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J.-P. Demailly T. Peternell
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Transcendental Methods in Algebraic Geometry

Cetraro, 1994

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Lectures given at the 3rd Session of the
Centro Internazionale Matematico Estivo
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held in Cetraro, Italy, July 4–12, 1994

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Authors

Jean-Pierre Demailly
Institut Fourier
Université de Grenoble I
F-38402 Saint-Martin-d'Hères, France

Thomas Peternell
Mathematisches Institut
Universität Bayreuth
Universitätsstrasse 30
D-95447 Bayreuth, Germany

Gang Tian
Massachusetts Institute of Technology
1105 Massachusetts Avenue
Cambridge, MA 02139, USA

Andrej N. Tyurin
Steklov Institute of Mathematics
ul. Vavilov 42
117996 Moscow, Russia

Editors

Fabrizio Catanese
Dipartimento di Matematica
Università di Pisa
Via Buonarroti, 2
I-56127 Pisa, Italy

Ciro Ciliberto
Dipartimento di Matematico
Università di Tor Vergata
V. della Ricerca Scientifica
I-00133 Rome, Italy

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Preface

The Third 1994 C.I.M.E. Session "Transcendental Methods in Algebraic Geometry" took place from July 4 to July 12 in the beautiful location of the Grand Hotel San Michele, Cetraro (Cosenza).

Already in the prehistory of algebraic geometry we find the theory of elliptic and Abelian integrals, which is directly linked with Riemann's topological approach to creating the concept of a manifold. Later on, from Poincaré's use of potential theory for the study of function theory on complex tori to Hodge's theory of harmonic integrals and the vanishing theorems of Kodaira and others, we see that the transcendental approach puts many algebraic geometric questions on a firm basis.

In doing so, it establishes deep and surprising results, which can often be stated simply, invigorating a century-long tradition of manifold and fruitful relations with other disciplines. In a surprising way we see close analogies displayed between apparently distant methodologies, thus concretely augmenting the unified edifice of mathematics. It was one of the purposes of the 1994 course to look at the recent developments relating algebraic geometry to complex analysis, complex differential geometry, and differential topology as further manifestations of the core of algebraic geometry: a core which, although nourished by a myriad of subtle and intricate problems, has as its lifeblood the crucial interplay with a host of other subjects, be they physics, topology, algebra, analysis, differential geometry, or arithmetic.

From this point of view, the courses given by Demailly, Peternell, Tian, and Tyurin covered a very wide spectrum, each offering not only a broad view of recent developments and new results published here for the first time, but also opening wide perspectives still in the earliest stages of exploration. The beautiful texts of the four courses reproduced here give us ample justification for dispensing with further historical and mathematical description.

We would just like to recall that, as in the ancient Greek dramas, unity of place (lecturers and participants brought close together in the "golden cage" of San Michele), unity of action (there were only courses and problem sessions), and unity of time (one of the features of C.I.M.E.) contributed to the success of the course. This success was in large part due not only to the excellent lecturers but also to the brightness and knowledge of the participants: the variety of their cultural interests was for us very impressive, as well as their devotion to science amidst such tempting scenery.

The organizers: Fabrizio Catanese and Ciro Ciliberto

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Jean-Pierre Demailly
Université de Grenoble I, Institut Fourier

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0. Introduction

Transcendental methods of algebraic geometry have been extensively studied since a very long time, starting with the work of Abel, Jacobi and Riemann in the nineteenth century. More recently, in the period 1940-1970, the work of Hodge, Hirzebruch, Kodaira, Atiyah revealed still deeper relations between complex analysis, topology, PDE theory and algebraic geometry. In the last ten years, gauge theory has proved to be a very efficient tool for the study of many important questions: moduli spaces, stable sheaves, non abelian Hodge theory, low dimensional topology ...

Our main purpose here is to describe a few analytic tools which are useful to study questions such as linear series and vanishing theorems for algebraic vector bundles. One of the early success of analytic methods in this context is Kodaira's use of the Bochner technique in relation with the theory of harmonic forms, during the decade 1950-60. The idea is to represent cohomology classes by harmonic forms and to prove vanishing theorems by means of suitable a priori curvature estimates. The prototype of such results is the Akizuki-Kodaira-Nakano theorem (1954): if X is a nonsingular projective algebraic variety and L is a holomorphic line bundle on X with positive curvature, then $H^q(X, \Omega^p_X \otimes L) = 0$ for $p+q > \dim X$ (throughout the paper we set $\Omega_X^p = \Lambda^p T_X^\star$ and $K_X = \Lambda^n T_X^\star$, $n = \dim X$, viewing these objects either as holomorphic bundles or as locally free $\mathcal{O}_X$-modules). It is only much later that an algebraic proof of this result has been proposed by Deligne-Illusie, via characteristic p methods, in 1986.

A refinement of the Bochner technique used by Kodaira led about ten years later to fundamental L^2 estimates due to Hörmander [Hör65], concerning solutions of the Cauchy-Riemann operator. Not only vanishing theorems are proved, but more precise information of a quantitative nature is obtained about solutions of $\overline{\partial}$-equations. The best way of expressing these L^2 estimates is to use a geometric setting first considered by Andreotti-Vesentini [AV65]. More explicitly, suppose that we have a holomorphic line bundle L is equipped with a hermitian metric of weight $e^{-2\varphi}$, where φ is a (locally defined) plurisubharmonic function; then explicit bounds on the L^2 norm $\int_X |f|^2 e^{-2\varphi}$ of solutions is obtained. The result is still more useful if the plurisubharmonic weight φ is allowed to have singularities. Following Nadel [Nad89], one defines the *multiplier ideal sheaf* $\mathcal{I}(\varphi)$ to be the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-2\varphi}$ is locally summable. Then $\mathcal{I}(\varphi)$ is a coherent algebraic sheaf over X and $H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0$ for all $q \geq 1$ if the curvature of L is positive (as a current). This important result can be seen as a generalization of the Kawamata-Viehweg vanishing theorem ([Kaw82], [Vie82]), which is one of the cornerstones of higher dimensional algebraic geometry (especially of Mori's minimal model program).

