

CALCULUS

WITH

ANALYTIC

GEOMETRY

1

JOHN M. H. OLMSTED

VOLUME I

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WITH ANALYTIC GEOMETRY

John M. H. Olmsted

Southern Illinois University



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TO CYNTHIA

PREFACE

A perennial problem facing teachers of mathematics is the determination of the proper time for introducing the student to the “delta-and-epsilon” type of mathematical rigor, sometimes referred to informally as “epsilononics.” Experience in recent years has convinced many people that much of the subject matter of “epsilononics” can be effectively taught at a much earlier stage in a student’s academic career than had at one time seemed possible. However, the fact remains that most students do have difficulty when first confronted with the limit concept. What is the best solution for the problem posed above? Should a considerable period of time be invested in a purely intuitive approach before the student is asked to face the harsh realities of mathematical analysis? Or is it possible to present “epsilononics” in such a fashion and with such a timing that the student can assimilate the essential ideas in a natural way and with a reasonably correct intuitive understanding?

For purposes of discussion let it be agreed that the limit concept is a complicated and difficult one for most students new to calculus. Explicitly or implicitly the formulation of a limit statement uses not only a universal quantifier and an existential quantifier but also an implication involving two inequalities (or the equivalent in terms of a neighborhood and a deleted neighborhood). One of the principal assumptions determining the character of the present book is that the most appropriate groundwork for the study of limits is a substantial amount of detailed and careful work with inequalities, implications, and quantifiers *before* these are all combined into the type of compound statement needed for limits. For this reason, and also because of the many variants that involve “neighborhoods of infinity,” the treatment of limits has been placed as late as possible, in Chapter 10. This postponement of limits is made possible, in part, by the earlier discussion, in Chapter 9, of continuity (free from the complications of infinities and deleted neighborhoods), but more especially by the still earlier focusing of attention on such global concepts as the Riemann integral, in Chapters 6 and 7, and uniform continuity, in Chapter 8.

Placing the introduction of the definite integral before that of the derivative has a well-recognized historical background, but it also has a sound logical justification. When the Riemann integral is approached by means of step-functions the definition of integrability involves only two quantifiers and a single inequality, and the definition of the Riemann integral itself is simultaneously the supremum of one set and the infimum of another. The avoidance, at this stage, of implications between pairs of

inequalities provides a natural arena for students to get a mathematical workout with quantifiers. Then later, when it is time to struggle with limit statements, the quantified portions have become familiar objects and are no longer a barrier.

It is possible that some readers of this preface are not disposed at first to agree with the proposition that the global concept of uniform continuity is a simpler one than the local concept of ordinary continuity. Undoubtedly, if one wishes only to talk intuitively in terms of “approach” or “nearness” with accompanying “hand-waving” gestures, uniform continuity *is* more complicated. However, it is when one expresses himself in exact terms that this concept becomes simpler. More precisely, it is nearly always simpler to find an explicit δ in terms of ϵ for a function on a set than for a function at a point. An example may help clarify this statement. For the “squaring” function $x \rightarrow x^2$ on the compact interval $[3, 5]$ a suitable δ can be found quite simply to be $\epsilon/10$, whereas for this same function at the point 3, either δ turns out to be something like $\min(1, \epsilon/7)$ or else ϵ must be artificially limited in size. In fine, the position adopted in this book is that to concentrate on relatively simple global concepts before turning attention to point-wise continuity and limits makes good sense, and is preferable in the long run to the more traditional intuitive approach which necessitates an often painful relearning process later.

The dominant philosophy of this book emphasizes *concepts* and *structure*, with the dual objectives of developing an appreciation for a truly beautiful and well-conceived subject and exploiting the great potentials of these concepts and structures. The concept of *vector space* should serve to illustrate what is meant. Vector spaces are introduced first for function spaces, with examples such as bounded functions, step-functions, and polynomials. It then becomes natural to develop the Riemann integral as a positive linear functional on the vector space of integrable functions. Then later, many of the standard theorems of continuous functions and limits find simple expression when cast in the language of vector spaces and algebras of functions. Furthermore, these same ideas become tools when it is shown that the Riemann integral can be obtained as the limit of a sum. Later in the book, generous attention is again devoted to vector spaces and algebras, this time in the context of matrices and vectors in two and three dimensions. Eigenvalue techniques are used effectively to simplify transformations of equations of the second degree.

