Topics in

FUNCTIONAL ANALYSIS AND APPLICATIONS

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S. Kesavan
School of Mathematics
Tata Institute of Fundamental Research
Bangalore, India

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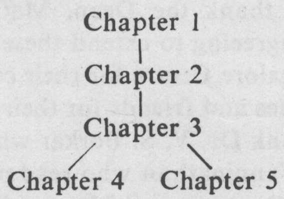
Preface

With the discovery of the theory of distributions, the role of Functional Analysis in the study of partial differential equations has become increasingly important. Not only is a proper functional analytic setting important for the theoretical study of the well-posedness of initial and boundary value problems but also for the construction of good numerical schemes for the computation of approximate solutions. Numerical methods like the Finite Element Method draw heavily upon the results of Functional Analysis both for the construction of the schemes as well as their error analysis.

It is thus clear that applied mathematicians must have a good background in Functional Analysis and its applications to partial differential equations. Such courses are absent from the curricula of most Indian universities. One of the main reasons for this is the lack of suitable textbooks on which such a course could be based and it is this gap that the present book is expected to fill. Indeed the material covered in this book is not original and can be found distributed amongst several treatises, texts or papers. The difficulty faced by teachers is the task of assembling material suitable for an introductory course from an abundant literature. The present book aims to provide such an introductory course on the functional analytic methods used in the study of partial differential equations and is based on lectures given by myself at the Tata Institute of Fundamental Research in the past few years.

The prerequisites for the use of this book are basic courses on Analysis (theory of measure and integration, L^p-spaces, etc.), Topology and Functional Analysis (Banach and Hilbert spaces, strong and weak topologies, compact operators etc.). Apart from these requirements, every effort has been made to keep the treatment as self-contained as possible. The interdependence of the various chapters is as follows:



The first chapter covers the main aspects of the theory of distributions and the Fourier transform. The notion of distribution solutions to partial differential equations is introduced. The second chapter studies the important properties of Sobolev spaces. These spaces form a natural functional analytic framework for the study of weak solutions of elliptic boundary value problems which is the topic discussed in the third chapter. Here we study the existence, uniqueness and regularity of weak solutions of linear elliptic boundary value problems. The general theory is illustrated by several classical examples from physics and engineering. We also study maximum principles and eigenvalue problems. The fourth chapter is devoted to the study of evolution equations, i.e. initial and initial-boundary value problems, using the theory of semigroups of linear operators on a Banach space. After a brief introduction to the abstract theory, illustrations via some standard partial differential equations of physics are provided. The last chapter provides an introduction to the study of semi-linear elliptic boundary value problems from the point of fixed point theorems, approximation methods and variational principles.

Comments at the end of each chapter provide additional information or results not given in the text and also give important bibliographic references. Each chapter is also provided with a selection of exercises which are designed to fill gaps in the proofs of some theorems or prove additional theorems or to construct examples or counter examples to notions introduced in the text.

As the Tamil poet Avvai put it, one's knowledge and one's ignorance roughly bear the ratio of a fistful of soil to the volume of the Earth. In the same way it must be remembered that this book is just an introduction to the study of the modern theory of partial differential equations and is by no means an exhaustive treatment of the subject. The literature is very vast and references to the important works and the frontiers of current research are indicated wherever possible.

The material covered in the text is more or less a prerequisite for any student aspiring for a good research career in applied mathematics whether it be from the theoretical or computational point of view. This book could thus be used for an M.Phil. or a pre-Ph.D. course in the applied mathematics curriculum in Indian universities. In case the M.Sc. curriculum provides a strong base in Analysis, Functional Analysis and Topology, it could also be used for an elective course at that level.

The preparation of this manuscript was possible due to the excellent facilities available at the Bangalore Centre of the Tata Institute of Fundamental Research and I thank the Dean, Mathematics Faculty of this Institute for generously agreeing to extend these facilities for this purpose and the Staff of the Bangalore Centre for their cooperation. I would also like to thank my colleagues and friends for their help and encouragement. In particular I wish to thank Dr. V. S. Borkar who egged me on to embark on this project, Dr. M. Vanninathan who read portions of the manuscript and helped me improve the same and Messrs. A. Patnaik, B. K. Ravi and A. S. Vasudevamurthy who, in various ways, helped me during the preparation of the manuscript. Finally I thank Ms. N. N. Shanthakumary for her neat and careful typing of the manuscript. I am grateful to the

personnel of Wiley Eastern Ltd. for their cooperation in bringing out this volume. To my family I owe the moral support extended throughout the execution of this project.

