/\omega}$. Consequently, if the eigenvalues of the Floquet matrix of an S-periodic trajectory are $\lambda_1, \ldots, \lambda_m$ then its Floquet multipliers are $e^{S\lambda_1}, \ldots, e^{S\lambda_m}$.

Remark 3. An equilibrium point is always Floquet. A periodic trajectory is Floquet if and only if its monodromy operator has a real logarithm (the Floquet theorem, see Arnol'd [12, § 26]). The question is now prompted under what conditions the system of differential equations (1.3) near an invariant torus with parallel dynamics can be reduced to the Floquet form (1.2). It is known, see [5, 12], that for $m = 1$ and arbitrary n reducibility to the Floquet form does take place for *Diophantine* tori. This problem will be treated below in detail (see § 1.5.1).

[1] This is not always the case. No neighborhood of the central circle on the Möbius strip is diffeomorphic to the cylinder $\mathrm{S}^1 \times \mathrm{R}$

For $m \geq 2$, non-pathological examples exist where reducibility does not hold. So, in general reducibility involves an assumption, which turns out to be satisfied typically when sufficiently many parameters are present. For more details here see, e.g., [5, 42, 62, 64, 148, 149, 162, 167, 168, 170, 171, 184].

In examples, reducibility often holds due to a normalization (averaging), see, e.g., Arnol'd [12] and Broer & Vegter [56, 66]. For Hamiltonian vector fields, reducibility is often implied by the presence of sufficiently many additional integrals in involution [184, 256].

1.1.2 Contexts

We next point briefly at the contexts to be explored, for detailed definitions and more up-to-date references see Section 1.3 below. The most important contexts are the Hamiltonian, the volume preserving and the reversible ones, as well as the general "dissipative" one. A Lie algebra of systems is often involved in the definition of the context, this being a natural way to express the "preservation of a structure".

In the sequel, several refinements of these contexts will be considered. Also the central question mentioned before, regarding the minimal number of parameters necessary to obtain the persistence of Diophantine invariant tori, will be addressed.

The Hamiltonian context. With respect to quasi-periodicity the most well known context is Hamiltonian (conservative), defined by the preservation of a symplectic form. This context is notorious for its strong relation to mechanics [1, 5, 13, 17, 130, 182, 183, 201, 335]. The classical result roughly states that in this context, it is a typical property to have many so called Lagrangian quasi-periodic invariant tori, seen from the measure-theoretical point of view. This implies that non-ergodicity is a typical property as well [219].

This "classical" KAM theory was initiated by Kolmogorov [179] in 1954 (see also [180] and [14, 235] as well) and further developed by Arnol'd [4, 5], Moser [243], and many others. For details we refer to, e.g., [13, 20, 30, 91–93, 103, 122, 129, 130, 133, 201, 248, 277, 278, 282, 306, 307, 335]. For a review and a large bibliography, also see Bost [43]. A further refinement of this theory is given in the sequel.

The dissipative context. Another important context is the "dissipative" one, where no structure at all is present. Here the notion of a quasi-periodic attractor comes up as a quasi-periodic torus that is isolated in the phase space. Although quasi-periodicity itself is not considered to be chaotic, this type of dynamics is a possible transient stage between order and chaos, cf. Ruelle & Takens [296, 297], see also [32]. In the dissipative context, parameters are needed in order to have the persistence of quasi-periodic motions. In a suitable class of families of dissipative systems, it is typical to have many parameter values with a quasi-periodic motion, again in the sense of the measure theory.

This part of the KAM theory was first developed by Moser [244, 246] and later on taken up by Broer, Huitema & Takens [62, 162]. Also these results will be explored further below.

The volume preserving context. The volume preserving context shows up, for example, when describing the velocity field of an incompressible fluid. With respect to quasi-periodicity, it has been studied by Moser [244, 246] and later on by Broer & Braaksma [46, 53]. It turns out that the case of codimension 1 Diophantine tori is much related to the above Hamiltonian one, while the case of codimension no less than 2 is very similar to the general dissipative one. This theory was also taken up in [62, 162] and will be presented and extended in the sequel.

The reversible context. Reversible systems are compatible with some involution G, that takes motions to motions while reversing the time parametrization. This concept often comes up in physical systems and is also known for some great similarities with the Hamiltonian case (in particular, regarding quasi-periodicity), see a list of such similarities in [320]. The reversible KAM theory was initiated by Moser [244, 246, 248] and Bibikov & Pliss [35, 36, 38, 40] and later on extended by Scheurle [310–312], Parasyuk [262], Pöschel [278], Arnol'd & Sevryuk [11, 22, 315, 316, 318], and others. Recently the theory has taken a lot of interest, leading to new developments that we will come back to later.

