OXFORD LECTURE SERIES IN MATHEMATICS
AND ITS APPLICATIONS · 26

# Dynamics of Viscous Compressible Fluids

Eduard Feireisl

# Dynamics of Viscous Compressible Fluids

Eduard Feireisl

*Mathematical Institute of the Academy of Sciences of the Czech Republic*

OXFORD
UNIVERSITY PRESS

OXFORD
UNIVERSITY PRESS
Great Clarendon Street, Oxford OX2 6DP
Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide in
Oxford New York
Auckland Bangkok Buenos Aires Cape Town Chennai
Dar es Salaam Delhi Hong Kong Istanbul Karachi Kolkata
Kuala Lumpur Madrid Melbourne Mexico City Mumbai Nairobi
São Paulo Shanghai Taipei Tokyo Toronto
Oxford is a registered trade mark of Oxford University Press
in the UK and in certain other countries
Published in the United States
by Oxford University Press Inc., New York

First published 2004

A catalogue record for this title is available from the British Library
Library of Congress Cataloging in Publication Data
(Data available)
ISBN 0 19 852838 8 (Hbk)
10 9 8 7 6 5 4 3 2 1
Typeset by Newgen Imaging Systems (P) Ltd., Chennai, India
Printed in Great Britain
*on acid-free paper by*
Biddles Ltd. www.biddles.co.uk

# PREFACE

Since the time of Euler, partial differential equations have been used to describe the movement of continuous media like fluids or vibration of solids. In its origins, hydrodynamics was a largely mathematical science for which much of the theories of partial differential equations of the eighteenth century were developed. As applications and experimental studies grew more numerous enhanced even more by a spectacular development of modern supercomputers, the rigorous mathematical theory gave ground to numerical studies and computer experiments commonly called applied mathematics of recent days. The Navier–Stokes equations are the most frequently used as a model of linearly viscous incompressible fluids (liquids) while the Euler equations play the same role for inviscid incompressible or compressible fluids (gases). Given the number of numerical results and successful practical applications, it comes as a striking fact how much less is known about the solutions to these equations at a purely theoretical level. The existence (or non-existence) of global-in-time regular solutions to the incompressible Navier–Stokes equations when the ambient physical space is three dimensional is one of the most challenging open problems of the modern theory of partial differential equations. The Euler equations of a compressible fluid form a nonlinear hyperbolic system for which any large data existence problem seems widely open even in the class of distributional solutions. Moreover, fluids modeled by these equations may exhibit very complicated chaotic or self-organized structures commonly denoted as "turbulent" phenomena, the understanding of which represents one of the main challenges of modern mathematical physics. It is hardly conceivable that any real progress in this direction could be made without answering the basic questions of well-posedness of the underlying equations—existence, uniqueness, stability, and continuous dependence of solutions on the data.

This book is designed as a contribution to the mathematical theory of viscous compressible fluids. In accordance with the basic principles of classical continuum mechanics, the state of a fluid at a given time is fully characterized by three macroscopic quantities—the density, the velocity, and the temperature. These satisfy a system of partial differential equations expressing the conservation of mass, momentum, and energy. In the situation when the only source of internal energy change is of purely mechanical origin; that is, when both conduction of heat and its generation by dissipation of mechanical energy can be neglected, the temperature changes can be expressed in terms of the density, and the original system reduces to the Navier–Stokes equations of a compressible barotropic fluid governing the time evolution of the density and momentum. In this model, the pressure of the fluid is given as an explicit function of the

density. One can go even further assuming the fluid is incompressible, which amounts to the hypothesis that the density is constant. The resulting system of the incompressible Navier–Stokes equations contains the velocity as the only unknown variable and represents probably the best known model problem in mathematical fluid dynamics.

The existence of global-in-time weak solutions for the incompressible Navier–Stokes equations was established by Leray. His notion of weak solution (1934) preceded both the introduction of the Sobolev spaces (1936) and the generalized derivatives (Schwartz 1944). A comparable theory for viscous compressible barotropic fluids has been developed only recently by P.-L. Lions in "Mathematical topics in fluid dynamics, II", Oxford University Press 1998. The major discovery made by Lions are some unexpected properties of a quantity commonly termed as effective viscous pressure. In spite of being strongly nonlinear, the effective viscous pressure behaves almost like it was a weakly continuous function of the density. Such a behavior is of course reminiscent of the quantities studied in the theory of compensated compactness but looking for a straightforward explanation in terms of this theory would be more misleading than elucidating. Another important ingredient of Lions' approach is the concept of renormalized solution developed in the framework of a joint programme with DiPerna (1989).