Finally, I wish to dedicate this book to the memory of my late father.

Bangalore,
August 1988

S. KESAVAN

Notations

I. Notations in Euclidean Spaces

$\mathbf{R}$	stands for the real line.
$\mathbf{R}^n$	stands for the n-dimensional Euclidean space over $\mathbf{R}$.
$\mathbf{C}$	stands for the set of complex numbers.
e_i	stands for the ith standard basis vector in $\mathbf{R}^n$.
x	$=(x_1 \ldots x_n)$ is a vector in $\mathbf{R}^n$ with coordinates x_i, $1 \leqslant i \leqslant n$.
$\|x\|$	is the Euclidean norm of $x \in \mathbf{R}^n$.
$\text{dist}(x, A)$	is the Euclidean distance of $x \in \mathbf{R}^n$ from a subset $A \subset \mathbf{R}^n$.
Ω	stands for an open set in $\mathbf{R}^n$.
$\bar{\Omega}$	stands for its closure in $\mathbf{R}^n$.
$\Gamma = \partial\Omega$	stands for the boundary of Ω.
$\Omega' \subset \subset \Omega$	means that Ω' is a relatively compact open subset of Ω.

II. Function Spaces

$C(\Omega)$	is the space of continuous functions on Ω.
$C(\bar{\Omega})$	is the space of continuous functions on $\bar{\Omega}$.
$C^k(\Omega)$	is the space of k times continuously differentiable functions on Ω.
$C^k(\bar{\Omega})$	is the space of functions in $C^k(\Omega)$ which together with all derivatives possess continuous extensions to $\bar{\Omega}$.
$C^\infty(\Omega)$	$= \bigcap_{k=0}^{\infty} C^k(\Omega)$.
$C^\infty(\bar{\Omega})$	$= \bigcap_{k=0}^{\infty} C^k(\bar{\Omega})$.
$\mathcal{D}(\Omega)$	is the space of functions in $C^\infty(\Omega)$ with compact support in Ω $(\mathcal{D} = \mathcal{D}(\mathbf{R}^n))$.
$\mathcal{D}'(\Omega)$	is the space of distributions on Ω $(\mathcal{D}' = \mathcal{D}'(\mathbf{R}^n))$.
$\mathcal{E}(\Omega)$	$=C^\infty(\Omega)$ $(\mathcal{E} = \mathcal{E}(\mathbf{R}^n))$.
$\mathcal{E}'(\Omega)$	is the space of distributions with compact support in Ω $(\mathcal{E}' = \mathcal{E}'(\mathbf{R}^n))$.
$\mathcal{S}$	is the Schwartz space of rapidly decreasing functions in $\mathbf{R}^n$.
$\mathcal{S}'$	is the space of tempered distributions on $\mathbf{R}^n$.

$W^{m,p}(\Omega)$ is the Sobolev space of order m for $1 \leqslant p \leqslant \infty$ with norm $\|\cdot\|_{m,p,\Omega}$ and semi-norm $|\cdot|_{m,p,\Omega}$.

$W_0^{m,p}(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ in $W^{m,p}(\Omega)$.

$W^{m,2}(\Omega)$ $=H^m(\Omega)$ with norm $\|\cdot\|_{m,\Omega}$ and semi-norm $|\cdot|_{m,\Omega}$.

$W_0^{m,2}(\Omega)$ $=H_0^m(\Omega)$.

$W^{0,p}(\Omega)$ $=L^p(\Omega)$ with norm $|\cdot|_{0,p,\Omega}$.

$W^{0,2}(\Omega)$ $=L^2(\Omega)$ with norm $|\cdot|_{0,\Omega}$.

