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E. Frenkel E. Previato

# Integrable Systems and Quantum Groups

## Montecatini Terme, 1993

Editors: M. Francaviglia, S. Greco

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# Integrable Systems and Quantum Groups

Lectures given at the 1st Session of the
Centro Internazionale Matematico Estivo
(C.I.M.E.) held in Montecatini Terme, Italy,
June 14–22, 1993

Editors: M. Francaviglia, S. Greco

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## FOREWORD

"Integrable Systems" form a classical subject having its origins in Physics and is deeply related with almost all the most important domains of Mathematics. Intriguing and often surprising are in fact the interplays of Integrable Systems with Algebraic Geometry, which started perhaps with Jacobi, to be then forgotten and discovered again in more recent years, with a fantastic impulse to both Theoretical Physics and Pure Mathematics. A very recent related topic is the theory of so-called "Quantum Groups", which is nowadays generating a further incredible amount of fruitful interplay between Physics and Mathematics, especially in the domains proper to Geometry and Algebra.

The scientific organisers of this CIME Session have for several years been enjoing these fascinating interrelations, also promoting and working in a research project supported by CNR and involving geometers and mathematical physicists from Genova, Milano, Torino and Trieste. Thus, the idea of a CIME session on this subject was quite ripe and necessary to the mathematical community when the proposal was worked out and promptly accepted by CIME.

Of course it was immediately clear to us that covering all the subjects was an impossible task. However, we were lucky enough to obtain the collaboration of four outstanding main lecturers, coming from different research experiences and animated by different points of view, but sharing the interest in the crosspoint of these disciplines and extremely able to develop their lectures with the most fascinating interdisciplinary attitude. The Course resulted then in a stimulating and exciting experience, not only for the participants but also for us. We really hope that the reader will find the same excitement as we did in following these lectures, even if a book cannot reproduce the unique atmosphere that existed in the school and surrounded it, in a special place where quiet and friendship were at the basis of a fruitful interaction.

The Session was held in fact in Montecatini Terme, at Villa "La Querceta", from June 14 to June 22, 1993. It consisted of the main four courses of six lectures each, accompanied by a number of interesting seminars concerning special topics and/or recent research announcements.

These proceedings contain the expanded versions of the four main courses. It took some time to collect them and recast them in their final form, but we believe that the importance and completeness of these texts made this longer delay worthwile. Unfortunately the lack of space has made impossible to include also the seminars, which shall be just listed on page VIII. For an outline of the contents of this volume we refer the reader to the Tables of Contents and the Introductions of the four sets of lectures themselves.

We are thankful to CIME for having given us the possibility of living such an exciting experience; we especially acknowledge the patience and the help of our colleagues Roberto Conti and Pietro Zecca. Special thanks are also due to our friend and collaborator Franco Magri, who was a source of inspiration for this session and helped us to construct its scientific structure.

Mauro Francaviglia
Silvio Greco

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M. NIEDERMAIER, Diagonalization Problem of Conserved Charges

C. REINA, A Borel-Weil-Bott Approach to Representations of Quantum Groups

A. STOLIN, Rational Solutions of Classical Yang-Baxter Equations and Frobenius Lie Algebras

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# Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles

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[2]Partially supported by a Rackham Fellowship, University of Michigan, 1993

# 1 Introduction

The purpose of these notes is to present an algebro-geometric point of view on several interrelated topics, all involving integrable systems in symplectic-algebro-geometric settings. These systems range from some very old examples, such as the geodesic flow on an ellipsoid, through the classical hierarchies of $KP-$ and $KdV$-types, to some new systems which are often based on moduli problems in algebraic geometry.