The major thrust of this book is to develop further the mathematical theory of viscous compressible fluids pursuing two main goals:

- Global existence results for the full system of the Navier–Stokes equations with large data supplemented with a suitable set of constitutive equations.
- Optimal existence results for the barotropic flows with respect to the available *a priori* estimates.

To this end, we introduce two new tools: (i) an oscillations defect measure—to obtain a more precise description of possible oscillations of the density component in a sequence of (approximate) solutions; (ii) a renormalized limit of a sequence of bounded integrable functions—to cope with possible concentrations in the temperature.

The material is organized in the following manner. Chapter 1 is devoted to a review of the underlying physical theory. Besides the basic notions of reference configurations, kinematics, constitutive equations, and balance laws, this part includes an account of general pressure–density–temperature relations including those arising in low energy nuclear physics and astrophysics. In particular, some examples of non-monotone pressure–density constitutive laws are presented.

The basic mathematical concepts used in the book are resumed in Chapters 2 and 3. In order to underline the physical background of the function spaces, the exact definitions introduced in Chapter 2 are followed by the energy estimates deduced in Chapter 3 directly for any smooth solution of the underlying equations (*a priori* estimates). Several other fundamental concepts are also treated at length: average continuity of weak solutions, renormalized solutions of the continuity equation, and instantaneous values of the state variables.

The concept of a variational solution is introduced in Chapter 4. Each equation of the full system, that means, the equation of continuity, the momentum equation, and the energy equation, is treated separately. The basic facts of the theory of renormalized solutions of the continuity equation are reviewed. Moreover, the renormalized solutions of the thermal energy are introduced as a suitable tool to deal with possible concentrations of temperature. The discussion on the variational solutions is further developed in Chapter 5, where more delicate estimates of the pressure and temperature necessary for future analysis are obtained.

Chapter 6 is central to the book. Given the rather poor *a priori* estimates available, the methods of weak convergence play a decisive role in the mathematical theory to be developed in this book. Both "classical" problems of this approach—the presence of oscillations and concentrations in sequences of approximate solutions—are present. The well-known results of the theory of compensated compactness are used in order to cope with possible density oscillations. More specifically, the fundamental properties of the effective viscous pressure discovered by P.-L. Lions are discussed together with an alternative proof of "weak continuity" of this quantity via the famous div-curl lemma. Next, the concept of oscillation defect measure is introduced, and its relation to the propagation of oscillations and the renormalized continuity equation is established. Furthermore, the whole machinery is applied to the crucial problem of propagation of density oscillations in a sequence of solutions, in particular, it is shown that the oscillations decay in time at a uniform rate independent of the choice of initial data provided the pressure is a monotone function of the density. The weak sequential stability (compactness) of the set of weak solutions is established for optimal values of the "adiabatic" exponent. In particular, the physically interesting case of the monoatomic gas in the isentropic regime in three space dimensions can be treated—a problem left open in the up to now available theory. Possible concentrations in the temperature are treated via the method of renormalization (rescaling). A "renormalized" formulation of the thermal energy equation introduced in Chapter 4 is supplemented with the concept of a renormalized limit, usefulness of which being demonstrated on the problem of weak sequential stability and the study of possible concentrations of the temperature in the thermal energy equation.

Chapter 7 contains a complete proof of the existence of global-in-time variational solutions for the full system of the Navier–Stokes equations of a viscous compressible and heat conducting fluid under suitable restrictions imposed on the constitutive laws. These restrictions are by no means optimal but, on the other hand, they seem to be in a good agreement with the underlying physical theory. Probably the most questionable hypothesis seems to be the necessity of the viscosity coefficients to be constant in all temperature regimes. On the other hand, as a byproduct of our approach, we derive "optimal" existence results for the barotropic flows with respect to the available (known) *a priori* estimates.

