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Sergei Yu. Pilyugin

# The Space of Dynamical Systems with the $C^0$-Topology

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# Preface

A standard object of interest in the global qualitative theory of dynamical systems is the space of smooth dynamical systems with the $C^1$-topology. In recent years many deep and important results were obtained in the theory of structural stability. These results are mostly based on the following fundamental fact : we may consider a $C^1$-small perturbation of a smooth dynamical system as a perturbation in a neighborhood of any trajectory which does not change essentially the corresponding "first approximation" linear system. It is known for a long time (beginning with works of A.Lyapunov and H.Poincaré) that under some intrinsic conditions on the "first approximation" system perturbations of this sort do not change the local structure of a neighborhood of a trajectory.

The situation becomes quite different if we study $C^0$-small perturbations of a system. It is easy to understand that arbitrarily $C^0$-small perturbation can result in a complete change of the qualitative behaviour of trajectories in a neighborhood of a fixed trajectory. Nevertheless the theory of $C^0$-small perturbations of dynamical systems which was developed intensively over the last 20 years contains now many interesting results.

It was an intention of the author to give the reader an initial perspective of the theory. So we are going to give in this book an introduction to some of the main methods of the theory and to formulate its principal results.

Of course, this book is a reflection of scientific interests of the author, hence we pay more attention to problems which are close to the author's own works. This book is an introduction rather than a monograph. That's why the author tried to simplify and to "visualize" some proofs. Due to this reason some technically complicated proofs (mostly connected with applications of the theory of systems with hyperbolic structure) are omitted, and the reader is referred to the original papers. Sometimes we give only an "explanation" of the main ideas instead of a complete proof (as for example in the case of the $C^0$-density Theorem of M.Shub).

The book consists of 5 chapters and 3 Appendices. Chapter 0 contains practically no theorems. It gives an introduction to the language of the theory and surveys some results we need later. In section 0.1 we define spaces of dynamical systems : the space $Z(M)$ of continuous discrete dynamical systems on a smooth closed manifold $M$ – the main object in this book, and some other spaces we work with. We devote section 0.2 to the space $M^*$ - the space of compact subsets of $M$ with different topologies. We describe also some properties of semi-continuous set-valued maps in this section. In section 0.3 we prove two variants of the $C^0$-closing Lemma. In section 0.4 we give a survey of basic results of the theory of smooth dynamical systems with hyperbolic structure. We describe the set of diffeomorphisms which satisfy the STC (the strong transversality condition). It is known now that this set coincides with the set of structurally stable systems. We try to explain in this book that diffeomorphisms which satisfy the STC play the crucial role not only in the theory of structural stability but also in the theory of $C^0$-small perturbations of dynamical systems.

Chapter 1 is devoted to generic properties of systems in $Z(M)$. We study tolerance stability in section 1.1. A counterexample constructed by W.White to the Tolerance Stability Conjecture is described. We prove some results of F.Takens connected with this conjecture. Pseudoorbits are considered in section 1.2. We define the POTP (the pseudo orbit tracing property) and some of its generalizations. The genericity of weak shadowing for systems in $Z(M)$ is established. We give also a proof by F.Takens of a variant of the Tolerance Stability Conjecture (with extended orbits instead of orbits).

Various types of prolongations are studied in section 1.3. We describe results of V.Dobrynsky and A.Sharkovsky which show that a generic system in $Z(M)$ has the following property : the set of points such that their positive trajectories are stable with respect to permanent perturbations is residual in $M$. It is shown also that a generic system in $Z(M)$ has the property : for any point of $M$ its prolongation with respect to the initial point, its prolongation with respect to the system, and its chain prolongation coincide.

We study various sets of returning points : the nonwandering set, the set of weakly nonwandering points, the chain-recurrent set in section 1.4. In this section we discuss also filtrations. We prove a theorem of M.Shub and S.Smale : if a system has a fine sequence of filtrations then it has no $C^0$ $\Omega$-explosions.

Chapter 2 is devoted to topological stability. We describe general properties of topologically stable systems in section 2.1. It is shown that if a dynamical system $\phi$ is topologically stable then $\phi$ is tolerance-$Z(M)$-stable and $\phi$ has the POTP. We prove also the following result obtained by P.Walters and A.Morimoto : if a system $\phi$ is expansive and has the POTP then $\phi$ is topologically stable.