III. General Remarks

1. In any normed linear space $B(x; a)$ will stand for the open or closed ball (depending on the context) centered at x and of radius a.
2. In any estimate or inequality the quantity C will denote a generic positive constant and need not necessarily be the same constant as in the preceding calculations.

Contents

ONE

Distributions

1.1 INTRODUCTION

The main aim of this book is to develop the basic tools of functional analysis which will be useful in the study of partial differential equations and to illustrate their use via examples. As a first step towards this, the notion of differentiable functions must be generalized. When we study a partial differential equation, we understand—in the classical sense—that a solution must be differentiable at least as many times as the order of the equation and that it must satisfy the equation everywhere in space (and time). However such a point of view is very restrictive and several interesting equations which model physical phenomena will fail to possess such solutions and thus we will be prevented from studying mathematically such physical situations. Let us consider a few examples.

Example 1.1.1 Consider the following equation:

$$u_t + uu_x = 0,\ x \in \mathbf{R},\ t > 0 \tag{1.1.1}$$

where the subscripts denote differentiation with respect to the corresponding independent variable. This equation is known as *Burger's Equation* and is closely related to a class of partial differential equations known as *hyperbolic conservation laws*. Let $u(x, t)$ be a 'smooth' solution of (1.1.1) satisfying an initial condition of the form

$$u(x, 0) = u_0(x),\ x \in \mathbf{R} \tag{1.1.2}$$

where $u_0(x)$ is a given function of x. Let us now define a curve $x = x(t)$ in the x-t plane by means of the ordinary differential equation

$$\frac{dx}{dt}(t) = u(x(t), t). \tag{1.1.3}$$

Along such a curve we have

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = u_t + uu_x = 0$$

since u satisfies (1.1.1). Hence along each such curve, u = a constant. It then follows from (1.1.3) that such curves are straight lines. These are called 'characteristic curves' and the curve through the point $(x_0, 0)$ on the real line will have the form

$$x = x_0 + ct,\ c = u_0(x_0) \tag{1.1.4}$$

and all along this curve $u(x, t) = u_0(x_0)$.

Now consider a smooth initial function u_0 as in figure 1.

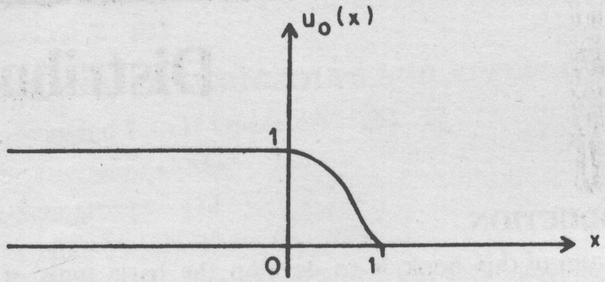


Fig. 1

The corresponding characteristic curves are shown in figure 2.

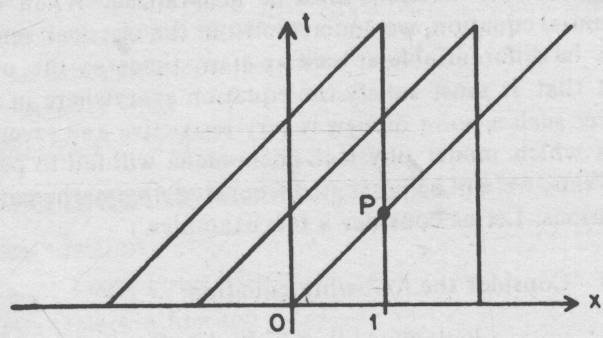


Fig. 2

It is seen from this figure that, for instance, at the point P two characteristic curves meet and hence the value of u at P is not even well defined. Thus except for a short time, we cannot even expect the function to be continuous! If we only wish to study classical solutions, such equations cannot be tackled and we will not be able to study interesting physical phenomena such as *shock waves* (which are very important in aeronautics). We are thus led to the need of generalizing the notion of a solution of partial differential equations which should eventually include discontinuous functions being recognized as solutions (albeit in a weak sense). ∎