In the dictionary between analytic geometry and algebraic geometry, the ideal $\mathcal{I}(\varphi)$ plays a very important role, since it directly converts an analytic object into an algebraic one, and, simultaneously, takes care of the singularities in a very efficient way. Another analytic tool used to deal with singularities is the theory of positive currents introduced by Lelong [Lel57]. Currents can be seen as generalizations of algebraic cycles, and many classical results of intersection theory still apply to currents. The concept of Lelong number of a current is the analytic analogue of the concept of multiplicity of a germ of algebraic variety. Intersections of cycles correspond to wedge products of currents (whenever these products are defined). A convenient measure of local positivity of a holomorphic line can be defined in this context: the *Seshadri constant* of a line bundle at a point is the largest possible Lelong number for a singular metric of positive curvature assuming an isolated singularity at the given point (see [Dem90]). Seshadri constants can also be given equivalent purely algebraic definitions. We refer to Ein-Lazarsfeld [EL92] and Ein-Küchle-Lazarsfeld [EKL94] for very interesting new results concerning Seshadri constants.

One of our main motivations has been the study of the following conjecture of Fujita: if L is an ample (i.e. positive) line bundle on a projective n-dimensional algebraic variety X, then $K_K + (n+2)L$ is very ample. A major result obtained by Reider [Rei88] is a proof of the Fujita conjecture in the case of surfaces (the case of curves is easy). Reider's approach is based on Bogomolov's inequality for stable vector bundles and the results obtained are almost optimal. Unfortunately, it seems difficult to extend Reider's original method to higher dimensions. In the analytic approch, which works for arbitrary dimensions, one tries to construct a suitable (singular) hermitian metric on L such that the the ideal $\mathcal{I}(\varphi)$ has a given 0-dimensional subscheme of X as its zero variety. As we showed in [Dem93b], this can be done essentially by solving a complex Monge-Ampère equation

$$(id'd''\varphi)^n = \text{linear combination of Dirac measures},$$

via the Aubin-Calabi-Yau theorem ([Aub78], [Yau78]). The solution φ then assumes logarithmic poles and the difficulty is to force the singularity to be an isolated pole; this is the point where intersection theory of currents is useful. In this way, we can prove e.g. that $2K_X + L$ is very ample under suitable numerical conditions for L. Alternative algebraic techniques have been developed recently by Kollár [Kol92], Ein-Lazarsfeld [EL93], Fujita [Fuj93] and [Siu94a, b]. The basic idea is to apply the Kawamata-Viehweg vanishing theorem, and to use the Riemann-Roch formula instead of the Monge-Ampère equation. The proofs proceed with careful inductions on dimension, together with an analysis of the base locus of the linear systems involved. Although the results obtained in low dimensions are slightly more precise than with the analytic method, it is still not clear whether the range of applicability of the methods are exactly the same. Because it fits well with our approach, we have included here a simple algebraic method due to Y.T. Siu [Siu94a], showing that $2K_X + mL$ is very ample for $m \geq 2 + \binom{3n+1}{n}$.

Our final concern in these notes is a proof of the effective Matsusaka big theorem obtained by [Siu93]. Siu's result is the existence of an effective value m_0 depending only on the intersection numbers L^n and $L^{n-1} \cdot K_X$, such that mL is very ample for $m \geq m_0$. The basic idea is to combine results on the very ampleness of $2K_X + mL$ together with the theory of holomorphic Morse inequalities ([Dem85b]). The Morse inequalities are used to construct sections of $m'L - K_X$ for m' large. Again this step can be made algebraic (following suggestions by F. Catanese and R. Lazarsfeld), but the analytic formulation apparently has a wider range of applicability.

These notes are essentially written with the idea of serving as an analytic toolbox for algebraic geometry. Although efficient algebraic techniques exist, our feeling is that the analytic techniques are very flexible and offer a large variety of guidelines for more algebraic questions (including applications to number theory which are not discussed here). We made a special effort to use as little prerequisites and to be as self-contained as possible; hence the rather long preliminary sections dealing with basic facts of complex differential geometry. The reader wishing to have a presentation of the algebraic approach to vanishing theorems and linear series is referred to the excellent notes written by R. Lazarsfeld [Laz93]. In the last years, there has been a continuous and fruitful interplay between the algebraic and analytic viewpoints on these questions, and I have greatly benefitted from observations and ideas contained in the works of J. Kollár, L. Ein, R. Lazarsfeld and Y.T. Siu. I would like to thank them for their interest in my work and for their encouragements.

1. Preliminary Material

1.A. Dolbeault Cohomology and Sheaf Cohomology

Let X be a $\mathbb{C}$-analytic manifold of dimension n. We denote by $\Lambda^{p,q}T_X^\star$ the bundle of differential forms of bidegree (p,q) on X, i.e., differential forms which can be written as

$$u = \sum_{|I|=p,|J|=q} u_{I,J} dz_I \wedge d\overline{z}_J.$$

Here $(z_1,\ldots,z_n)$ denote arbitrary local holomorphic coordinates, $I=(i_1,\ldots,i_p)$, $J=(j_1,\ldots,j_q)$ are multiindices (increasing sequences of integers in the range $[1,\ldots,n]$, of lengths $|I|=p$, $|J|=q$), and

$$dz_I := dz_{i_1}\wedge\ldots\wedge dz_{i_p}, \qquad d\overline{z}_J := d\overline{z}_{j_1}\wedge\ldots\wedge d\overline{z}_{j_q}.$$