The interplay between algebraic geometry and integrable systems goes back quite a way. It has been known at least since Jacobi that many integrable systems can be solved explicitly in terms of *theta functions.* (There are numerous examples, starting with various *spinning tops* and the *geodesic flow on an ellipsoid.*) Geometrically, this often means that the system can be mapped to the total space of a family of Jacobians of some curves, in such a way that the flows of the system are mapped to linear flows along the Jacobians. In practice, these curves tend to arise as the spectrum (hence the name *'spectral'* curves) of some parameter-dependent operator; they can therefore be represented as branched covers of the parameter space, which in early examples tended to be the Riemann sphere $\mathbf{CP}^1$.

In *Hitchin's system,* the base $\mathbf{CP}^1$ is replaced by an arbitrary (compact, non-singular) Riemann surface $\Sigma$. The cotangent bundle $T^*\mathcal{U}_\Sigma$ to the moduli space $\mathcal{U}_\Sigma$ of stable vector bundles on $\Sigma$ admits two very different interpretations: on the one hand, it parametrizes certain *Higgs bundles,* or vector bundles with a (canonically) twisted endomorphism; on the other, it parametrizes certain *spectral data,* consisting of torsion-free sheaves (generically, line bundles) on spectral curves which are branched covers of $\Sigma$. In our three central chapters (4,5,6) we study this important system, its extensions and variants. All these systems are linearized on Jacobians of spectral curves.

We also study some systems in which the spectral curve is replaced by a higher-dimensional geometric object: a *spectral variety* in Chapter 9, an algebraic *Lagrangian subvariety* in Chapter 8, and a *Calabi-Yau manifold* in Chapter 7. Our understanding of some of these wild systems is much less complete than is the case for the curve-based ones. We try to explain what we know and to point out some of what we do not. The Calabi-Yau systems seem particularly intriguing. Not only are the tori (on which these systems are linearized) not Jacobians of curves, they are in general not even abelian varieties. There are some suggestive relations between these systems and the conjectural mirror-symmetry for Calabi-Yaus.

The first three chapters are introductory. In Chapter 2 we collect the basic notions of *symplectic geometry* and *integrable systems* which will be needed, including some information about *symplectic reduction.* (An excellent further reference is [AG].) In Chapter 3 we work out in some detail the classical theory of geodesic flow on an ellipsoid, which is integrable via hyperelliptic theta functions. We think of this both as a beautiful elementary and explicit example and as an important special case of the much more powerful results which follow. (Our presentation follows [Kn, Re, D5]). Some of our main algebro-geometric objects of study are introduced in Chapter 4: vector bundles and their moduli spaces, spectral curves, and the *'spectral systems'* constructed from them. In particular, we consider the *polynomial matrix system* [AHH, B1] (which contains the geodesic flow on an ellipsoid as special case) and *Hitchin's system* [H1, H2].

Each of the remaining five chapters presents in some detail a recent or current research topic. Chapter 5 outlines constructions (from [Ma1, Bn, Ty1]) of the Poisson structure on the spectral system of curves. This is possible whenever the twisting line bundle $K$ is a non-negative twist $\omega_\Sigma(D)$ of the canonical bundle $\omega_\Sigma$, and produces an algebraically completely integrable Hamiltonian system. Following [Ma1] we emphasize the deformation-theoretic construction, in which the Poisson structure on an open subset of the system is obtained via symplectic reduction from the cotangent bundle $T^*\mathcal{U}_{\Sigma,D}$ of the moduli space $\mathcal{U}_{\Sigma,D}$ of stable bundles with a *level-D structure.*

In Chapter 6 we explore the relation between these spectral systems and the $KP$-hierarchy and its variants (multi-component $KP$, Heisenberg flows, and their $KdV$-type subhierarchies). These hierarchies are, of course, a rich source of geometry: The Krichever construction (e.g. [SW]) shows that any Jacobian can be embedded in $KP$-space, and these are the only finite-dimensional orbits [Mul, AdC, Sh]. Following [AB, LM1] we describe some "multi-Krichever" constructions which take spectral data to the spaces of the $KP$, $mcKP$ and Heisenberg systems. Our main new result is that the flows on the spectral system which are obtained by pulling back the $mcKP$ or Heisenberg flows via the corresponding Krichever maps are *Hamiltonian* with respect to the Poisson structure constructed in Chapter 5. In fact, we write down explicitly the Hamiltonians for these $KP$ flows on the spectral system, as residues of traces of meromorphic matrices. (Some related results have also been obtained recently in [LM2].)