Example 1.1.2 Let $\Omega \subset \mathbf{R}^2$ be a bounded open set. If Ω is the region occupied by a thin membrane fixed along the boundary $\partial\Omega$ and acted upon by a vertical force, then the displacement in the vertical direction is given by a function $u(x)$, $x \in \Omega$, which satisfies a partial differential equation of the form

$$\left.\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}\right\} \tag{1.1.5}$$

where Δ is the *Laplace operator* defined by

$$\Delta u = u_{xx} + u_{yy}. \tag{1.1.6}$$

This is a partial differential equation involving the second order derivatives of the function u. However, in mechanics, what is more important is that u minimizes the *strain energy* functional

$$J(v) = \frac{1}{2}\int_\Omega \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] dx\,dy - \int_\Omega fv\,dx\,dy \tag{1.1.7}$$

amongst all 'admissible displacements' v. A fact that strikes us immediately is that no second derivatives are involved in the definition of J! Thus in looking for the equilibrium state of the membrane the space of 'admissible displacements' need not involve functions which are twice differentiable and in fact for several 'reasonable' data f, it is not correct to do so. Nevertheless we would like to know the connection between the problem (1.1.5) and the minimizer u of the functional J. In other words though u may not be twice differentiable in the classical sense we would like to say that it still satisfies (1.1.5) in a weak sense.

Many computational schemes to approximate the solution of (1.1.5) stem from the variational characterization described above. ∎

The above examples are but few instances which motivate the need of generalizing the notion of a solution of a partial differential equation, which in turn motivates the need of generalizing the notion of differentiable functions. In other words, we will study a larger class of objects—called **Distributions**—on which we can define a (generalized) derivative and wherein the usual rules of calculus will hold. Further, for smooth functions, this new notion of a derivative must coincide with the usual one.

A rough idea as to how to set about realizing this class can be obtained from the following discussion.

Let $f \in L^2(\mathbb{R})$, the space of square integrable functions on $\mathbb{R}$. It can be shown that the space $\mathcal{D}$ of infinitely differentiable functions with compact support in $\mathbb{R}$ is dense in $L^2(\mathbb{R})$. (By the support of a function $\phi : \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) we mean the set

$$K = \overline{\{x \in \mathbb{R} \mid \phi(x) \neq 0\}} \tag{1.1.8}$$

which is always closed by definition.) Thus, since $L^2(\mathbb{R})$ is a Hilbert space, f is completely known once its innerproduct with each element of $\mathcal{D}$ is known, i.e. when all the numbers

$$\int_\Omega f\phi,\ \phi \in \mathcal{D}$$

are known. Now assume that f is continuously differentiable, with derivative f'. By integration by parts, we have

$$\int_{\mathbb{R}} f'\phi = -\int_{\mathbb{R}} f\phi'. \tag{1.1.9}$$

Notice now that the right-hand side of (1.1.9) does not involve the derivative of f! Also notice that the operations $\phi \to \int_{\mathbb{R}} f\phi$ and $\phi \to \int_{\mathbb{R}} f\phi'$ are linear on $\mathcal{D}$. Hence if we can define a suitable topology on $\mathcal{D}$ which makes these operations continuous, we can define f as a *continuous linear functional* on $\mathscr{D}$ and define f' via the right-hand-side of (1.1.9) *even when f is not differentiable* as long as the integrals make sense. This is the procedure we will follow in the next few sections.

1.2 TEST FUNCTIONS AND DISTRIBUTIONS

Let ϕ be a real (or complex) valued continuous function defined on an open set in $\mathbf{R}^n$. The **support** of ϕ, written as supp (ϕ), is defined as the closure (in $\mathbf{R}^n$) of the set on which ϕ is non-zero (cf. (1.1.8)). If this closed set is compact as well, then ϕ is said to be of compact support. The set of all infinitely differentiable (i.e. C^∞) functions defined on $\mathbf{R}^n$ with compact support is a vector space which will henceforth be denoted by $\mathscr{D}(\mathbf{R}^n)$ or, simply, $\mathscr{D}$. We will now show that this class of functions is quite rich.