Results of Z.Nitecki on topological stability of diffeomorphisms with hyperbolic structure are described in section 2.2. We show that a hyperbolic set is locally topologically stable. After that we apply Smale's techniques for constructing filtrations to prove that if a diffeomorphism $\phi$ satisfies the Axiom A and the no-cycle condition then $\phi$ is topologically $\Omega$-stable. We formulate ( without a proof) the main result of Z.Nitecki : if a diffeomorphism $\phi$ satisfies the STC then $\phi$ is topologically stable.

K.Yano characterized topologically stable dynamical systems on the circle. The main statement of section 2.3 is the following theorem of K.Yano : a system $\phi$ on $S^1$ is topologically stable if and only if $\phi$ is topologically conjugate to a Morse-Smale diffeomorphism. We describe in this section an example constructed by K.Yano of a dynamical system which has the POTP but is not topologically stable.

Section 2.4 is devoted to the $C^0$-density theorem of M.Shub : any diffeomorphism $\phi$ can be isotoped to a diffeomorphism satisfying the STC by an isotopy which is arbitrarily small in the $C^0$-topology. We do not give a complete proof of this result but describe its main ideas in the most "visible" case $\dim M = 2$. In section 2.5 we formulate (without proofs) two results : a theorem of M.Hurley who described the chain-recurrent set of a topologically stable diffeomorphism and a theorem of J.Lewowich who applied Lyapunov type functions in the theory of topological stability.

We study $C^0$-small perturbations of attractors in Chapter 3. Basic properties of attractors are described in section 3.1. Section 3.2 is devoted to stability of attractors under $C^0$-small perturbations of the system with respect to different metrics on $M^*$. It is shown that a generic system $\phi$ in $Z(M)$ has the property : any attractor of $\phi$ is stable with respect to $R_0$. This result is a generalization of a theorem of M.Hurley which considers stability of attractors with respect to the Hausdorff metric $R$. We prove also the following result of the author : if an attractor is stable with respect to $R_0$ then its boundary is Lyapunov stable.

Lyapunov stable sets and quasi-attractors in generic systems are considered in section 3.3. We prove a theorem of M.Hurley : if $\dim M \leq 3$ then the union of basins of chain-transitive quasi-attractors of a generic dynamical system is a residual subset of $M$. The second main result of this section was obtained by the author : for a generic dynamical system any Lyapunov stable set is a quasi-attractor.

In section 3.4 we study stability of attractors with the STC on the boundary. The main theorem of this section was proved by O.Ivanov and the author : if $I$ is an attractor of a diffeomorphism $\phi$ which satisfies the STC on the boundary of $I$ then $I$ is Lipschitz stable with respect to the Hausdorff metric $R$. Section 3.5 is devoted to stability of attractors for Morse-Smale diffeomorphisms. We describe results of V.Pogonysheva. It is shown that an attractor $I$ of a Morse-Smale diffeomorphism is stable with respect to metric $R_2$ on $M^*$ if and only if $I = \overline{\text{Int} I}$.

In Chapter 4 we study limit sets of domains and describe some results obtained by the author and V.Pogonysheva. Section 4.1 is devoted to Lyapunov stability of limit sets. It is shown, for example, that a generic system in $Z(M)$ has the following property : given $x \in M$ there exists a countable set $B(x)$ such that for any $r \in (0, +\infty) \setminus B(x)$ the $\omega$-limit set of the ball of radius $r$ centered at $x$ is Lyapunov stable. We investigate also the process of "iterating of taking limit sets of neighborhoods". It is shown that for a generic system this process "approximately stops" after the first step.

Section 4.2 is devoted to limit sets for diffeomorphisms which satisfy the STC. It is shown that if a diffeomorphism $\phi$ satisfies the STC then given $x \in M$ there is a finite set $C(x)$ such that for any $r \in (0, +\infty) \setminus C(x)$ the $\omega$-limit set of the ball of radius $r$ centered at $x$ is an attractor. We prove a result of V.Pogonysheva which gives sufficient conditions for the stability of the $\omega$-limit set of a domain $G$ with respect to both the set $G$ and to the system.

Appendices A,B of the book are devoted to two important technical results we use in previous chapters. Appendix A contains a proof of the following statement : for a diffeomorphism $\phi$ which satisfies the STC there is a constant $L$ such that any $\delta$-trajectory of $\phi$ with small $\delta$ is $L\delta$-traced by a real trajectory. In Appendix B we investigate attractors with the STC on the boundary. The structure of the boundary of the attractor in this case is described ; it is shown, for example, that the boundary is an attractor itself.