Another novelty allowed by the present method is the possibility to consider general, not necessarily monotone, pressure–density constitutive equations arising in applications.

This book is intended to be a compact and self-contained presentation of the most recent results of the mathematical theory of viscous compressible fluids including some applications to more specific problems. In order to place the text in better perspective, each chapter is concluded with a section of historical notes including references to all important and relatively new results. However, the cited works have been chosen on the basis of selectiveness rather than completeness to keep the bibliography concise. The results presented in the book are by no means the last word on the subject but they rather indicate possible directions of future research.

E. Feireisl
September, 2003

# ACKNOWLEDGEMENT

The work of the author was supported by Grant 201/02/0854 of GA ČR. He is also indebted to Sonja Goj and Josef Málek for careful reading of the manuscript.

# CONTENTS

# 1

# PHYSICAL BACKGROUND

An investigation into the behavior of a *fluid* in motion may be undertaken from either a *microscopic* or a *macroscopic* point of view. If we attempt to use the microscopic description, the position of each atom (molecule) of the fluid as well as its velocity at a given time must be specified. In order to capture completely the behavior of such a system, it is necessary to deal with a large number of equations describing the motion of each individual particle.

The macroscopic approach pursued in this book reduces the number of variables to a few related to the average effects of action of many molecules. These effects are perceived by our senses and can be measured by instruments.

From the macroscopic point of view, a fluid is always understood as a *continuum* occupying at a given time $t \in R$ a certain spatial domain $\Omega$ in the $N$-dimensional *Euclidean space* $R^N$. The *state* of the fluid is identified through certain observable macroscopic properties such as the *density* $\varrho$, the *velocity field* $\mathbf{u}$, and the *temperature* $\vartheta$. These quantities will be assumed to have the same value for a given state regardless of how the system arrived at that state. Accordingly, the time evolution of a fluid is described through a system of partial differential equations, with the *time* $t \in R$ and the *spatial position* $\mathbf{x} \in \Omega$ as independent variables, and the state functions $\varrho = \varrho(t, \mathbf{x})$, $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$, and $\vartheta = \vartheta(t, \mathbf{x})$ as unknowns.

## 1.1 Kinematics, description of motion

### *1.1.1 Motions*

A *motion* of a body in continuum mechanics is described by a family of one-to-one mappings

$$\mathbf{X}(t, \cdot) : \Omega \to \Omega, \quad t \in I,$$

where $I \subset R$ is a *time interval*, and $\Omega \subset R^N$ is a spatial domain occupied by the body. *Continuum hypothesis* requires $\mathbf{X}(t, \cdot)$ to be a diffeomorphism for any fixed time $t \in I$. It is convenient to choose a *reference configuration* $\mathbf{X}(t_I, \mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \Omega$ at a certain time $t_I \in I$. Accordingly, the curve $\mathbf{X}(t, \mathbf{x})$, $t \in I$, represents the trajectory of a particle occupying at the time $t_I$ a spatial position $\mathbf{x} \in \Omega$. Thus the motion is visualized as mapping parts of space onto parts of space. The reference configuration is introduced to allow us to use *Euclidean geometry* of the ambient space.

There are several ways to describe a motion but all of them are equivalent provided the motion is smooth. Since we are concerned with the dynamics of fluids, it is convenient to use the *spatial description*, usually called *Eulerian*, where the time $t \in I$ and the place $\mathbf{x} \in \Omega$ in the ambient space play the role of independent variables.

A smooth motion $\mathbf{X}$ is completely determined by a *velocity field* $\mathbf{u} : I \times \Omega \to R^N$ through a system of equations:

$$\frac{\partial \mathbf{X}(t,\mathbf{x})}{\partial t} = \mathbf{u}(t, \mathbf{X}(t,\mathbf{x})), \qquad \mathbf{X}(t_I, \mathbf{x}) = \mathbf{x} \quad \text{for } \mathbf{x} \in \Omega, \ t \in I. \tag{1.1}$$