Lemma 1.2.1 Let $f: \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} \exp(-x^{-2}), & x > 0 \\ 0 & , x \leqslant 0. \end{cases} \tag{1.2.1}$$

Then f is a C^∞ function.

Proof: We only need to check the smoothness at $x = 0$. As $x \uparrow 0$ all derivatives are zero. As $x \downarrow 0$, the derivatives are finite linear combinations of terms of the form $x^{-l} \exp(-x^{-2})$, l an integer greater than or equal to zero.

A simple application of *l*'Hospital's Rule shows that these terms tend to zero as $x \downarrow 0$. ▮

We can use the above lemma to construct examples of elements of $\mathscr{D}$.

Example 1.2.1 Consider the function

$$\phi(x) = \begin{cases} \exp(-a^2/(a^2 - x^2)), & |x| < a \\ 0, & |x| \geqslant a \end{cases} \tag{1.2.2}$$

Then a simple application of the preceding lemma shows that $\phi \in \mathscr{D}(\mathbf{R})$ and supp $(\phi) = [-a, a]$.

More generally, define

$$\phi(x) = \begin{cases} \exp(-a^2/(a^2 - |x|^2)), & |x| < a \\ 0 & , |x| \geqslant a \end{cases} \tag{1.2.3}$$

where $|x|^2 = \sum_{i=1}^{n} |x_i|^2$, $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$. Then $\phi \in \mathscr{D}(\mathbf{R}^n)$ with supp (ϕ) = Ball centre 0 and radius a (denoted $B(0; a)$). ▮

Example 1.2.2 This is a slight, but very useful, variation of the previous example. Let $\epsilon > 0$ and set

$$\rho_\epsilon(x) = \begin{cases} k\epsilon^{-n} \exp(-\epsilon^2/(\epsilon^2 - |x|^2)), & |x| < \epsilon \\ 0 & , |x| \geqslant \epsilon \end{cases} \tag{1.2.4}$$

where

$$k^{-1} = \int_{|x| \leqslant 1} \exp(-1/(1 - |x|^2))\, dx. \tag{1.2.5}$$

Then by (1.2.3), we know that $\rho_\epsilon \in D(\mathbb{R}^n)$ with supp $(\rho_\epsilon) = B(0; \epsilon)$, the ball centre 0 and radius ϵ. Further $\rho_\epsilon \geqslant 0$ and

$$\int_{\mathbb{R}^n} \rho_\epsilon(x)\, dx = 1. \tag{1.2.6}$$

For,

$$\int_{\mathbb{R}^n} \rho_\epsilon(x) dx = \frac{k}{\epsilon^n} \int_{|x| \leqslant \epsilon} (-\epsilon^2/(\epsilon^2 - |x|^2))\, dx$$

$$= k \int_{|x| \leqslant 1} \exp(-1/(1 - |x|^2))\, dx = 1.$$

Thus the functions ρ_ϵ, $\epsilon \to 0$, have smaller supports, but preserve the volume contained under the graph. As $\epsilon \to 0$, these functions are concentrated at the origin. They will be used repeatedly in the sequel and are called **mollifiers.** ▮

A family of sets $\{E_i\}_{i \in I}$ in $\mathbb{R}^n$ is said to be *locally finite* if for every point x, there exists a neighbourhood of x which intersects only a finite number of the E_i. We now quote, without proof, a very important theorem. For complete details, see Appendix 1.

Theorem 1.2.1 (*Locally Finite C^∞ Partition of Unity*). Let Ω be an open set in $\mathbf{R}^n$ and let $\Omega = \bigcup_{i \in I} \Omega_i$, Ω_i, open. Then there exist C^∞ functions ϕ_i defined on Ω such that

(i) supp $(\phi_i) \subset \Omega_i$

(ii) $\{\text{supp}\, (\phi_i)\}_{i \in I}$ is locally finite

(iii) $0 \leqslant \phi_i(x) \leqslant 1$, for all $i \in I$, and

(iv) $\sum_{i \in I} \phi_i \equiv 1$. ▮

Remark 1.2.1 Since given any $x \in \Omega$, there exists a neighbourhood which will intersect only a finite number of the sets $\{\text{supp}\, (\phi_i)\}$, it follows that $\phi_i(x) = 0$ for all but finitely many i. Thus the sum in (iv) above is in fact a finite sum and is thus well defined. The name *partition of unity* is self-explanatory: the constant function 1 is partitioned into C^∞ functions whose support can be controlled. ▮

Corollary Let K be a compact set in $\mathbf{R}^n$. Then there exists a $\phi \in \mathcal{D}(\mathbf{R}^n)$ such that $\phi \equiv 1$ on K.