The starting point for Chapter 7 is an attempt to understand the condition for a given family of complex tori to admit a symplectic structure and thus become an ACIHS. We find that the condition is a symmetry on the derivatives of the period map, which essentially says that the periods are obtained as partials of some field of symmetric cubic tensors on the base. In the rest of this Chapter we apply this idea to an analytically (not algebraically) integrable system constructed from any family of Calabi-Yau 3-folds. Some properties of this system suggest that it may be relevant to a purely hodge-theoretic reformulation of the mirror-symmetry conjectures. (This chapter is based on [DM].)

Chapter 8 is devoted to the construction of symplectic and Poisson structures in some inherently non-linear situations, vastly extending the results of Chapter 5. The basic space considered here is the moduli space parametrizing line-bundle-like sheaves supported on (variable) subvarieties of a given symplectic space $X$. It is shown that when the subvarieties are Lagrangian, the moduli space itself becomes symplectic. The spectral systems considered in Chapter 5 can be recovered as the case where $X$ is the total space of $T^*\Sigma$ and the Lagrangian subvarieties are the spectral curves. (A fuller version of these results will appear in [Ma2].)

In the final chapter we consider extensions of the spectral system to allow a higher-dimensional base variety $S$, an arbitrary reductive group $G$, an arbitrary representation $\rho : G \to AutV$, and values in an arbitrary vector bundle $K$. (Arbitrary reductive groups $G$ were considered, over a curve $S = \Sigma$ with $K = \omega_\Sigma$, by Hitchin [H2], while the case $K = \Omega_S$ over arbitrary base $S$ is Simpson's [Sim1]). We replace spectral curves by various kinds of spectral covers, and introduce the cameral cover, a version of the Galois-closure of a spectral cover which is independent of $K$ and $\rho$. It comes with an action of $W$, the Weyl group of $G$. We analyze the decomposition, under the action of $W$, of the cameral and spectral Picard varieties, and identify the distinguished Prym in there. This is shown to correspond, up to certain shifts and twists, to the fiber of the Hitchin map in this general setting, i.e. to moduli of Higgs bundles with a given $\tilde{S}$. Combining this with some obvious remarks about existence of Poisson structures, we find that the moduli spaces of K-valued Higgs bundles support algebraically completely integrable systems. Our presentation closely follows that of [D4]

It is a pleasure to express our gratitude to the organizers, Mauro Francaviglia and Silvio Greco, for the opportunity to participate in the CIME meeting and to publish these notes here. During the preparation of this long work we benefited from many enjoyable conversations with M. Adams, M. Adler, A. Beauville, R. Bryant, C. L. Chai, I. Dolgachev, L. Ein, B. van Geemen, A. Givental, M. Green, P. Griffiths, N. Hitchin, Y. Hu, S. Katz, V. Kanev, L. Katzarkov, R. Lazarsfeld, P. van Moerbeke, D. Morrison, T. Pantev, E. Previato and E. Witten.

# 2 Basic Notions

We gather here those basic concepts and elementary results from symplectic and Poisson geometry, completely integrable systems, and symplectic reduction which will be helpful throughout these notes. Included are a few useful examples and only occasional proofs or sketches. To the reader unfamiliar with this material we were hoping to impart just as much of a feeling for it as might be needed in the following chapters. For more details, we recommend the excellent survey [AG].

## 2.1 Symplectic Geometry

Symplectic structure

A symplectic structure on a differentiable manifold $M$ of even dimension $2n$ is given by a non-degenerate closed 2-form $\sigma$. The non degeneracy means that either of the following equivalent conditions holds.