Applying the spatial *gradient operator* $\nabla_x \equiv (\partial_{x_1}, \dots, \partial_{x_N})$ to both sides of (1.1) we get

$$\frac{\partial \nabla_x \mathbf{X}(t,\mathbf{x})}{\partial t} = \nabla_x \mathbf{u}(t, \mathbf{X}(t,\mathbf{x})) \nabla_x \mathbf{X}(t,\mathbf{x}), \quad \nabla_x \mathbf{X}(t_I, \cdot) = \mathbb{I},$$

from which we deduce

$$\frac{\partial}{\partial t}[\det \nabla_x \mathbf{X}(t,\mathbf{x})] = \mathrm{div}_x \mathbf{u}(t, \mathbf{X}(t,\mathbf{x}))[\det \nabla_x \mathbf{X}(t,\mathbf{x})],$$

where $\mathrm{div}_x \mathbf{u} \equiv \mathrm{trace}[\nabla_x \mathbf{u}]$ denotes the *divergence operator* (cf. Chapter 1 of [117]).

The same relation can be written in an equivalent form as a *transport equation*

$$\partial_t J + \mathrm{div}_x(J\mathbf{u}) = 2J\mathrm{div}_x \mathbf{u} \quad \text{in } I \times \Omega \tag{1.2}$$

for the quantity $J(t, \mathbf{X}(t,\mathbf{x})) \equiv |\det \nabla_x \mathbf{X}(t,\mathbf{x})|$ termed *specific volume*.

### *1.1.2 Mass transport*

In general, *mass* may be thought of as a family of non-negative measures $\{M_t\}$, $t \in I$, on $\Omega$ obeying the *principle of mass conservation*:

$$M_{t_1}[\mathbf{X}(t_1, B)] = M_{t_2}[\mathbf{X}(t_2, B)] \tag{1.3}$$

for any Borel set $B \subset \Omega$ and $t_1, t_2 \in I$. Accordingly, the distribution of mass at each time $t \in I$ is uniquely determined by the motion $\mathbf{X}$ and the reference distribution $M_{t_I}$.

We shall assume that $M$ is absolutely continuous with respect to the standard Lebesgue measure; that means the mass distribution is characterized by a *density* function $\varrho = \varrho(t,\mathbf{x})$, which is non-negative and locally integrable on $\Omega$.

Consequently, the relation (1.3) may be rewritten as

$$\int_{\mathbf{X}(t_1,B)} \varrho(t_1,\mathbf{x})\mathrm{d}\mathbf{x} = \int_{\mathbf{X}(t_2,B)} \varrho(t_2,\mathbf{x})\,\mathrm{d}\mathbf{x} \quad \text{for any } t_1, t_2 \in I$$

or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{X}(t,B)} \varrho(t,\mathbf{x})\,\mathrm{d}x = 0.$$

If the motion is smooth, the same equation may be expressed with the help of the *convection theorem* (see Theorem 3.1 of Chapter 1 in [95]) in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_B \varrho(t,\mathbf{x})\mathrm{d}\mathbf{x} + \int_{\partial B} \varrho(t,\mathbf{x})\,\mathbf{u}(t,\mathbf{x})\cdot\mathbf{n}\,\mathrm{d}\sigma = 0 \tag{1.4}$$

for any bounded domain $B \subset \Omega$ with a sufficiently smooth boundary $\partial B$ that the outer normal vector $\mathbf{n} = \mathbf{n}(\mathbf{x})$ may be defined for any $\mathbf{x} \in \partial B$.

If the density $\varrho$ is smooth, one can use Green's theorem to deduce the *continuity equation:*

$$\partial_t \varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0 \quad \text{in } I \times \Omega, \tag{1.5}$$

which is a mathematical formulation of the physical *principle of mass conservation.*

If $\varrho$ is strictly positive, division of equation (1.5) by $\varrho^2$ yields

$$\partial_t \varrho^{-1} + \mathrm{div}_x(\varrho^{-1}\mathbf{u}) = 2\varrho^{-1}\,\mathrm{div}_x\mathbf{u}$$

which is nothing other than equation (1.2). This yields a relation between the fluid density $\varrho$ and the *specific volume* $J$, namely,

$$J(t,\mathbf{x}) = \frac{\varrho(t_I, \mathbf{X}^{-1}(t,\mathbf{x}))}{\varrho(t,\mathbf{x})} \quad \text{for } t \in I,\ \mathbf{x} \in \Omega.$$

In other words, equations (1.2) and (1.5) are equivalent provided both $\varrho$ and $J$ are smooth, and $\varrho$ strictly positive. The regions of zero density are usually associated with regions of cavitation, the density being positive elsewhere.