In Appendix C we study families of pseudotrajectories generated by numerical methods. We prove a theorem obtained by R.Corless and the author. It shows that for any diffeomorphism $\phi$ satisfying the STC there exist numerical methods of arbitrary accuracy such that $\phi$ has trajectories which are not weakly traced by approximate trajectories obtained using these methods.

We usually do not give in this book any special references to statements included in basic university courses of mathematics. The list of references is far from being complete. It contains only those books and research papers which are directly mentioned in the text.

Many conversations with colleagues were very important for the author during the preparation of this book. Special thanks are to D.Anosov, V.Arnold, R.Corless, Yu.Il'yashenko, O.Ivanov, V.Pliss, V.Pogonysheva, R.Russell, G.Sell, T.Eirola and to Zhang Zhi-fen.

# Contents

# 0. Definitions and Preliminary Results

## 0.1 Spaces of Dynamical Systems

In this book we study dynamical systems on closed smooth manifolds. Some results we prove have analogues for continuous dynamical systems on metric compact sets but we don't pay attention to possible generalizations of this kind.

This section is mostly devoted to fix the language and basic notation. Prerequisites for reading this book are basic courses on Dynamical Systems (see [Pa4, Pi8] for example) and on Differentiable Manifolds ([Hil, Mu2] for example).

Throughout the book $M$ is a $C^\infty$-smooth closed (that is compact and without boundary) manifold of dimension $n$. We fix a Riemannian metric $d$ on $M$. We denote by $T_xM$ the tangent space of $M$ at $x \in M$, and by $TM$ the tangent bundle of $M$. For $v \in T_xM$ we denote by $|v|$ the norm generated on $T_xM$ by $d$. With fixed Riemannian metric $d$ for $x \in M$ we can define the exponential map $exp_x$ being a diffeomorphism of class $C^\infty$ of a neighborhood of 0 at $T_xM$ onto a neighborhood of $x$ in $M$. As $M$ is compact there is $r > 0$ such that for any $x \in M$ the map $exp_x$ is a diffeomorphism of $\{v \in T_xM : |v| < r\}$ onto its image.

We study discrete dynamical systems generated by homeomorphisms $\phi : M \to M$. We do not distinguish between a homeomorphism $\phi$ and the dynamical system it generates.

Let us denote by

$$O(x, \phi) = \{\phi^k(x) : k \in \mathbf{Z}\}$$

the trajectory (orbit) of a point $x \in M$ in a dynamical system $\phi$. Sometimes we write $O(\phi)$ (if it is not important to mark the initial point) or $O(x)$ (if we work with a fixed dynamical system) instead of $O(x, \phi)$.

We denote

$$O^+(x, \phi) = \{\phi^k(x) : k \in \mathbf{Z}, k \geq 0\},$$
$$O^-(x, \phi) = \{\phi^k(x) : k \in \mathbf{Z}, k \leq 0\}.$$

We also use the following notation. If $x \in M$, $k_1, k_2 \in \mathbf{Z}, k_1 \leq k_2$ we write

$$O_{k_1}^{k_2}(x, \phi) = \{\phi^k(x) : k_1 \leq k \leq k_2\}.$$

In this case $k_1 = -\infty$ or $k_2 = +\infty$ are admissible, so that

$$O_{-\infty}^{k_2}(x, \phi) = \{\phi^k(x) : -\infty < k \leq k_2\},$$

and $O^-(x,\phi) = O^0_{-\infty}(x,\phi)$, for example. We also sometimes write $O^+(\phi), O^+(x)$ instead of $O^+(x,\phi)$. We use the same notation for trajectories of sets: if $X \subset M$ then we write

$$O(X,\phi) = \{\phi^k(X) : k \in \mathbf{Z}\},$$

and so on.

As usually we say that a point $x \in M$ is a *periodic point* of period $m$ for a dynamical system $\phi$ if

$$\phi^m(x) = x; \phi^k(x) \neq x \text{ for } 0 < k \leq m-1$$

(if $\phi(x) = x$ we say that $x$ is a fixed point). We denote by $\mathrm{Per}(\phi)$ the set of periodic points of $\phi$, and by $\mathrm{Fix}(\phi)$ the set of fixed points of $\phi$.