- $\sigma^n$ is a nowhere vanishing volume form.
- Contraction with $\sigma$ induces an isomorphism $\rfloor\sigma : TM \to T^*M$
- For any non-zero tangent vector $v \in T_mM$ at $m \in M$, there is some $v' \in T_mM$ such that $\sigma(v, v') \neq 0$.

**Examples 2.1**

1. Euclidean space

   The standard example of a symplectic manifold is Euclidean space $\mathbb{R}^{2n}$ with $\sigma = \Sigma dp_i \wedge dq_i$, where $p_1, \cdots, p_n$, $q_1, \cdots, q_n$ are linear coordinates. Darboux's theorem says that any symplectic manifold is locally equivalent to this example (or to any other).

2. Cotangent bundles

   For any manifold $X$, the cotangent bundle $M := T^*X$ has a natural symplectic structure. First, $M$ has the tautological 1-form $\alpha$, whose value at $(x, \theta) \in T^*X$ is $\theta$ pulled back to $T^*M$. If $q_1, \cdots, q_n$ are local coordinates on $X$, then locally $\alpha = \Sigma p_i dq_i$ where the $p_i$ are the fiber coordinates given by $\partial/\partial q_i$. The differential

   $$\sigma := d\alpha$$

   is then a globally defined closed (even exact) 2-form on $M$. It is given in local coordinates by $\Sigma dp_i \wedge dq_i$, hence is non-degenerate.

3. Coadjoint orbits

   Any Lie group $G$ acts on its Lie algebra $\mathfrak{g}$ (adjoint representation) and hence on the dual vector space $\mathfrak{g}^*$ (coadjoint representation). Kostant and Kirillov noted that for any $\xi \in \mathfrak{g}^*$, the coadjoint orbit $\mathcal{O} = G\xi \subset \mathfrak{g}^*$ has a natural symplectic structure. The tangent space to $\mathcal{O}$ at $\xi$ is given by $\mathfrak{g}/\mathfrak{g}_\xi$, where $\mathfrak{g}_\xi$ is the stabilizer of $\xi$:

   $$\mathfrak{g}_\xi := \{x \in \mathfrak{g} \mid ad_x^*\xi = 0\} = \{x \in \mathfrak{g} \mid (\xi, [x, y]) = 0 \quad \forall\, y \in \mathfrak{g}\}.$$

Now $\xi$ determines an alternating bilinear form on $\mathbf{g}$

$$x, y \longmapsto (\xi, [x, y]),$$

which clearly descends to $\mathbf{g}/\mathbf{g}_\xi$ and is non-degenerate there. Varying $\xi$ we get a non-degenerate 2-form $\sigma$ on $\mathcal{O}$. The Jacobi identity on $g$ translates immediately into closedness of $\sigma$.

## Hamiltonians

To a function $f$ on a symplectic manifold $(M, \sigma)$ we associate its *Hamiltonian vector field* $v_f$, uniquely determined by

$$v_f \rfloor \sigma = df.$$

A vector field $v$ on $M$ is Hamiltonian if and only if the 1-form $v \rfloor \sigma$ is exact. We say $v$ is *locally Hamiltonian* if $v \rfloor \sigma$ is closed. This is equivalent to saying that the flow generated by $v$ preserves $\sigma$. Thus on a symplectic surface ($n = 1$), the locally Hamiltonian vector fields are the area-preserving ones.

**Example:** (Geodesic flow)

A Riemannian metric on a manifold $X$ determines an isomorphism of $M := TX$ with $T^*X$; hence we get on $M$ a natural symplectic structure together with a $C^\infty$ function $f =$ (squared length). The geodesic flow on $X$ is the differential equation, on $M$, given by the Hamiltonian vector field $v_f$. Its integral curves are the geodesics on $M$.