## 1.2 Balance laws

### *1.2.1 Equations of motion*

The inertial nature of a fluid is expressed when we apply to a small element *Newton's second law of motion* to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{X}(t,B)} (\varrho\mathbf{u})(t,\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbf{X}(t,B)} \varrho(t,\mathbf{x})\mathbf{f}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\partial\mathbf{X}(t,B)} \mathbf{t}(t,\mathbf{x},\mathbf{n})\,\mathrm{d}\sigma, \tag{1.6}$$

or, by virtue of the convection theorem,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_B (\varrho\mathbf{u})(t,\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\partial B} (\varrho\mathbf{u})(t,\mathbf{x})\,\mathbf{u}(t,\mathbf{x})\cdot\mathbf{n}\,\mathrm{d}\mathbf{x} = \int_B \varrho(t,\mathbf{x})\,\mathbf{f}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\partial B} \mathbf{t}(t,\mathbf{x},\mathbf{n})\,\mathrm{d}\sigma \tag{1.7}$$

where the right-hand side is the resultant force acting on a volume element $B \subset \Omega$. In accordance with the *Euler–Cauchy stress principle*, the resultant force can be written as the sum of an *external force* with density $\mathbf{f}$ and a *simple traction* represented by the vector $\mathbf{t}$.

The stress principle in continuum mechanics is put to use through two fundamental *laws of Cauchy* (see e.g. Section 2.6 of Chapter 2 in [117]):

- There is a *stress tensor* $\mathbb{T} = \mathbb{T}(t,\mathbf{x})$ such that

$$\mathbf{t}(t,\mathbf{x},\mathbf{n}) = \mathbb{T}(t,\mathbf{x})\mathbf{n}.$$

- The stress tensor $\mathbb{T}$ is symmetric.

Accordingly, equation (1.7) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_B (\varrho\mathbf{u})(t,\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\partial B} (\varrho\mathbf{u})(t,\mathbf{x})\mathbf{u}(t,\mathbf{x})\cdot\mathbf{n}\,\mathrm{d}\mathbf{x} = \int_B \varrho(t,\mathbf{x})\,\mathbf{f}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\partial B} \mathbb{T}(t,\mathbf{x})\mathbf{n}\,\mathrm{d}\sigma. \tag{1.8}$$

If all quantities are smooth, we can apply Green's theorem to obtain the *momentum equation*:

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) = \mathrm{div}_x\mathbb{T} + \varrho\mathbf{f} \quad \text{in } I\times\Omega, \tag{1.9}$$

where the symbol $\otimes$ stands for the *tensor product*, $[\mathbf{u}\otimes\mathbf{u}]_{i,j} \equiv u_i u_j$.

### *1.2.2 Total energy balance*

Up to this point, emphasis has been put on purely mechanical aspects of motion. However, common experience makes it plain that mechanical action does not always give rise to mechanical effects alone.

Taking the scalar product of (1.9) with $\mathbf{u}$ and using equation (1.5) we get the *mechanical energy equation*

$$\partial_t\left(\tfrac{1}{2}\varrho|\mathbf{u}|^2\right) + \mathrm{div}_x\left(\tfrac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u}\right) - \mathrm{div}_x(\mathbb{T}\mathbf{u}) = -\mathbb{T}:\nabla_x\mathbf{u} + \varrho\mathbf{f}\cdot\mathbf{u} \tag{1.10}$$

with the specific *kinetic energy* $\frac{1}{2}|\mathbf{u}|^2$. Here, the symbol $\mathbb{A} : \mathbb{B}$ stands for the *scalar product*,

$$\mathbb{A} : \mathbb{B} = \sum_{i,j=1}^{N} A_{i,j} B_{i,j}.$$

Equation (1.10) contains a non-conservative term $\mathbb{T} : \nabla_x \mathbf{u}$ which is responsible for changes of *internal energy* $e$. In accordance with the *first law of thermodynamics*, the inertial contribution (1.10) to the *total energy* of the body has to be matched by appropriate changes of its *internal energy* associated with restoring forces (e.g., with compressibility of the fluid), and energy dissipation into heat.