We say that a point $x \in M$ is a *nonwandering point* of $\phi$ if given a neighborhood $U$ of $x$ and a number $m_0 > 0$ we can find a number $m \in \mathbf{Z}, |m| \geq m_0$ such that

$$\phi^m(U) \cap U \neq \emptyset.$$

Denote by $\Omega(\phi)$ the set of nonwandering points of $\phi$.

For a point $x \in M$ and for a dynamical system $\phi$ we define $\alpha_x$, the $\alpha$-limit set of $O(x,\phi)$, and $\omega_x$, the $\omega$-limit set of $O(x,\phi)$, as follows:

$$\alpha_x(\phi) = \{\lim_{k\to\infty} \phi^{t_k}(x) : \lim_{k\to\infty} t_k \to -\infty\},$$

$$\omega_x(\phi) = \{\lim_{k\to\infty} \phi^{t_k}(x) : \lim_{k\to\infty} t_k \to +\infty\}.$$

Sometimes we write $\alpha_x, \omega_x$ instead of $\alpha_x(\phi), \omega_x(\phi)$. The following statement is standard (see [Pi8], for example).

**Lemma** 0.1.1.
*(a) the set $\Omega(\phi)$ is compact and $\phi$-invariant;*
*(b) $x \in \Omega(\phi)$ if and only if there exist sequences*

$$x_k \to x,\ t_k \to +\infty \text{ as } k \to \infty$$

*such that*

$$\phi^{t_k}(x_k) \to x \text{ as } k \to \infty;$$

*(c) for any $x \in M$ we have $\alpha_x(\phi) \cup \omega_x(\phi) \subset \Omega(\phi)$.*

Take two discrete dynamical systems $\phi, \psi$ on $M$ and let

$$\rho_0(\phi,\psi) = \max_{x\in M}(d(\phi(x),\psi(x)), d(\phi^{-1}(x),\psi^{-1}(x))).$$

It is easy to see that $\rho_0$ is a metric on the space of dynamical systems. The main object in this book is the space $Z(M)$ of continuous discrete dynamical systems on $M$ with the $C^0$-topology induced by the metric $\rho_0$. Standard considerations show that $(Z(M),\rho_0)$ is a complete metric space.

For a set $A \subset M$, for a dynamical system $\phi \in Z(M)$, and for a number $\epsilon > 0$ we denote

$$N_\epsilon(A) = \{x \in M : d(x, A) < \epsilon\}$$

(here as usually $d(x, A) = \inf_{y \in A} d(x, y)$ ),

$$N_\epsilon(\phi) = \{\psi \in Z(M) : \rho_0(\phi, \psi) < \epsilon\}.$$

We consider not only continuous but also differentiable dynamical systems on $M$ generated by diffeomorphisms of $M$. To introduce the $C^r$-topology on the space of diffeomorphisms of class $C^r, 1 \le r < +\infty$, we can proceed as follows.

Consider a smooth map $f : M \to N$, where $M, N$ are smooth manifolds.

The map $f$ is said to be an *immersion* if the derivative

$$Df(p) : T_pM \to T_{f(p)}N$$

is injective for all $p \in M$.

The map $f$ is said to be an *embedding* if $f$ is an injective immersion which has a continuous inverse $f^{-1} : f(M) \subset N \to M$.

The classical Whitney's Theorem [Hi1] states that if $\dim M = n$ then there exists an embedding

$$f : M \to \mathbf{R}^{2n+1}. \tag{0.1}$$

Fix a finite covering of $M$ by open sets $V_1, \dots, V_m$ such that each $\overline{V_i}$ is contained in the domain of a local chart $(\xi_j, U_j)$ of $M$.