## Poisson structures

The association $f \mapsto v_f$ gives a map of sheaves

$$v : C^\infty(M) \longrightarrow V(M) \tag{1}$$

from $C^\infty$ functions on the symplectic manifold $M$ to vector fields. Now $V(M)$ always has the structure of a Lie algebra, under commutation of vector fields. The symplectic structure on $M$ determines a Lie algebra structure on $C^\infty(M)$ such that $v$ becomes a morphism of (sheaves of) Lie algebras. The operation on $C^\infty(M)$, called *Poisson bracket*, is

$$\{f, g\} := (df, v_g) = -(dg, v_f) = \frac{n df \wedge dg \wedge \sigma^{n-1}}{\sigma^n}.$$

More generally, a *Poisson structure* on a manifold $M$ is a Lie algebra bracket $\{\,,\,\}$ on $C^\infty(M)$ which acts as a derivation in each variable:

$$\{f, gh\} = \{f, g\}h + \{f, h\}g, \quad f, g, h \in C^\infty(M).$$

Since the value at a point $m$ of a given derivation acting on a function $g$ is a linear function of $d_m g$, we see that a Poisson structure on $M$ determines a global 2-vector

$$\psi \in H^0(M, \overset{2}{\wedge} TM).$$

or equivalently a skew-symmetric homomorphism

$$\Psi : T^*M \longrightarrow TM.$$

Conversely, any 2-vector $\psi$ on $M$ determines an alternating bilinear bracket on $C^\infty(M)$, by

$$\{f, g\} := (df \wedge dg, \psi),$$

and this acts as a derivation in each variable. An equivalent way of specifying a Poisson structure is thus to give a global 2-vector $\psi$ satisfying an integrability condition (saying that the above bracket satisfies the Jacobi identity, hence gives a Lie algebra).

We saw that a symplectic structure $\sigma$ determines a Poisson bracket $\{\ ,\ \}$. The corresponding homomorphism $\Psi$ is just $(\rfloor\sigma)^{-1}$; the closedness of $\sigma$ is equivalent to integrability of $\psi$. Thus, a Poisson structure which is (i.e. whose 2-vector is) everywhere non-degenerate, comes from a symplectic structure.

A general Poisson structure can be degenerate in two ways: first, there may exist non-constant functions $f \in C^\infty(M)$, called *Casimirs*, satisfying

$$0 = df \rfloor \psi = \Psi(df),$$

i.e.

$$\{f, g\} = 0 \text{ for all } g \in C^\infty(M).$$

This implies that the rank of $\Psi$ is less than maximal everywhere. In addition, or instead, rank $\Psi$ could drop along some strata in $M$. For even $r$, let

$$M_r := \{m \in M | rank(\Psi) = r\}.$$

Then a basic result [We] asserts that the $M_r$ are submanifolds, and they are canonically foliated into *symplectic leaves*, i.e. $r$-dimensional submanifolds $Z \subset M_r$ which inherit a symplectic structure. (This means that the restriction $\psi_{|Z}$ is the image, under the inclusion $Z \hookrightarrow M_r$, of a two-vector $\psi_Z$ on $Z$ which is everywhere nondegenerate, hence comes from a symplectic structure on $Z$.) These leaves can be described in several ways:

- The image $\Psi(T^*M_r)$ is an involutive subbundle of rank $r$ in $TM_r$; the $Z$ are its integral leaves.
- The leaf $Z$ through $m \in M_r$ is $Z = \{z \in M_r | f(m) = f(z)$ for all Casimirs $f$ on $M_r\}$.
- Say that two points of $M$ are $\psi$-connected if there is an integral curve of some Hamiltonian vector field passing through both. The leaves are the equivalence classes for the equivalence relation generated by $\psi$-connectedness.