Let $\mathcal{E}^{p,q}$ be the sheaf of germs of complex valued differential (p,q)-forms with C^∞ coefficients. Recall that the exterior derivative d splits as $d=d'+d''$ where

$$d'u = \sum_{|I|=p,|J|=q,1\le k\le n} \frac{\partial u_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J,$$

$$d''u = \sum_{|I|=p,|J|=q,1\le k\le n} \frac{\partial u_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J$$

are of type $(p+1,q)$, $(p,q+1)$ respectively. The well-known Dolbeault-Grothendieck lemma asserts that any d''-closed form of type (p,q) with $q>0$ is locally d''-exact (this is the analogue for d'' of the usual Poincaré lemma for d, see e.g. Hörmander 1966). In other words, the complex of sheaves $(\mathcal{E}^{p,\bullet},d'')$ is exact in degree $q>0$; in degree $q=0$, $\operatorname{Ker} d''$ is the sheaf Ω_X^p of germs of holomorphic forms of degree p on X.

More generally, if F is a holomorphic vector bundle of rank r over X, there is a natural d'' operator acting on the space $C^\infty(X,\Lambda^{p,q}T_X^\star\otimes F)$ of smooth (p,q)-forms with values in F; if $s=\sum_{1\le\lambda\le r} s_\lambda e_\lambda$ is a (p,q)-form expressed in terms of a local holomorphic frame of F, we simply define $d''s := \sum d''s_\lambda\otimes e_\lambda$, observing that the holomorphic transition matrices involved in changes of holomorphic frames do not affect the computation of d''. It is then clear that the Dolbeault-Grothendieck lemma still holds for F-valued forms. For every integer $p=0,1,\ldots,n$, the *Dolbeault Cohomology* groups $H^{p,q}(X,F)$ are defined to be the cohomology groups of the complex of global (p,q) forms (graded by q):

$$H^{p,q}(X,F) = H^q\big(C^\infty(X,\Lambda^{p,\bullet}T_X^\star\otimes F)\big). \tag{1.1}$$

Now, let us recall the following fundamental result from sheaf theory (De Rham-Weil isomorphism theorem): let $(\mathcal{L}^\bullet,d)$ be a resolution of a sheaf $\mathcal{A}$ by acyclic sheaves, i.e. a complex of sheaves $(\mathcal{L}^\bullet,\delta)$ such that there is an exact sequence of sheaves

$$0\longrightarrow\mathcal{A}\xrightarrow{j}\mathcal{L}^0\xrightarrow{\delta^0}\mathcal{L}^1\longrightarrow\cdots\longrightarrow\mathcal{L}^q\xrightarrow{\delta^q}\mathcal{L}^{q+1}\longrightarrow\cdots,$$

and $H^s(X,\mathcal{L}^q)=0$ for all $q\ge0$ and $s\ge1$. Then there is a fonctorial isomorphism

$$H^q(\Gamma(X,\mathcal{L}^\bullet)) \longrightarrow H^q(X,\mathcal{A}). \tag{1.2}$$

We apply this to the following situation: let $\mathcal{E}(F)^{p,q}$ be the sheaf of germs of C^∞ sections of $\Lambda^{p,q}T_X^\star \otimes F$, Then $(\mathcal{E}(F)^{p,\bullet}, d'')$ is a resolution of the locally free $\mathcal{O}_X$-module $\Omega_X^p \otimes \mathcal{O}(F)$ (Dolbeault-Grothendieck lemma), and the sheaves $\mathcal{E}(F)^{p,q}$ are acyclic as modules over the soft sheaf of rings $\mathcal{C}^\infty$. Hence by (1.2) we get

(1.3) Dolbeault Isomorphism Theorem (1953). *For every holomorphic vector bundle F on X, there is a canonical isomorphism*

$$H^{p,q}(X,F) \simeq H^q(X,\Omega_X^p \otimes \mathcal{O}(F)). \qquad \square$$

If X is projective algebraic and F is an algebraic vector bundle, Serre's GAGA theorem [Ser56] shows that the algebraic sheaf cohomology group $H^q(X,\Omega_X^p \otimes \mathcal{O}(F))$ computed with algebraic sections over Zariski open sets is actually isomorphic to the analytic cohomology group. These results are the most basic tools to attack algebraic problems via analytic methods. Another important tool is the theory of plurisubharmonic functions and positive currents originated by K. Oka and P. Lelong in the decades 1940-1960.

1.B. Plurisubharmonic Functions

Plurisubharmonic functions have been introduced independently by Lelong and Oka in the study of holomorphic convexity. We refer to [Lel67, 69] for more details.

(1.4) Definition. *A function $u : \Omega \longrightarrow [-\infty,+\infty[$ defined on an open subset $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic (psh for short) if*

a) *u is upper semicontinuous ;*

b) *for every complex line $L \subset \mathbb{C}^n$, $u_{\restriction \Omega \cap L}$ is subharmonic on $\Omega \cap L$, that is, for all $a \in \Omega$ and $\xi \in \mathbb{C}^n$ with $|\xi| < d(a, \complement\Omega)$, the function u satisfies the mean value inequality*

$$u(a) \le \frac{1}{2\pi}\int_0^{2\pi} u(a+e^{i\theta}\,\xi)\,d\theta.$$

The set of psh functions on Ω is denoted by Psh(Ω).

We list below the most basic properties of psh functions. They all follow easily from the definition.