Consider a smooth map $\chi : \mathbf{R}^n \to \mathbf{R}^q$, let $x = (x_1, \dots, x_n)$ be coordinates of $\mathbf{R}^n$. As usually we denote for $p = (p_1, \dots, p_n)$ with $p_i \in \mathbf{Z}, p_i \ge 0$

$$\frac{\partial^p \chi}{\partial x^p} = \frac{\partial^p \chi}{\partial x_1^{p_1} \dots \partial x_n^{p_n}}, |p| = p_1 + \dots + p_n.$$

Take two diffeomorphisms $\phi, \psi$ of $M$. Take $i \in \{1, \dots, m\}$ , suppose that $\overline{V_i} \subset U_j$, and let

$$\tilde{V}_i = \xi_j(V_i) \subset \mathbf{R}^n,$$

$$\tilde{\phi}_i = f \circ \phi \circ \xi_j^{-1}, \tilde{\psi}_i = f \circ \psi \circ \xi_j^{-1}$$

(here $f$ is the embedding (0.1)). If $\phi, \psi$ are diffeomorphisms of class $C^r, 1 \le r < +\infty$, we can introduce the number

$$\rho_r(\phi, \psi) = \max_{1 \le i \le m} \sup_{x \in \tilde{V}_i} \sum_{0 \le |p| \le r} \| \frac{\partial^p (\tilde{\phi}_i - \tilde{\psi}_i)}{\partial x^p}(x) \|$$

(as everywhere in this book for a linear operator $A$ we consider the operator norm

$$\|A\| = \sup_{|y|=1} |Ay|.)$$

It is easy to see that $\rho_r$ is a metric on the space of diffeomorphisms of class $C^r$ of $M$. This metric induces the $C^r$-topology (as $M$ is compact the topology is independent on the choice of $V_1, \dots, V_m$). We denote by $\mathrm{Diff}^r(M)$ the corresponding topological space. It is evident that for any $r \geq 1$ the topology of $\mathrm{Diff}^r(M)$ is not coarser than the topology on the space of $C^r$-diffeomorphisms induced by the topology of $Z(M)$.

We denote by CLD$(M)$ the closure of $\mathrm{Diff}^1(M)$ in $Z(M)$. We shall use below the following important statement obtained independently by J.Munkres [Mu1] and by J.Whitehead [Wh].

**Theorem** 0.1.1. *If* $\dim M \leq 3$ *then* $Z(M) = CLD(M)$.

Note that it is shown in [Mu1] that if $\dim M > 3$ then there exist homeomorphisms which are not $C^0$-approximated by diffeomorphisms.

We do not discuss in this book analogues for flows of the results we describe. Nevertheless sometimes we work with flows - for example we apply techniques of flows to prove variants of the $C^0$-closing Lemma (Lemmas 0.3.1,0.3.3). Besides, it is very convenient to visualize some constructions using flows (see Sect. 0.1).

So we describe main notation we use below. We consider vector fields $X$ which are Lipschitz on $M$ in the following standard sense. Denote by $\pi$ the projection $TM \to M$, that is for $(x, v) \in TM$ with $v \in T_xM$ we have $\pi(x, v) = x$. As usually we define a vector field $X$ on $M$ as a map $X : M \to TM$ such that $\pi \circ X(x) = x$ for any $x$. That means that we can write $X(p) = (p, X_p)$. Let $\tilde{d}$ be a metric on $TM$ generated by $d$. We say that $X$ is Lipschitz on $M$ if there is a constant $L > 0$ such that for any $p_1, p_2 \in M$ we have

$$\tilde{d}(X(p_1), X(p_2)) \leq L d(p_1, p_2).$$

It is easy to show that if a vector field $X$ on $M$ is of class $C^1$ then $X$ is Lipschitz on $M$.

It is well-known that if a vector field $X$ is Lipschitz then the corresponding system of differential equations

$$\frac{dx}{dt} = X(x)$$

generates a flow $\Phi : \mathbf{R} \times M \to M$ with $\Phi(0, x) = x$.

For two flows $\Phi_1, \Phi_2$ we let

$$\rho_0(\Phi_1, \Phi_2) = \max_{p \in M, t \in [-1,1]} d(\Phi_1(t, p), \Phi_2(t, p)).$$

It is easy to see that $\rho_0$ is a metric on the space of flows. Below we say that the topology induced by $\rho_0$ on the space of flows is the standard $C^0$-topology.

## 0.2 The Space $M^*$

We denote everywhere in this book by $X^*$ the set of compact subsets of a topological space $X$. In this section we mostly pay attention to $M^*$ - the set of compact subsets of our manifold $M$.