The *total energy*

$$\varrho\Big(\tfrac{1}{2}|\mathbf{u}|^2 + e\Big)$$

is a conserved quantity satisfying an integral identity

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{X}(t,B)} \varrho(t,\mathbf{x})\big(\tfrac{1}{2}|\mathbf{u}(t,\mathbf{x})|^2 + e(t,\mathbf{x})\big)\,\mathrm{d}\mathbf{x}\\ = \int_{\partial\mathbf{X}(t,B)} (\mathbb{T}(t,\mathbf{x})\mathbf{u}(t,\mathbf{x}) - \mathbf{q}(t,\mathbf{x}))\cdot\mathbf{n}\,\mathrm{d}\sigma\\ + \int_{\mathbf{X}(t,B)} \varrho(t,\mathbf{x})\,\mathbf{f}(t,\mathbf{x})\cdot\mathbf{u}(t,\mathbf{x})\,\mathrm{d}\mathbf{x},\end{aligned} \tag{1.11}$$

where $e$ denotes the *specific internal energy*, and $\mathbf{q}$ is the *energy flux* directly related to the transfer of heat.

Similarly, as above, one can use the convection theorem and Green's formula to deduce the *energy equation*:

$$\partial_t\big(\varrho\big(\tfrac{1}{2}|\mathbf{u}|^2 + e\big)\big) + \mathrm{div}_x\big(\varrho\big(\tfrac{1}{2}|\mathbf{u}|^2 + e\big)\mathbf{u}\big) + \mathrm{div}_x\mathbf{q} = \mathrm{div}_x(\mathbb{T}\mathbf{u}) + \varrho\mathbf{f}\cdot\mathbf{u}. \tag{1.12}$$

The *continuity equation* (1.5), the *momentum equation* (1.9), and the *energy equation* (1.12) represent the most general system governing the time evolution of a body in continuum mechanics.

The total energy balance (1.12) may be split into (1.10) and the *internal energy equation*:

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e\mathbf{u}) + \mathrm{div}_x\mathbf{q} = \mathbb{T} : \nabla_x\mathbf{u}. \tag{1.13}$$

However, we must be careful to remember that the mechanical energy equation (1.10) was derived from the laws of mechanics, and that the total energy

equation (1.12) and the internal energy equation (1.13) are two distinct equations with a proper physical meaning. In particular, it is worth noting that the dissipation function $\mathbb{T} : \nabla_x \mathbf{u}$, which appears in both (1.10) and (1.13) but *not* in (1.12), was derived from the differential form of Newton's second law expressed through (1.9), that is, under the assumption of smoothness of the motion. We will come back to this issue in Chapter 4 when a variational formulation of these equations will be discussed.

## 1.3 Constitutive equations

As already pointed out at the beginning of this chapter, the principal assumption adopted in this book is that the *state* of a body in motion at a given instant $t \in I$ is completely characterized by three macroscopic quantities: the density $\varrho$, the velocity $\mathbf{u}$, and the *temperature* $\vartheta$. The physical properties of a particular material will be reflected through *constitutive equations* relating the state variables to other quantities appearing in the system (1.5), (1.9), (1.12)—the *stress tensor* $\mathbb{T}$, the *specific internal energy* $e$, and the *energy flux* $\mathbf{q}$.

A well accepted mathematical definition of a *fluid* reads as follows: when a shear stress is applied to any fluid, the fluid will deform continuously so long as the shear stress is active (see e.g. [61]). Equivalently, one can say that a fluid does not suppport shear stress when in equilibrium. Accordingly, the *stress tensor* $\mathbb{T}$ of a general fluid obeys *Stokes' law*

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}, \tag{1.14}$$

where $p$ is a scalar function termed *pressure*, and $\mathbb{S}$ denotes the *viscous stress tensor*, which characterizes the measure of resistance of the fluid to flow. As is to be expected, viscosity represents the mechanism by which the mechanical energy is transported into heat. The quantity $\mathbb{T} : \nabla_x \mathbf{u}$ appearing as an internal energy source on the right-hand side of (1.13) reads

$$\mathbb{T} : \nabla_x \mathbf{u} = \mathbb{S} : \nabla_x \mathbf{u} - p \operatorname{div}_x \mathbf{u},$$

where the former term, called the *dissipative function*, stands for a real (irreversible) *dissipation* of the mechanical energy into heat while the latter represents the energy change due to the work of compression.