In order to achieve the goals enumerated above, the first two chapters contain an introduction to sets and logic. Truth sets serve to explain exactly what an implication is. The quantifier symbols \forall and \exists are used widely, and their role in the formulation of negations is discussed carefully and used frequently. Both the nature of a proof and the meaning of a counterexample are given considerate attention. The groundwork laid in the first two chapters forms the basis for the extensive use of both sets and logic throughout the book. It is felt that the time spent initially in forming a sound foundation for the ideas that are so essential to calculus is well worth the investment, and is amply justified by the ultimate dividends in the form of understanding and intellectual satisfaction.

Alternative formulations are sometimes included. For example, several limit statements are expressed in terms of deltas and epsilons, neighborhoods and deleted neighborhoods, and mappings. These alternatives help provide a broad and varied

texture for the underlying flow of ideas, as well as emphasizing a basic unity that pervades many apparently dissimilar settings.

Much attention is given to applications. For example, Chapter 7 is devoted entirely to such applications of the Riemann integral as area, volume, work, and distance. Applications of the derivative are scattered through most of the chapters. Differential equations find their first use in Chapter 18, where they are applied to problems involving gravitation, orthogonal trajectories, radioactive decay, bacterial growth, cooling, and mixing. The final Chapter 34 returns to the subject of differential equations, concentrating on linear equations with constant coefficients and their applications. Extremal problems receive unusually thorough consideration, with full attention given to sufficiency conditions and endpoint extrema.

Area and volume are introduced axiomatically, each being a positive additive function on a ring of sets and satisfying a certain completeness axiom. For a function of a single real variable these axioms suffice for establishing the theorem that the ordinate set of a nonnegative Riemann-integrable function f on a compact interval has area equal to the integral of f .

Elliptic notation (for example, “the function $e^{2x} \sin 3x$ ”) is discussed clearly so that the student learns to understand and use both the strict f and the elliptic notation. This is important since both notations abound in the mathematical literature of the world.

Special attention should be called to the character and arrangement of honors sections, suitable for enrichment content in honors courses. These honors sections are planned so as to permit an honors course running concurrently with a standard course to cover the standard as well as the extra material, and also to permit exploratory reading and extra assignments for interested and capable students who are not registered in a special honors class. In order that a student may be able to transfer either out of or into an honors sequence between terms it is desirable that the honors course keep pace with the standard course to which it corresponds. This is made especially feasible for courses using this book by the uniform arrangement of the honors sections, which constitute exactly twenty percent of the sections, as follows: Every chapter contains a multiple of ten sections, including sections of exercises; every section whose number ends with the digit 9 is an honors section, and every section whose number ends with the digit 0 is the accompanying honors section of exercises. All honors material is identified by the letter H.

The division of the book into two volumes is done simply to reduce the bulk of the alternative of a single tome. The place of division between Chapters 21 and 22 achieves a local minimum for cross-references between the two volumes. The use of a table of integrals is limited to the second volume.

The book is designed for courses of three semesters — or four or five quarters — meeting four or five days a week. Prerequisite is a standard high school mathematical preparation, including trigonometry and college algebra. Review material in trigonometry and such topics as mathematical induction is included.

Over 6000 exercises of all levels of difficulty are available, for practice, for challenge, and for individual exploration. These exercises form an essential part of the book, and are designed to fortify and deepen, as well as guarantee, learning. An-

swers to nearly all problems are given in the back portions of the two volumes. Illustrative examples are liberally provided.

The author wishes to express his appreciation for the extensive aid and numerous suggestions given by Professor R. W. Brink. He is also indebted to many others for their helpful comments, given both informally in conversation and more specifically on paper.

J. M. H. O.

Carbondale, Illinois

CONTENTS

Preface.....	vii
--------------	-----

1 SETS, RELATIONS, AND FUNCTIONS

101	Introduction.....	1
102	Sets, members, and subsets.....	2
103	Universes and complements.....	4
104	Unions; Venn diagrams.....	5
105	Intersections; combinations.....	7
106	Venn diagrams for statements.....	9
107	Set-builder notation.....	10
108	Exercises.....	11
H109	Infinite collections.....	13
H110	Exercises.....	14
111	Introduction to variables and functions.....	15
112	Cartesian products and relations.....	16
113	Functions.....	18
114	Ellipsis.....	23
115	Exercises.....	23
116	The real axis and the Euclidean plane.....	24
117	Graphs.....	27
118	Exercises.....	31
H119	Equivalence relations.....	32
H120	Exercises.....	34