We begin with description of the standard *Hausdorff metric* $R$ on $M^*$. Take $A, B \in M^*$ and let

$$r(A,B) = \max_{x \in A} d(x,B).$$

Take $A, B \in M^*$ and if $A, B \neq \emptyset$ let

$$R(A,B) = \max(r(A,B), r(B,A)).$$

Let for any $A \in M^*, A \neq \emptyset$

$$R(\emptyset, A) = \text{diam} M = \max_{x,y \in M} d(x,y),$$

and let

$$R(\emptyset, \emptyset) = 0.$$

It is easy to show that $R$ is a metric on $M^*$. It is evident that for any $A$ we have $R(A,A) = 0$ and that $R(A,B) = 0$ implies $A = B$. To prove the triangle inequality take $A, B, C \in M^*$. Consider $x \in A, y \in B$, then

$$d(x,C) \leq d(x,y) + d(y,C) \leq d(x,y) + R(B,C),$$

hence

$$d(x,C) \leq \min_{y \in B} d(x,y) + R(B,C) = d(x,B) + R(B,C),$$

and

$$d(x,C) \leq R(A,B) + R(B,C).$$

Therefore,

$$r(A,C) \leq R(A,B) + R(B,C).$$

The same reasons show that

$$r(C,A) \leq R(A,B) + R(B,C),$$

so we obtain that

$$R(A,C) \leq R(A,B) + R(B,C).$$

It follows that $R$ is a metric in $M^*$. It is shown in [K] that $(M^*, R)$ is a complete metric space. Later if we write $M^*$ we have in mind the space $(M^*, R)$.

We work in this book also with the following metrics $R_0, R_1, R_2$ on $M^*$ (we use for $R_0$ the original notation from the paper [Pi3] in which this metric was introduced; $R_2$ was introduced in [Pol] and was denoted $R_1$ there). Take $A, B \in M^*$ and let

$$R_0(A,B) = \max(R(A,B), R(\overline{M \setminus A}, \overline{M \setminus B})),$$

$$R_1(A,B) = \max(R(A,B), R(\overline{\text{IntA}}, \overline{\text{IntB}})),$$

$$R_2(A,B) = \max(R_0(A,B), R_1(A,B)).$$

It is evident that for $S = R_0, R_1, R_2$ we have $S(A,B) \geq 0, S(A,A) = 0$, and that $S(A,B) = 0$ implies $A = B$.

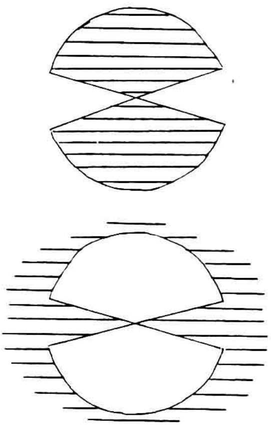


Figure 0.1

If $\rho_1, \rho_2 : M^* \times M^* \to \mathbf{R}$ satisfy the triangle inequality and $\rho(A, B) = \max(\rho_1(A, B), \rho_2(A, B))$ then $\rho$ also satisfies the triangle inequality. Indeed, take $A, B, C \in M^*$. If $\rho(A, C) = \rho_1(A, C)$ then

$$\rho(A, C) \le \rho_1(A, B) + \rho_1(B, C) \le$$

$$\max(\rho_1(A, B), \rho_2(A, B)) + \max(\rho_1(B, C), \rho_2(B, C)) = \rho(A, B) + \rho(B, C).$$

If $\rho(A, C) = \rho_2(A, C)$ the reasons are the same. Hence, $R, R_0, R_1, R_2$ are metrics on $M^*$.

Let us show that $R, R_0, R_1, R_2$ induce different topologies on $M^*$. To show this, consider $M = S^2$ and take a coordinate neighborhood being homeomorphic to $\mathbf{R}^2$ with coordinates $(x, y)$.

Let $A = \{(x, y) : x^2 + y^2 \le 1\}$ and for $m \ge 1$

$$A_m = A \bigcap \{(x, y) : |x| \ge \frac{|y|}{m}\}$$

(see Fig. 0.1). It is evident that

$$\lim_{m\to\infty} R(A, A_m) = 0, R_0(A, A_m) \equiv 1,$$

$$\lim_{m\to\infty} R_1(A, A_m) = 0.$$

Now let

$$A = \{(x, y) : x^2 + y^2 \le 1\} \bigcup \{(x, y) : 1 \le x \le 2, y = 0\},$$

and for $m \ge 1$

$$A_m = A \bigcup \{(x, y) : 0 \le x \le 2, \frac{x-2}{m} \le y \le \frac{2-x}{m}\}$$