**Example 2.2** The Kostant-Kirillov symplectic structures on coadjoint orbits of a Lie algebra $\mathbf{g}$ extend to a Poisson structure on the dual vector space $\mathbf{g}^*$. For a function $F \in C^\infty(\mathbf{g}^*)$ we identify its differential $d_\xi F$ at $\xi \in \mathbf{g}*$ with an element of $\mathbf{g} = \mathbf{g}^{**}$. We then set:

$$\{F, G\}(\xi) := (\xi, [d_\xi F, d_\xi G]).$$

This is a Poisson structure, whose symplectic leaves are precisely the coadjoint orbits. The rank of $\mathbf{g}$ is, by definition, the smallest codimension $\ell$ of a coadjoint orbit. The Casimirs are the ad-invariant functions on $\mathbf{g}^*$. Their restrictions to the largest stratum $\mathbf{g}^*_{\dim \mathbf{g} - \ell}$ foliate this stratum, the leaves being the *regular* (i.e. largest dimensional) coadjoint orbits.

## 2.2 Integrable Systems

We say that two functions $h_1, h_2$ on a Poisson manifold $(M, \psi)$ *Poisson commute* if their Poisson bracket $\{h_1, h_2\}$ is zero. In this case the integral flow of the Hamiltonian vector field of each function $h_i$, $i = 1, 2$ is tangent to the level sets of the other. In other words, $h_2$ is a conservation law for the Hamiltonian $h_1$ and the Hamiltonian flow of $h_2$ is a symmetry of the Hamiltonian system associated with $(M, \psi, h_1)$ (the flow of the Hamiltonian vector field $v_{h_1}$ on $M$).

A map $f : M \to B$ between two Poisson manifolds is a *Poisson map* if pullback of functions is a Lie algebra homomorphism with respect to the Poisson bracket

$$f^*\{F, G\}_B = \{f^*F, f^*G\}_M.$$

Equivalently, if $df(\psi_M)$ equals $f^*(\psi_B)$ as sections of $f^*(\overset{2}{\wedge} T_B)$. If $H : M \to B$ is a Poisson map with respect to the trivial (zero) Poisson structure on $B$ we will call $H$ a *Hamiltonian map.* Equivalently, $H$ is Hamiltonian if the Poisson structure $\psi$ vanishes on the pullback $H^*(T^*B)$ of the cotangent bundle of $B$ (regarding the latter as a subbundle of $(T^*M, \psi)$). In particular, the rank of the differential $dH$ is less than or equal to $\dim M - \frac{1}{2}\operatorname{rank}(\psi)$ at every point. A Hamiltonian map pulls back the algebra of functions on $B$ to a commutative Poisson subalgebra of the algebra of functions on $M$.

The study of a Hamiltonian system $(M, \psi, h)$ simplifies tremendously if one can extend the Hamiltonian function $h$ to a Hamiltonian map $H : M \to B$ of maximal rank $\dim M - \frac{1}{2}\operatorname{rank}(\psi)$. Such a system is called a completely integrable Hamiltonian system. The Hamiltonian flow of a completely integrable system can often be realized as a linear flow on tori embedded in $M$. The fundamental theorem in this case is Liouville's theorem (stated below).

**Definition 2.3** 1. Let $V$ be a vector space, $\sigma \in \overset{2}{\wedge} V^*$ a (possibly degenerate) two form. A subspace $Z \subset V$ is called *isotropic (coisotropic)* if it is contained in (contains) its symplectic complement. Equivalently, $Z$ is isotropic if $\sigma$ restricts to zero on $Z$. If $\sigma$ is nondegenerate, a subspace $Z \subset V$ is called *Lagrangian* if it is both isotropic and coisotropic. In this case $V$ is even (say $2n$) dimensional and the Lagrangian subspaces are the $n$ dimensional isotropic subspaces.

2. Let $(M, \sigma)$ be a symplectic manifold. A submanifold $Z$ is *isotropic* (respectively *coisotropic, Lagrangian*) if the tangent subspaces $T_zZ$ are, for all $z \in Z$.