2 ELEMENTARY LOGIC IN MATHEMATICS

201	Statements, formulas, and truth sets.....	35
202	Negations, disjunctions, and conjunctions.....	37
203	Exercises.....	41
204	Implications, converses, and contrapositives.....	42

205	Exercises	49
206	Quantifiers; for all; there exists	50
207	Composite propositions	52
208	Exercises	54
H209	The conditional and biconditional	55
H210	Exercises	57

3 THE REAL NUMBER SYSTEM

301	Introduction; binary operations	58
302	Axioms of a field	59
303	Exercises	64
304	Ordered fields	64
305	Exercises	70
306	Natural numbers	72
307	Mathematical induction	74
308	Exercises	77
H309	Integral domains and rings	79
H310	Exercises	80
311	Integers and rational numbers	82
312	Intervals; infinity symbols	84
313	Absolute values; neighborhoods	86
314	Solving inequalities	88
315	Exercises	91
316	Completeness	93
317	Density of the sets of rational numbers and irrational numbers	97
318	Exercises	101
H319	The Dedekind property	101
H320	Exercises	102

4 LINEAR EQUATIONS AND INEQUALITIES

401	Graphs; vertical and horizontal lines	104
402	Slope of a line segment	109
403	Slope of a line; parallel lines	112
404	Tangent formula for slope; perpendicular lines	115
405	Exercises	119
406	Point-slope, two-point, and slope-intercept equations	120
407	The general equation of the first degree	123
408	Exercises	128
H409	Geometry in the Cartesian plane	129
H410	Exercises	131

411	Linear inequalities; systems	132
412	Exercises	135
413	Midpoint, points of division; parametric form	136
414	Exercises	138
415	The distance formula; line and a point; circles	140
416	Exercises	144
417	Locus problems	147
418	Exercises	150
H419	Locus problems with parameters	150
H420	Exercises	151

5 COMBINATIONS OF FUNCTIONS

501	Composites	154
502	One-to-one mappings and inverses	156
503	Exercises	159
504	Sums, products, differences, and quotients; ordering	161
505	Exercises	165
506	Vector spaces of functions	167
507	Bounded functions	170
508	Exercises	173
H509	Meets and joins; lattices of functions	174
H510	Exercises	176
511	Abstract vector spaces; linear functionals; positivity	176
512	The sigma functional	179
513	Exercises	181
514	Step-functions	182
515	Exercises	186
516	Monotonic functions	188
517	Algebras of functions	192
518	Exercises	195
H519	Abstract algebras	196
H520	Exercises	197

6 THE DEFINITE INTEGRAL

601	The idea of area; ordinate sets	198
602	The integral of a step-function	201
603	The integral as a positive linear functional on \mathcal{S}	204
604	Exercises	205
605	Riemann-integrability; vector space structure	206
606	Integrability of monotonic functions	210
607	Products and absolute values of integrable functions	212

608	Exercises	212
H609	The algebra and lattice of integrable functions.	213
H610	Exercises	214
611	The Riemann integral.	214
612	The integral as a linear functional.	217
613	The integral as a positive linear functional.	218
614	Exercises	220
615	Integrability on subintervals.	221
616	Additivity on adjacent intervals.	223
617	Other theorems.	225
618	Exercises	229
H619	Integration of positive integral powers.	230
H620	Exercises.	233

7 APPLICATIONS OF THE DEFINITE INTEGRAL

701	Area as a positive additive function	236
702	Areas of ordinate sets	239
703	Sets between two graphs.	243
704	Exercises	245
705	Rectangular coordinates in space.	248
706	Volume as a positive additive function.	250
707	Volumes of revolution by cylinders and washers.	251
708	Exercises	256
H709	Two deferred proofs.	257
H710	Exercises	259
711	Volumes of revolution by cylindrical shells.	260
712	Volumes of known cross section area.	265
713	Exercises	269
714	Force and work	271
715	Distance and velocity	273
716	Exercises	274
717	Informal review of trigonometry	275
718	Exercises	279
H719	First moments and centroids.	280
H720	Exercises	284

8 UNIFORM CONTINUITY

801	Uniform continuity	285
802	Negation of uniform continuity.	290

803	Vector space structure.....	295
804	Algebra structure.....	297
805	Exercises.....	299
806	Composite functions.....	300
807	Integrability of uniformly continuous functions.....	303
808	Exercises.....	305
H809	Lattice structure.....	305
H810	Exercises.....	306