### *1.3.1 Viscous dissipation*

There are two sources of *viscous dissipation* in a fluid: (i) the *shear (or "genuine") viscosity* resulting in the departure of the tensor $\mathbb{S}$ from its isotropic form, (ii) the *bulk viscosity* related to irreversibility due to delays in attaining thermodynamic equilibrium. The bulk viscosity results from the molecular motion relative to the macroscopic velocity $\mathbf{u}$, and it is set at zero in the case of a monoatomic gas (see Chapter 1 of [80]). Accordingly, the viscous stress tensor $\mathbb{S}$ can be written as a

sum of two orthogonal components

$$\mathbb{S} = \Big(\mathbb{S} - \frac{1}{N}\ \mathrm{trace}[\mathbb{S}]\,\mathbb{I}\Big) + \frac{1}{N}\ \mathrm{trace}[\mathbb{S}]\,\mathbb{I}, \tag{1.15}$$

where the former represents the shear viscosity while the latter corresponds to the bulk viscosity. Here, orthogonal means with respect to the standard scalar product $\mathbb{A} : \mathbb{B}$ of tensors $\mathbb{A}$, $\mathbb{B}$.

In accordance with the principle of material frame indifference, the viscous stress tensor $\mathbb{S}$ must depend on the *symmetric part* $\mathbb{D}_x$ of the velocity gradient,

$$\mathbb{D}_x(\mathbf{u}) \equiv \tfrac{1}{2}(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t),$$

its invariants, and possibly other scalar state variables like $\varrho$ and $\vartheta$. If the fluid is *isotropic* and

$$\mathbb{S} = \mathcal{F}(\mathbb{D}_x(\mathbf{u})),$$

then

$$\mathbb{O}\,\mathbb{S}\,\mathbb{O}^t = \mathcal{F}(\mathbb{O}\,\mathbb{D}_x(\mathbf{u})\,\mathbb{O}^t) \quad \text{for any unitary matrix } \mathbb{O} \in SO(N)$$

(see e.g. Chapter 2 of [117]).

An important class of fluids that occupies a central place in mathematical theory is represented by *linearly viscous* (Navier–Stokes, *Newtonian*) fluids for which the *viscous stress tensor* $\mathbb{S}$ depends linearly on the symmetric part of the velocity gradient $\mathbb{D}_x(\mathbf{u})$. In accordance with the general principles delineated above, the only admissible form of $\mathbb{S}$ reads

$$\mathbb{S} = 2\mu\ \mathbb{D}_x(\mathbf{u}) + \lambda\,\mathrm{trace}[\mathbb{D}_x(\mathbf{u})]\ \mathbb{I}, \tag{1.16}$$

where $\mu$ and $\lambda$ are called *viscosity coefficients* (see Section 1.3 of Chapter 1 in [22]).

From the physical point of view, it is more natural to use the representation (1.15), which can be written as

$$\mathbb{S} = 2\mu\Big(\mathbb{D}_x(\mathbf{u}) - \frac{1}{N}\mathrm{div}_x\mathbf{u}\ \mathbb{I}\Big) + \zeta\ \mathrm{div}_x\mathbf{u}\ \mathbb{I},$$

where $\mu$ is the *shear viscosity coefficient* and $\zeta \equiv \lambda + (2/N)\mu$ the *bulk viscosity coefficient*. While $\mu$ should be positive for any "genuinely" viscous fluid, $\zeta$ may vanish, as is the case for a monoatomic gas.

The viscosity coefficients are quantitities that may depend on the values of other state variables such as $\varrho$ and $\vartheta$. Experiments show that the viscosity of fluids is quite sensitive to changes in temperature $\vartheta$. From this point of view,