Proof: We consider a relatively compact open set U containing K. Now consider the covering of $\mathbf{R}^n$ consisting of $\{U, \mathbf{R}^n \setminus K\}$ and the partition of unity subordinate to this cover. Let ϕ and ψ be non-negative C^∞ functions with $\phi + \psi \equiv 1$, supp $(\phi) \subset U$ and supp $(\psi) \subset \mathbf{R}^n \setminus K$. Thus $\psi \equiv 0$ on K and hence $\phi \equiv 1$ on K. Also supp $(\phi) \subset U \subset \bar{U}$ which is compact. Thus $\phi \in \mathcal{D}\, (\mathbf{R}^n)$. ▮

The function ϕ constructed above is called a **cut-off function** with respect to the compact set K.

We have thus established that the class $\mathcal{D}$, also called the space of **test-functions,** is well endowed with functions. If Ω is any open set in $\mathbb{R}^n$, we can still talk of the space of C^∞ functions with compact support, the support being contained in Ω. This space will be denoted by $\mathcal{D}(\Omega)$.

We will now provide $\mathcal{D}(\Omega)$ with a topology which will make it a topological vector space. In fact, for the development of the theory, we will not need a complete description of the topology; we will only need to know what are convergent sequences in $\mathcal{D}(\Omega)$. Hence we will abstain, for the moment, from describing this topology and defer this task to Appendix 2. We will just define convergent sequences in $\mathcal{D}(\Omega)$.

Definition 1.2.1 A sequence of functions $\{\phi_m\}$ in $\mathcal{D}(\Omega)$ is said to converge to 0 if there exists a *fixed* compact set $K \subset \Omega$ such that supp $(\phi_m) \subset K$ for all m and ϕ_m and all its derivatives converge *uniformly* to zero on K. ▮

As indicated in Section 1.1, we will generalize the notion of a function by considering linear functionals on $\mathcal{D}(\Omega)$ which are continuous with respect to the above mentioned topology.

Definition 1.2.2 A linear functional T on $\mathcal{D}(\Omega)$ is said to be a **distribution** on Ω if whenever $\phi_m \to 0$ in $\mathcal{D}(\Omega)$, we have $T(\phi_m) \to 0$. ▮

The space of distributions, which is the dual of the space of test-functions, is denoted by $\mathcal{D}'(\Omega)$. In case $\Omega = \mathbb{R}^n$, the symbol $\mathcal{D}'$ will also be used. We now proceed to give several examples of distributions.

Example 1.2.3 A function $f: \Omega \to \mathbb{R}$ (or $\mathbb{C}$) is said to be **locally integrable** if for every compact set $K \subset \Omega$,

$$\int_K |f| < +\infty. \tag{1.2.7}$$

For instance any continuous function is locally integrable. Another example of a locally integrable function (on $\mathbb{R}^2$) is r^{-1} where $r = |x|$. If B is the ball of radius ϵ centred at the origin, then

$$\int_B \frac{1}{r} = \int_0^\epsilon \int_0^{2\pi} \frac{1}{r}\, r\, d\theta\, dr,$$

which is finite.

Given a locally integrable function f, define $T_f : \mathcal{D}(\Omega) \to \mathbb{R}$ (or $\mathbb{C}$) by

$$T_f(\phi) = \int_\Omega f\phi. \tag{1.2.8}$$

Clearly T_f is a linear functional on $\mathcal{D}$ and it is easy to verify that it is a distribution.

If f and g are two locally integrable functions such that $f = g$ a.e. then it is obvious that $T_f = T_g$. In particular if $f = 0$ a.e., it defines the zero