**Example 2.4** For every manifold $X$, the fibers of the cotangent bundle $T^*X$ over points of $X$ are Lagrangian submanifolds with respect to the standard symplectic structure. A section of $T^*X$ over $X$ is Lagrangian if and only if the corresponding 1-form on $X$ is closed.

We will extend the above definition to Poisson geometry:

**Definition 2.5** 1. Let $U$ be a vector space, $\psi$ an element of $\overset{2}{\wedge} U$. Let $V \subset U$ be the image of the contraction $\rfloor\, \psi : U^* \to U$. Let $W \subset U^*$ be its kernel. $W$ is called the null space of $\psi$. $\psi$ is in fact a nondegenerate element of $\overset{2}{\wedge} V$ giving rise to a symplectic form $\sigma \in \overset{2}{\wedge} V^*$ (its inverse). A subspace $Z \subset U$ is *Lagrangian* with respect to $\psi$ if $Z$ is a Lagrangian subspace of $V \subset U$ with respect to $\sigma$. Equivalently, $Z$ is Lagrangian if $(U/Z)^*$ is both an isotropic and a coisotropic subspace of $U^*$ with respect to $\psi \in \overset{2}{\wedge} U \cong \overset{2}{\wedge} (U^*)^*$.

2. Let $(M, \psi)$ be a Poisson manifold, assume that $\psi$ has constant rank (this condition will be relaxed in the complex analytic or algebraic case). A submanifold $Z \subset M$ is *Lagrangian* if the tangent subspaces $T_z Z$ are, for all $z \in Z$. Notice that the constant rank assumption implies that each connected component of $Z$ is contained in a single symplectic leaf.

**Theorem 2.6** *(Liouville). Let $M$ be an $m$-dimensional Poisson manifold with Poisson structure $\psi$ of constant rank $2g$. Suppose that $H : M \to B$ is a proper submersive Hamiltonian map of maximal rank, i.e, dim $B = m - g$. Then*

i) *The null foliation of $M$ is induced locally by a foliation of $B$ (globally if $H$ has connected fibers).*

ii) *The connected components of fibers of $H$ are Lagrangian compact tori with a natural affine structure.*

iii) *The Hamiltonian vector fields of the pullback of functions on $B$ by $H$ are tangent to the level tori and are translation invariant (linear).*

**Remark 2.7** *:* If $H$ is not proper, but the Hamiltonian flows are complete, then the fibers of $H$ are generalized tori (quotients of a vector space by a discrete subgroup, not necessarily of maximal rank).

Sketch of proof of Liouville's theorem:

i) Since $H$ is a proper submersion the connected components of the fibers of $H$ are smooth compact submanifolds. Since $H$ is a Hamiltonian map of maximal rank $m - g$, the pullback $H^*(T_B^*)$ is isotropic and coisotropic and hence $H$ is a Lagrangian fibration. In particular, each connected component of a fiber of $H$ is contained in a single symplectic leaf.

ii),iii) Let $A_b$ be a connected component of the fiber $H^{-1}(b)$. Let $0 \to T_{A_b} \to T_{M|_{A_b}} \xrightarrow{dH} (T_b B) \otimes \mathcal{O}_{A_b} \to 0$ be the exact sequence of the differential of $H$. Part i) implies that the null subbundle $W_{|A_b} := Ker[\Psi : T^*M \to TM]_{|A_b}$ is the pullback of a subspace $W_b$ of $T_b^* B$. Since $H$ is a Lagrangian fibration, the Poisson structure induces a surjective homomorphism $\phi_b : H^*(T_b^* B) \to T_{A_b}$ inducing a trivialization $\bar{\phi}_b : (T_b^* B / W_b) \otimes \mathcal{O}_{A_b} \xrightarrow{\sim} T_{A_b}$.

A basis of the vector space $T_b^* B / W_b$ corresponds to a frame of global independent vector fields on the fiber $A_b$ which commute since the map $H$ is Hamiltonian. Hence $A_b$ is a compact torus.

□