9 CONTINUITY

901	Continuity at a point.....	307
902	Exercises.....	313
903	Negation of continuity at a point.....	314
904	Vector space and algebra.....	315
905	Exercises.....	318
906	Composite functions; quotients.....	318
907	One-sided continuity; types of discontinuity.....	322
908	Exercises.....	326
H909	An integrable function discontinuous on \mathbb{Q}	327
H910	Exercises.....	329
911	Intermediate-value property.....	330
912	Inverses of monotonic functions.....	332
913	Roots and rational powers.....	334
914	Exercises.....	337
915	The Heine–Borel property.....	338
916	Continuity and uniform continuity.....	339
917	Boundedness and extrema.....	341
918	Exercises.....	343
H919	Proof of the Heine–Borel property.....	344
H920	Exercises.....	345

10 LIMITS

1001	Deleted neighborhoods and the limit concept.....	346
1002	Limit at a point.....	348
1003	One-sided limits.....	350
1004	Continuity and limits.....	352
1005	Exercises.....	354
1006	Limit theorems.....	355
1007	Limit of a composite function.....	361
1008	Exercises.....	362

H1009	The function $x \sin(1/x)$	363
H1010	Exercises.....	364
1011	Limits of infinity.....	364
1012	Infinite limits.....	370
1013	Exercises.....	375
1014	Curve sketching.....	377
1015	Exercises.....	378
1016	The integral as the limit of a sum.....	379
1017	Bliss's theorem.....	384
1018	Exercises.....	385
H1019	Proof of Bliss's theorem.....	386
H1020	Exercises.....	388

11 THE DERIVATIVE

1101	The idea of a tangent line.....	389
1102	Average and instantaneous velocity.....	392
1103	The derivative.....	396
1104	Exercises.....	400
1105	Laws of derivatives; polynomials; higher orders.....	400
1106	Exercises.....	404
1107	Differentiability and continuity; products and quotients.....	404
1108	Exercises.....	406
H1109	The derivative as a linear operator.....	407
H1110	Exercises.....	408
1111	Differentiation of powers.....	408
1112	Composite functions and the chain rule.....	410
1113	Exercises.....	413
1114	One-sided derivatives.....	414
1115	The law of the mean.....	415
1116	Monotonic differentiable functions.....	419
1117	Vertical tangents.....	421
1118	Exercises.....	424
H1119	Uniform continuity and bounded derivatives.....	425
H1120	Exercises.....	426
1121	The intermediate-value property for derivatives.....	427
1122	Differentiability of inverse functions.....	428
1123	Differentiation of rational powers.....	431
1124	Exercises.....	433
1125	The formula for differentiating the sine function.....	434
1126	The derivative of the cosine function.....	435
1127	Other trigonometric functions.....	436
1128	Exercises.....	437
H1129	Counterexamples.....	438
H1130	Exercises.....	441

12 DIFFERENTIALS AND IMPLICIT FUNCTIONS

1201	Differentials.....	443
1202	Identity, inverse, and composite functions.....	447
1203	Exercises.....	450
1204	Approximations by differentials.....	451
1205	Exercises.....	453
1206	An extension of the law of the mean.....	454
1207	Accuracy of an approximation; inequalities.....	457
1208	Exercises.....	458
H1209	A formula for the second derivative.....	459
H1210	Exercises.....	460
1211	Functions defined implicitly.....	460
1212	The use of differentials.....	462
1213	Second-order derivatives.....	463
1214	Exercises.....	464
1215	Curves defined parametrically.....	465
1216	Second-order derivatives.....	469
1217	The generalized law of the mean.....	470
1218	Exercises.....	472
H1219	Higher order derivatives.....	473
H1220	Exercises.....	476

13 CONIC SECTIONS

1301	Introduction.....	478
1302	The parabola.....	478
1303	Vertices different from the origin.....	482
1304	Exercises.....	484
1305	The ellipse.....	486
1306	Eccentricity.....	492
1307	Centers different from the origin.....	492
1308	Exercises.....	494
H1309	Similarity.....	496
H1310	Exercises.....	497
1311	The hyperbola.....	498
1312	Asymptotes.....	502
1313	Eccentricity.....	505
1314	Conjugate hyperbolas.....	505
1315	Centers different from the origin.....	505
1316	Exercises.....	508
1317	Locus problems.....	510
1318	Exercises.....	511
H1319	Diameters.....	512
H1320	Exercises.....	513