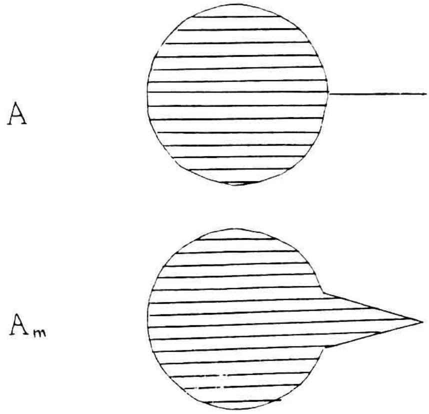


Figure 0.2

(see Fig. 0.2). It is evident that

$$\lim_{m\to\infty} R(A, A_m) = 0, \lim_{m\to\infty} R_0(A, A_m) = 0,$$

$$R_1(A, A_m) \equiv 1.$$

Now we describe a general process of constructing metrics on $M^*$. We are going to show that the metrics $R, R_0, R_1, R_2$ form the complete list of metrics given by this process.

Let for a set $A \subset M$

$$F_1(A) = M \setminus A, F_2(A) = \overline{A}.$$

Consider the following set of finite sequences

$$J = \{(i_1, \dots, i_m) : i_k \in \{1, 2\}\},$$

and let

$$F^{\emptyset}(A) = A,$$

for $j = \{i_1, \dots, i_m\} \in J$ let

$$F^j(A) = F_{i_1} \circ F_{i_2} \circ \dots \circ F_{i_m}(A).$$

Now take a finite subset $\tilde{J}$ of $J$ which has the properties:

(a) for any $j \in \tilde{J}, A \in M^*$ we have $F^j(A) \in M^*$;

(b) there exists $j \in \tilde{J}$ such that $R(F^j(.), F^j(.))$ is a metric on $M^*$.

Define for $A, B \in M^*$

$$R_{\tilde{J}}(A, B) = \max_{j\in\tilde{J}}(R(F^j(A), F^j(B))). \tag{0.2}$$

It follows from our previous considerations that any $R_{\tilde{J}}$ is a metric on $M^*$.

It is well-known (see [K]) that if we take $A \in M^*$ then for any $j = (i_1, \dots, i_m) \in J$ the set $F^j(A)$ coincides with one of the following sets:

$$A, \text{IntA}, \overline{\text{IntA}}, M \setminus A, \overline{M \setminus A}, \text{Int}(M \setminus A) \tag{0.3}$$

Only 3 sets in (0.3) are compact : $A, \overline{M \setminus A}, \overline{\text{IntA}}$. As

$$\overline{M \setminus A} = F_2 \circ F_1(A),$$

$$\overline{\text{IntA}} = \overline{M \setminus (\overline{M \setminus A})} = F_2 \circ F_1 \circ F_2 \circ F_1(A),$$

the metrics $R, R_0, R_1, R_2$ are obtained in the form (0.2) with $\tilde{J} = \{\emptyset\}, \tilde{J} = \{\emptyset, (2,1)\}, \tilde{J} = \{\emptyset, (2,1,2,1,)\}, \tilde{J} = \{\emptyset, (2,1), (2,1,2,1,)\}$, respectively.

Evidently there exist different compact sets $A, B$ such that $\overline{M \setminus A} = \overline{M \setminus B}$ and $\overline{\text{IntA}} = \overline{\text{IntB}}$, hence

$$R(\overline{M \setminus A}, \overline{M \setminus B}), R(\overline{\text{IntA}}, \overline{\text{IntB}}),$$

$$\max(R(\overline{M \setminus A}, \overline{M \setminus B}), R(\overline{\text{IntA}}, \overline{\text{IntB}}))$$

are not metrics on $M^*$. This proves that $\{R, R_0, R_1, R_2\}$ is the complete list of metrics obtained by the described process.

We shall consider also $M^{**}$ – the set of all compact subsets of $M^*$ (we remind that here $M^*$ means $(M^*, R)$). We denote by $\bar{R}$ the Hausdorff metric on $M^{**}$.

Let now $X$ be a topological space. A subset $A \subset X$ is called *residual* if $A$ contains a countable intersection of open dense sets in $X$. If $P$ is a property of elements of $X$ we say that this property is *generic* if the set $\{x \in X : x \text{ satisfies } P\}$ is residual in $X$. Sometimes in this case we say that a generic element of $X$ satisfies $P$.