(1.5) Basic properties.

a) Every function $u \in \mathrm{Psh}(\Omega)$ is subharmonic, namely it satisfies the mean value inequality on euclidean balls or spheres:

$$u(a) \le \frac{1}{\pi^n r^{2n}/n!}\int_{B(a,r)} u(z)\,d\lambda(z)$$

for every $a \in \Omega$ and $r < d(a, \complement\Omega$. Either $u \equiv -\infty$ or $u \in L^1_{\rm loc}$ on every connected component of Ω.

b) For any decreasing sequence of psh functions $u_k \in \mathrm{Psh}(\Omega)$, the limit $u = \lim u_k$ is psh on Ω.

c) Let $u \in \mathrm{Psh}(\Omega)$ be such that $u \not\equiv -\infty$ on every connected component of Ω. If (ρ_ε) is a family of smoothing kernels, then $u \star \rho_\varepsilon$ is C^∞ and psh on

$$\Omega_\varepsilon = \{x \in \Omega\,;\, d(x, \complement\Omega) > \varepsilon\},$$

the family $(u \star \rho_\varepsilon)$ is increasing in ε and $\lim_{\varepsilon\to 0} u \star \rho_\varepsilon = u$.

d) Let $u_1, \ldots, u_p \in \mathrm{Psh}(\Omega)$ and $\chi : \mathbb{R}^p \longrightarrow \mathbb{R}$ be a convex function such that $\chi(t_1, \ldots, t_p)$ is increasing in each t_j. Then $\chi(u_1, \ldots, u_p)$ is psh on Ω. In particular $u_1 + \cdots + u_p$, $\max\{u_1, \ldots, u_p\}$, $\log(e^{u_1} + \cdots + e^{u_p})$ are psh on Ω. □

(1.6) Lemma. *A function $u \in C^2(\Omega, \mathbb{R})$ is psh on Ω if and only if the hermitian form $Hu(a)(\xi) = \sum_{1\le j,k\le n} \partial^2 u/\partial z_j \partial\overline{z}_k(a)\, \xi_j \overline{\xi}_k$ is semipositive at every point $a \in \Omega$.*

Proof. This is an easy consequence of the following standard formula

$$\frac{1}{2\pi}\int_0^{2\pi} u(a + e^{\mathrm{i}\theta}\,\xi)\,d\theta - u(a) = \frac{2}{\pi}\int_0^1 \frac{dt}{t}\int_{|\zeta|<t} Hu(a+\zeta\xi)(\xi)\,d\lambda(\zeta),$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{C}$. Lemma 1.6 is a strong evidence that plurisubharmonicity is the natural complex analogue of linear convexity. □

For non smooth functions, a similar characterization of plurisubharmonicity can be obtained by means of a regularization process.

(1.7) Theorem. *If $u \in \mathrm{Psh}(\Omega)$, $u \not\equiv -\infty$ on every connected component of Ω, then for all $\xi \in \mathbb{C}^n$*

$$Hu(\xi) = \sum_{1\le j,k\le n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\,\xi_j\overline{\xi}_k \in \mathcal{D}'(\Omega)$$

is a positive measure. Conversely, if $v \in \mathcal{D}'(\Omega)$ is such that $Hv(\xi)$ is a positive measure for every $\xi \in \mathbb{C}^n$, there exists a unique function $u \in \mathrm{Psh}(\Omega)$ which is locally integrable on Ω and such that v is the distribution associated to u. □

In order to get a better geometric insight of this notion, we assume more generally that u is a function on a complex n-dimensional manifold X. If $\Phi : X \to Y$ is a holomorphic mapping and if $v \in C^2(Y, \mathbb{R})$, we have $d'd''(v \circ \Phi) = \Phi^\star d'd''v$, hence

$$H(v \circ \Phi)(a, \xi) = Hv\big(\Phi(a), \Phi'(a).\xi\big).$$

In particular Hu, viewed as a hermitian form on T_X, does not depend on the choice of coordinates $(z_1, \ldots, z_n)$. Therefore, the notion of psh function makes sense on any complex manifold. More generally, we have

(1.8) Proposition. *If $\Phi : X \longrightarrow Y$ is a holomorphic map and $v \in \mathrm{Psh}(Y)$, then $v \circ \Phi \in \mathrm{Psh}(X)$.* □

(1.9) Example. It is a standard fact that $\log|z|$ is psh (i.e. subharmonic) on $\mathbb{C}$. Thus $\log|f| \in \mathrm{Psh}(X)$ for every holomorphic function $f \in H^0(X, \mathcal{O}_X)$. More generally

$$\log\left(|f_1|^{\alpha_1} + \cdots + |f_q|^{\alpha_q}\right) \in \mathrm{Psh}(X)$$

for every $f_j \in H^0(X, \mathcal{O}_X)$ and $\alpha_j \geq 0$ (apply Property 1.5 d with $u_j = \alpha_j \log|f_j|$). We will be especially interested in the singularities obtained at points of the zero variety $f_1 = \ldots = f_q = 0$, when the α_j are rational numbers. □

(1.10) Definition. *A psh function $u \in \mathrm{Psh}(X)$ will be said to have analytic singularities if u can be written locally as*

$$u = \frac{\alpha}{2} \log\left(|f_1|^2 + \cdots + |f_N|^2\right) + v,$$

where $\alpha \in \mathbb{R}_+$, v is a locally bounded function and the f_j are holomorphic functions. If X is algebraic, we say that u has algebraic singularities if u can be written as above on sufficiently small Zariski open sets, with $\alpha \in \mathbb{Q}_+$ and f_j algebraic.

We then introduce the ideal $\mathcal{J} = \mathcal{J}(u/\alpha)$ of germs of holomorphic functions h such that $|h| \leq Ce^{u/\alpha}$ for some constant C, i.e.

$$|h| \leq C\left(|f_1| + \cdots + |f_N|\right).$$

This is a globally defined ideal sheaf on X, locally equal to the integral closure $\overline{\mathcal{I}}$ of the ideal sheaf $\mathcal{I} = (f_1, \ldots, f_N)$, thus $\mathcal{J}$ is coherent on X. If $(g_1, \ldots, g_{N'})$ are local generators of $\mathcal{J}$, we still have

$$u = \frac{\alpha}{2} \log\left(|g_1|^2 + \cdots + |g_{N'}|^2\right) + O(1).$$

If X is projective algebraic and u has analytic singularities with $\alpha \in \mathbb{Q}_+$, then u automatically has algebraic singularities. From an algebraic point of view, the singularities of u are in 1:1 correspondence with the "algebraic data" $(\mathcal{J}, \alpha)$. Later on, we will see another important method for associating an ideal sheaf to a psh function.