1.2 Occurrence of quasi-periodicity

This section is devoted to a few simple examples of dynamical systems, sometimes depending on parameters, where quasi-periodicity occurs in a persistent way. These examples are situated in various contexts and motivate the introduction of parameters. Also, a first idea is given of the Whitney-smoothness of families of Diophantine invariant tori. The examples given here all are in the world of oscillators, with forcings (both periodic and quasi-periodic), couplings, etc.

1.2.1 Quasi-periodic attractors

The first examples are within the dissipative context, involving quasi-periodic attractors, cf. Broer, Dumortier, van Strien & Takens [58, Ch. 4] and Broer [57] (see also Bogolyubov, Mitropol'skiĭ & Samoĭlenko [42]). We shall see that for the persistence of these attractors, the systems in question (vector fields) have to depend on "external" parameters. For simplicity we restrict ourselves to the case of 2-tori, where the discussion leads to circle maps, compare [12, § 11].

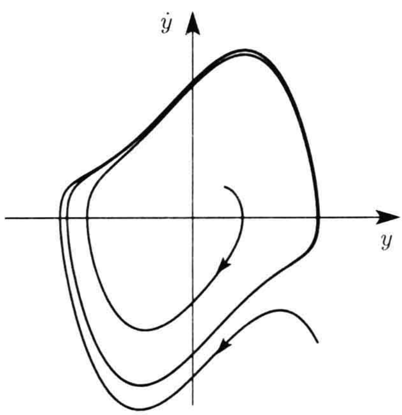

Figure 1.2: A hyperbolic periodic attractor (limit cycle) of the free oscillator.

Preliminaries

We give two examples, both being based on a nonlinear oscillator

$$\ddot{y} + c\dot{y} + ay + f(y, \dot{y}) = 0$$

with $y \in \mathrm{R}$, which is assumed to have a periodic attractor. Here a and c are real constants. For example, one may think of the "Van der Pol" form $f(y, \dot{y}) = by^2\dot{y}$ [58, Ch. 1], b being a real constant. In Figure 1.2, the corresponding phase portrait is shown in the $(y, \dot{y})$-plane.

Periodic forcing. In the first example we force this oscillator periodically, which leads to the following equation of motion:

$$\ddot{y} + c\dot{y} + ay + f(y, \dot{y}) = \epsilon g(y, \dot{y}, t) \tag{1.5}$$

with $g(y, \dot{y}, t + 2\pi) \equiv g(y, \dot{y}, t)$. For simplicity we assume the functions f and g to be analytic in all the arguments; ϵ is a small parameter "controlling" the forcing. In this example we consider the three-dimensional extended phase space $\mathrm{R}^2 \times \mathrm{S}^1$ with coordinates $(y, \dot{y}, t \bmod 2\pi)$, where we obtain a vector field determining the system of differential equations

$$\begin{aligned} \dot{y} &= z \\ \dot{z} &= -ay - cz - f(y, z) + \epsilon g(y, z, t) \\ \dot{t} &= 1. \end{aligned}$$

Coupling. In the second example we consider two "Van der Pol"-type oscillators, as before, with a coupling:

$$\begin{aligned} \ddot{y}_1 + c_1\dot{y}_1 + a_1 y_1 + f_1(y_1, \dot{y}_1) &= \epsilon g_1(y_1, y_2, \dot{y}_1, \dot{y}_2) \\ \ddot{y}_2 + c_2\dot{y}_2 + a_2 y_2 + f_2(y_2, \dot{y}_2) &= \epsilon g_2(y_1, y_2, \dot{y}_1, \dot{y}_2). \end{aligned} \tag{1.6}$$

Here $y_j \in \mathrm{R}$ while a_j, c_j are real constants for $j = 1, 2$. The functions f_j, g_j $(j = 1, 2)$ are again assumed to be analytic and ϵ is a small parameter. The phase space here is therefore R^4 with coordinates $(y_1, y_2, \dot{y}_1, \dot{y}_2)$, on which we obtain a vector field determining the system of differential equations

$$\begin{aligned} \dot{y}_j &= z_j \\ \dot{z}_j &= -a_j y_j - c_j z_j - f_j(y_j, z_j) + \epsilon g_j(y_1, y_2, z_1, z_2) \end{aligned}$$

for $j = 1, 2$.

A preliminary perturbation theory: The torus as an invariant manifold

The torus as an invariant manifold. First we consider the "unperturbed" case $\epsilon = 0$, where in both examples the situation is simple. Indeed, in the three- (respectively four-dimensional) phase space we find an attracting invariant 2-torus with suitable coordinates $(x_1, x_2$ modulo $2\pi)$, in which the differential equations afforded by the restricted vector field have the constant form $\dot{x}_1 = \omega_1$, $\dot{x}_2 = \omega_2$. From hyperbolicity of each limit cycle it follows that normally to this torus, the attraction already can be seen from the linear