The topological space $X$ is called a Baire space if every residual set is dense in it. A classical theorem of Baire [K] says that every complete metric space is a Baire space. Hence, $Z(M)$ is a Baire space, and $\text{CLD}(M)$ is also a Baire space.

Now we are going to prove a result which is very useful to establish genericity of some properties of dynamical systems in $Z(M)$. Let us begin with the following definition.

Let $X$ be a topological space, and let $(N, \rho)$ be a compact metric space. A map

$$\psi : X \to N^*$$

is called *upper semi-continuous* (respectively, *lower semi-continuous*) if for every $x \in X$ and $\delta > 0$ there is a neighborhood $W(x)$ of $x$ in $X$ such that for any $y \in W(x)$ we have

$$\psi(y) \subset N_\delta(\psi(x))$$

(respectively,

$$\psi(x) \subset N_\delta(\psi(y)) \text{ ).}$$

As previously, $N_\delta(\psi(x))$ is the $\delta$-neighborhood of $\psi(x)$.

Let $R$ be the Hausdorff metric on $N^*$. Fix $\delta > 0$. We say that the map $\psi$ is $\delta$-continuous at $x \in X$ if $x$ has a neighborhood $W(x)$ in $X$ such that for any $x', x'' \in W(x)$ we have

$$R(\psi(x'), \psi(x'')) < \delta.$$

Clearly $\psi$ is continuous at $x \in X$ if it is $\delta$-continuous at $x$ for any $\delta > 0$. Let $V_\delta$ be the set of points of $\delta$- continuity of $\psi$.

**Lemma** 0.2.1 [Ta1]. *If $\psi : X \to N^*$ is upper semi-continuous or lower semi-continuous then for any $\delta > 0$ the set $V_\delta$ is open and dense in $X$.*

*Proof.* . We consider the case of $\psi$ being upper semi-continuous, the case of $\psi$ being lower semi-continuous is treated analogously. It follows immediately from the definition that the set $V_\delta$ is open. Let us show that this set is dense.

Fix an open covering $U_1, \dots, U_k$ of $N$ such that diam $U_i < \delta, i = 1, \dots, k$. Let $K = \{1, \dots, k\}$ (we consider $K$ as a discrete topological space), and let $K^*$ be the set of subsets of $K$ (we are following our notation here taking into account that any subset of $K$ is compact).

Take $L \in K^*$ and consider the subset $N_L$ of $N^*$ : $A \in N^*$ is in $N_L$ if and only if

(a) $A \cap U_i \neq \emptyset$ for any $i \in L$;

(b) $A \subset \bigcup_{i \in L} U_i$.

It is evident that for any $L, N_L$ is open in $N^*$.

Now fix an open subset $W$ of $X$. Define

$$B_W = \{L \in K^* : \text{ there is } x \in W \text{ with } \psi(x) \in N_L\}.$$

As $K^*$ is finite and has a partial order (by the inclusion relation for subsets of $K$) we can find a minimal element $L_0 \in B_W$. This means that there is no $L \in B_W$ with $L \subset L_0, L \neq L_0$.

Let $x_0 \in W$ be such that $\psi(x_0) \in N_{L_0}$. We claim that $\psi$ is $\delta$-continuous at $x_0$. Indeed, as $\psi$ is upper semi-continuous we can find a neighborhood $W(x_0)$ such that for any $x' \in W(x_0)$ we have

$$\psi(x') \subset \bigcup_{i \in L_0} U_i.$$

As $L_0$ is minimal we see that for any $x' \in W(x_0)$

$$\psi(x') \cap U_i \neq \emptyset, i \in L_0.$$

Take $x', x'' \in W(x_0)$ and consider $y' \in \psi(x')$. There is $i \in L_0$ such that $y' \in U_i$. Find $y'' \in \psi(x'')$ such that $y'' \in U_i$. It follows from diam $U_i < \delta$ that $\rho(y', y'') < \delta$. This evidently implies

$$R(\psi(x'), \psi(x'')) < \delta.$$

So, $V_\delta$ is dense. This completes the proof.

**Corollary 0.1** *If $\psi : X \to N^*$ is upper semi-continuous or lower semi-continuous then the set of continuity points of $\psi$ is residual in $X$.*