(1.11) Exercise. Show that the above definition of the integral closure of an ideal $\mathcal{I}$ is equivalent to the following more algebraic definition: $\overline{\mathcal{I}}$ consists of all germs h satisfying an integral equation

$$h^d + a_1 h^{d-1} + \ldots + a_{d-1} h + a_d = 0, \qquad a_k \in \mathcal{I}^k.$$

Hint. One inclusion is clear. To prove the other inclusion, consider the normalization of the blow-up of X along the (non necessarily reduced) zero variety $V(\mathcal{I})$. □

1.C. Positive Currents

The reader can consult [Fed69] for a more thorough treatment of current theory. Let us first recall a few basic definitions. A *current* of degree q on an oriented differentiable manifold M is simply a differential q-form Θ with distribution coefficients. The space of currents of degree q over M will be denoted by $\mathcal{D}'^q(M)$. Alternatively, a current of degree q can be seen as an element Θ in the dual space $\mathcal{D}'_p(M) := \big(\mathcal{D}^p(M)\big)'$ of the space $\mathcal{D}^p(M)$ of smooth differential forms of degree $p = \dim M - q$ with compact support; the duality pairing is given by

$$(1.12)\qquad \langle \Theta, \alpha\rangle = \int_M \Theta\wedge\alpha, \qquad \alpha\in\mathcal{D}^p(M).$$

A basic example is the *current of integration* $[S]$ over a compact oriented submanifold S of M:

$$(1.13)\qquad \langle [S], \alpha\rangle = \int_S \alpha, \qquad \deg\alpha = p = \dim_{\mathbb{R}} S.$$

Then $[S]$ is a current with measure coefficients, and Stokes' formula shows that $d[S] = (-1)^{q-1}[\partial S]$, in particular $d[S] = 0$ if S has no boundary. Because of this example, the integer p is said to be the dimension of Θ when $\Theta\in\mathcal{D}'_p(M)$. The current Θ is said to be *closed* if $d\Theta = 0$.

On a complex manifold X, we have similar notions of bidegree and bidimension; as in the real case, we denote by

$$\mathcal{D}'^{p,q}(X) = \mathcal{D}'_{n-p,n-q}(X), \qquad n = \dim X,$$

the space of currents of bidegree (p,q) and bidimension $(n-p, n-q)$ on X. According to [Lel57], a current Θ of bidimension (p,p) is said to be (*weakly*) *positive* if for every choice of smooth $(1,0)$-forms $\alpha_1,\ldots,\alpha_p$ on X the distribution

$$(1.14)\qquad \Theta\wedge \mathrm{i}\alpha_1\wedge\overline{\alpha}_1\wedge\ldots\wedge\mathrm{i}\alpha_p\wedge\overline{\alpha}_p \quad \text{is a positive measure.}$$

(1.15) Exercise. If Θ is positive, show that the coefficients $\Theta_{I,J}$ of Θ are complex measures, and that, up to constants, they are dominated by the trace measure

$$\sigma_\Theta = \Theta\wedge\frac{1}{p!}\beta^p = 2^{-p}\sum\Theta_{I,I}, \qquad \beta = \frac{\mathrm{i}}{2}d'd''|z|^2 = \frac{\mathrm{i}}{2}\sum_{1\le j\le n} dz_j\wedge d\overline{z}_j,$$

which is a positive measure.
Hint. Observe that $\sum\Theta_{I,I}$ is invariant by unitary changes of coordinates and that the (p,p)-forms $\mathrm{i}\alpha_1\wedge\overline{\alpha}_1\wedge\ldots\wedge\mathrm{i}\alpha_p\wedge\overline{\alpha}_p$ generate $\Lambda^{p,p}T^\star_{\mathbb{C}^n}$ as a $\mathbb{C}$-vector space. □

A current $\Theta = \mathrm{i}\sum_{1\le j,k\le n}\Theta_{jk}dz_j\wedge d\overline{z}_k$ of bidegree $(1,1)$ is easily seen to be positive if and only if the complex measure $\sum\lambda_j\overline{\lambda}_k\Theta_{jk}$ is a positive measure for every n-tuple $(\lambda_1,\ldots,\lambda_n)\in\mathbb{C}^n$.

(1.16) Example. If u is a (not identically $-\infty$) psh function on X, we can associate with u a (closed) positive current $\Theta = \mathrm{i}\partial\overline{\partial}u$ of bidegree $(1,1)$. Conversely, every closed positive current of bidegree $(1,1)$ can be written under this form on any open subset $\Omega \subset X$ such that $H^2_{DR}(\Omega, \mathbb{R}) = H^1(\Omega, \mathcal{O}) = 0$, e.g. on small coordinate balls (exercise to the reader). □

It is not difficult to show that a product $\Theta_1 \wedge \ldots \wedge \Theta_q$ of positive currents of bidegree $(1,1)$ is positive whenever the product is well defined (this is certainly the case if all Θ_j but one at most are smooth; much finer conditions will be discussed in Section 2).

We now discuss another very important example of closed positive current. In fact, with every closed analytic set $A \subset X$ of pure dimension p is associated a current of integration

$$(1.17) \qquad \langle [A], \alpha \rangle = \int_{A_{\mathrm{reg}}} \alpha, \qquad \alpha \in \mathcal{D}^{p,p}(X),$$

obtained by integrating over the regular points of A. In order to show that (1.17) is a correct definition of a current on X, one must show that A_{reg} has locally finite area in a neighborhood of A_{sing}. This result, due to [Lel57] is shown as follows. Suppose that 0 is a singular point of A. By the local parametrization theorem for analytic sets, there is a linear change of coordinates on $\mathbb{C}^n$ such that all projections

$$\pi_I : (z_1, \ldots, z_n) \mapsto (z_{i_1}, \ldots, z_{i_p})$$

define a finite ramified covering of the intersection $A \cap \Delta$ with a small polydisk Δ in $\mathbb{C}^n$ onto a small polydisk Δ_I in $\mathbb{C}^p$. Let n_I be the sheet number. Then the p-dimensional area of $A \cap \Delta$ is bounded above by the sum of the areas of its projections counted with multiplicities, i.e.

$$\mathrm{Area}(A \cap \Delta) \le \sum n_I \mathrm{Vol}(\Delta_I).$$

The fact that $[A]$ is positive is also easy. In fact

$$i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha}_p = |\det(\alpha_{jk})|^2\, iw_1 \wedge \overline{w}_1 \wedge \ldots \wedge iw_p \wedge \overline{w}_p$$

if $\alpha_j = \sum \alpha_{jk} dw_k$ in terms of local coordinates $(w_1, \ldots, w_p)$ on A_{reg}. This shows that all such forms are ≥ 0 in the canonical orientation defined by $iw_1 \wedge \overline{w}_1 \wedge \ldots \wedge iw_p \wedge \overline{w}_p$. More importantly, Lelong [Lel57] has shown that $[A]$ is d-closed in X, even at points of A_{sing}. This last result can be seen today as a consequence of the Skoda-El Mir extension theorem. For this we need the following definition: a *complete pluripolar* set is a set E such that there is an open covering (Ω_j) of X and psh functions u_j on Ω_j with $E \cap \Omega_j = u_j^{-1}(-\infty)$. Any (closed) analytic set is of course complete pluripolar (take u_j as in Example 1.9).

(1.18) Theorem (Skoda [Sko81], El Mir [EM84], Sibony [Sib85]). *Let E be a closed complete pluripolar set in X, and let Θ be a closed positive current on $X \smallsetminus E$ such that the coefficients $\Theta_{I,J}$ of Θ are measures with locally finite mass near E. Then the trivial extension $\widetilde{\Theta}$ obtained by extending the measures $\Theta_{I,J}$ by 0 on E is still closed on X.*

Lelong's result $d[A] = 0$ is obtained by applying the Skoda-El Mir theorem to $\Theta = [A_{\mathrm{reg}}]$ on $X \smallsetminus A_{\mathrm{sing}}$.

Proof of Theorem 1.18. The statement is local on X, so we may work on a small open set Ω such that $E \cap \Omega = v^{-1}(-\infty)$, $v \in \mathrm{Psh}(\Omega)$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a convex increasing function such that $\chi(t) = 0$ for $t \leq -1$ and $\chi(0) = 1$. By shrinking Ω and putting $v_k = \chi(k^{-1}v \star \rho_{\varepsilon_k})$ with $\varepsilon_k \to 0$ fast, we get a sequence of functions $v_k \in \mathrm{Psh}(\Omega) \cap C^\infty(\Omega)$ such that $0 \leq v_k \leq 1$, $v_k = 0$ in a neighborhood of $E \cap \Omega$ and $\lim v_k(x) = 1$ at every point of $\Omega \smallsetminus E$. Let $\theta \in C^\infty([0,1])$ be a function such that $\theta = 0$ on $[0, 1/3]$, $\theta = 1$ on $[2/3, 1]$ and $0 \leq \theta \leq 1$. Then $\theta \circ v_k = 0$ near $E \cap \Omega$ and $\theta \circ v_k \to 1$ on $\Omega \smallsetminus E$. Therefore $\tilde{\Theta} = \lim_{k \to +\infty} (\theta \circ v_k)\Theta$ and

$$d'\tilde{\Theta} = \lim_{k \to +\infty} \Theta \wedge d'(\theta \circ v_k)$$

in the weak topology of currents. It is therefore sufficient to verify that $\Theta \wedge d'(\theta \circ v_k)$ converges weakly to 0 (note that $d''\tilde{\Theta}$ is conjugate to $d'\tilde{\Theta}$, thus $d''\tilde{\Theta}$ will also vanish).

Assume first that $\Theta \in \mathcal{D}'^{n-1,n-1}(X)$. Then $\Theta \wedge d'(\theta \circ v_k) \in \mathcal{D}'^{n,n-1}(\Omega)$, and we have to show that

$$\langle \Theta \wedge d'(\theta \circ v_k), \overline{\alpha} \rangle = \langle \Theta, \theta'(v_k) d'v_k \wedge \overline{\alpha} \rangle \underset{k \to +\infty}{\longrightarrow} 0, \qquad \forall \alpha \in \mathcal{D}^{1,0}(\Omega).$$

As $\gamma \mapsto \langle \Theta, \mathrm{i}\gamma \wedge \overline{\gamma} \rangle$ is a non-negative hermitian form on $\mathcal{D}^{1,0}(\Omega)$, the Cauchy-Schwarz inequality yields

$$\big|\langle \Theta, \mathrm{i}\beta \wedge \overline{\gamma} \rangle\big|^2 \leq \langle \Theta, \mathrm{i}\beta \wedge \overline{\beta} \rangle \; \langle \Theta, \mathrm{i}\gamma \wedge \overline{\gamma} \rangle, \qquad \forall \beta, \gamma \in \mathcal{D}^{1,0}(\Omega).$$

Let $\psi \in \mathcal{D}(\Omega)$, $0 \leq \psi \leq 1$, be equal to 1 in a neighborhood of $\mathrm{Supp}\,\alpha$. We find

$$\big|\langle \Theta, \theta'(v_k) d'v_k \wedge \overline{\alpha} \rangle\big|^2 \leq \langle \Theta, \psi \mathrm{i} d'v_k \wedge d''v_k \rangle \; \langle \Theta, \theta'(v_k)^2 \mathrm{i}\alpha \wedge \overline{\alpha} \rangle.$$

By hypothesis $\int_{\Omega \smallsetminus E} \Theta \wedge \mathrm{i}\alpha \wedge \overline{\alpha} < +\infty$ and $\theta'(v_k)$ converges everywhere to 0 on Ω, thus $\langle \Theta, \theta'(v_k)^2 \mathrm{i}\alpha \wedge \overline{\alpha} \rangle$ converges to 0 by Lebesgue's dominated convergence theorem. On the other hand

$$\mathrm{i}d'd''v_k^2 = 2v_k\,\mathrm{i}d'd''v_k + 2\mathrm{i}d'v_k \wedge d''v_k \geq 2\mathrm{i}d'v_k \wedge d''v_k,$$
$$2\langle \Theta, \psi \mathrm{i} d'v_k \wedge d''v_k \rangle \leq \langle \Theta, \psi \mathrm{i} d'd''v_k^2 \rangle.$$

As $\psi \in \mathcal{D}(\Omega)$, $v_k = 0$ near E and $d\Theta = 0$ on $\Omega \smallsetminus E$, an integration by parts yields

$$\langle \Theta, \psi \mathrm{i} d'd''v_k^2 \rangle = \langle \Theta, v_k^2 \mathrm{i} d'd''\psi \rangle \leq C \int_{\Omega \smallsetminus E} \|\Theta\| < +\infty$$

where C is a bound for the coefficients of $\mathrm{i}d'd''\psi$. Thus $\langle \Theta, \psi \mathrm{i} d'v_k \wedge d''v_k \rangle$ is bounded, and the proof is complete when $\Theta \in \mathcal{D}'^{n-1,n-1}$.

In the general case $\Theta \in \mathcal{D}'^{p,p}$, $p < n$, we simply apply the result already proved to all positive currents $\Theta \wedge \gamma \in \mathcal{D}'^{n-1,n-1}$ where $\gamma = \mathrm{i}\gamma_1 \wedge \overline{\gamma}_1 \wedge \ldots \wedge \mathrm{i}\gamma_{n-p-1} \wedge \overline{\gamma}_{n-p-1}$ runs over a basis of forms of $\Lambda^{n-p-1,n-p-1} T^\star_\Omega$ with constant coefficients (Lemma 1.4). Then we get $d(\tilde{\Theta} \wedge \gamma) = d\tilde{\Theta} \wedge \gamma = 0$ for all such γ, hence $d\tilde{\Theta} = 0$. □

(1.19) Corollary. *Let Θ be a closed positive current on X and let E be a complete pluripolar set. Then $\mathbf{1}_E\Theta$ and $\mathbf{1}_{X\smallsetminus E}\Theta$ are closed positive currents. In fact, $\widetilde{\Theta} = \mathbf{1}_{X\smallsetminus E}\Theta$ is the trivial extension of $\Theta_{\restriction X\smallsetminus E}$ to X, and $\mathbf{1}_E\Theta = \Theta - \widetilde{\Theta}$.* □

As mentioned above, any current $\Theta = id'd''u$ associated with a psh function u is a closed positive $(1,1)$-current. In the special case $u = \log|f|$ where $f \in H^0(X, \mathcal{O}_X)$ is a non zero holomorphic function, we have the important

(1.20) Lelong-Poincaré equation. *Let $f \in H^0(X, \mathcal{O}_X)$ be a non zero holomorphic function, $Z_f = \sum m_j Z_j$, $m_j \in \mathbb{N}$, the zero divisor of f and $[Z_f] = \sum m_j[Z_j]$ the associated current of integration. Then*

$$\frac{\mathrm{i}}{\pi}\partial\overline{\partial}\log|f| = [Z_f].$$

Proof (sketch). It is clear that $id'd''\log|f| = 0$ in a neighborhood of every point $x \notin \mathrm{Supp}(Z_f) = \bigcup Z_j$, so it is enough to check the equation in a neighborhood of every point of $\mathrm{Supp}(Z_f)$. Let A be the set of singular points of $\mathrm{Supp}(Z_f)$, i.e. the union of the pairwise intersections $Z_j \cap Z_k$ and of the singular loci $Z_{j,\mathrm{sing}}$; we thus have $\dim A \le n-2$. In a neighborhood of any point $x \in \mathrm{Supp}(Z_f) \smallsetminus A$ there are local coordinates $(z_1, \ldots, z_n)$ such that $f(z) = z_1^{m_j}$ where m_j is the multiplicity of f along the component Z_j which contains x and $z_1 = 0$ is an equation for Z_j near x. Hence

$$\frac{\mathrm{i}}{\pi}d'd''\log|f| = m_j\frac{\mathrm{i}}{\pi}d'd''\log|z_1| = m_j[Z_j]$$

in a neighborhood of x, as desired (the identity comes from the standard formula $\frac{\mathrm{i}}{\pi}d'd''\log|z| =$ Dirac measure δ_0 in $\mathbb{C}$). This shows that the equation holds on $X \smallsetminus A$. Hence the difference $\frac{\mathrm{i}}{\pi}d'd''\log|f| - [Z_f]$ is a closed current of degree 2 with measure coefficients, whose support is contained in A. By Exercise 1.21, this current must be 0, for A has too small dimension to carry its support (A is stratified by submanifolds of real codimension ≥ 4). □

(1.21) Exercise. Let Θ be a current of degree q on a real manifold M, such that both Θ and $d\Theta$ have measure coefficients ("normal current"). Suppose that $\mathrm{Supp}\,\Theta$ is contained in a real submanifold A with $\mathrm{codim}_{\mathbb{R}} A > q$. Show that $\Theta = 0$.
Hint: Let $m = \dim_{\mathbb{R}} M$ and let $(x_1, \ldots, x_m)$ be a coordinate system in a neighborhood Ω of a point $a \in A$ such that $A \cap \Omega = \{x_1 = \ldots = x_k = 0\}$, $k > q$. Observe that $x_j\Theta = x_j d\Theta = 0$ for $1 \le j \le k$, thanks to the hypothesis on supports and on the normality of Θ, hence $dx_j \wedge \Theta = d(x_j\Theta) - x_j d\Theta = 0$, $1 \le j \le k$. Infer from this that all coefficients in $\Theta = \sum_{|I|=q}\Theta_I dx_I$ vanish. □

We now recall a few basic facts of slicing theory (the reader will profitably consult [Fed69] and [Siu74] for further developments). Let $\sigma : M \to M'$ be a submersion of smooth differentiable manifolds and let Θ be a *locally flat* current on M, that is, a current which can be written locally as $\Theta = U + dV$ where U, V have L^1_{loc} coefficients. It is a standard fact (see Federer) that every current Θ such that both Θ and $d\Theta$ have measure coefficients is locally flat; in particular, closed positive currents are locally flat. Then, for almost every $x' \in M'$, there is a well defined *slice* $\Theta_{x'}$, which is the current on the fiber $\sigma^{-1}(x')$ defined by

$$\Theta_{x'} = U_{\restriction\sigma^{-1}(x')} + dV_{\restriction\sigma^{-1}(x')}.$$

The restrictions of U, V to the fibers exist for almost all x' by the Fubini theorem. The slices $\Theta_{x'}$ are currents on the fibers with the same degree as Θ (thus of dimension $\dim\Theta - \dim(\text{fibers})$). Of course, every slice $\Theta_{x'}$ coincides with the usual restriction of Θ to the fiber if Θ has smooth coefficients. By using a regularization $\Theta_\varepsilon = \Theta \star \rho_\varepsilon$, it is easy to show that the slices of a closed positive current are again closed and positive: in fact $U_{\varepsilon,x'}$ and $V_{\varepsilon,x'}$ converge to $U_{x'}$ and $V_{x'}$ in $L^1_{\text{loc}}(\sigma^{-1}(x'))$, thus $\Theta_{\varepsilon,x'}$ converges weakly to $\Theta_{x'}$ for almost every x'. Now, the basic slicing formula is

$$\int_M \Theta \wedge \alpha \wedge \sigma^\star\beta = \int_{x'\in M'} \Big(\int_{x''\in\sigma^{-1}(x')} \Theta_{x'}(x'') \wedge \alpha_{\restriction\sigma^{-1}(x')}(x'') \Big) \beta(x') \tag{1.22}$$

for every smooth form α on M and β on M', such that α has compact support and $\deg\alpha = \dim M - dimM' - \deg\Theta$, $\deg\beta = \dim M'$. This is an easy consequence of the usual Fubini theorem applied to U and V in the decomposition $\Theta = U + dV$, if we identify locally σ with a projection map $M = M' \times M'' \to M'$, $x = (x', x'') \mapsto x'$, and use a partitition of unity on the support of α.

To conclude this section, we discuss De Rham and Dolbeault cohomology theory in the context of currents. A basic observation is that the Poincaré and Dolbeault-Grothendieck lemma still hold for currents. Namely, if $(\mathcal{D}'^q, d)$ and $(\mathcal{D}'(F)^{p,q}, d'')$ denote the complex of sheaves of degree q currents (resp. of (p,q)-currents with values in a holomorphic vector bundle F), we still have De Rham and Dolbeault sheaf resolutions

$$0 \to \mathbb{R} \to \mathcal{D}'^\bullet, \qquad 0 \to \Omega^p_X \otimes \mathcal{O}(F) \to \mathcal{D}'(F)^{p,\bullet}.$$

Hence we get canonical isomorphisms

$$\begin{aligned} H^q_{\text{DR}}(M,\mathbb{R}) &= H^q\big((\Gamma(M,\mathcal{D}'^\bullet),d)\big), \\ H^{p,q}(X,F) &= H^q\big((\Gamma(X,\mathcal{D}'(F)^{p,\bullet}),d'')\big). \end{aligned} \tag{1.23}$$

In other words, we can attach a cohomology class $\{\Theta\} \in H^q_{\text{DR}}(M,\mathbb{R})$ to any closed current Θ of degree q, resp. a cohomology class $\{\Theta\} \in H^{p,q}(X,F)$ to any d''-closed current of bidegree (p,q). Replacing if necessary every current by a smooth representative in the same cohomology class, we see that there is a well defined cup product given by the wedge product of differential forms

$$\begin{aligned} H^{q_1}(M,\mathbb{R}) \times \ldots \times H^{q_m}(M,\mathbb{R}) &\longrightarrow H^{q_1+\ldots+q_m}(M,\mathbb{R}), \\ (\{\Theta_1\},\ldots,\{\Theta_1\}) &\longmapsto \{\Theta_1\} \wedge \ldots \wedge \{\Theta_m\}. \end{aligned}$$

In particular, if M is a compact oriented variety and $q_1 + \ldots + q_m = \dim M$, there is a well defined intersection number

$$\{\Theta_1\} \cdot \{\Theta_2\} \cdot \cdots \cdot \{\Theta_m\} = \int_M \{\Theta_1\} \wedge \ldots \wedge \{\Theta_m\}.$$

However, as we will see in the next section, the pointwise product $\Theta_1 \wedge \ldots \wedge \Theta_m$ need not exist in general.