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Benoit Fresse

Modules over Operads and Functors

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Preface

The notion of an operad was introduced 40 years ago in algebraic topology in order to model the structure of iterated loop spaces [6, 47, 60]. Since then, operads have been used fruitfully in many fields of mathematics and physics.

Indeed, the notion of an operad supplies both a conceptual and effective device to handle a variety of algebraic structures in various situations. Many usual categories of algebras (like the category of commutative and associative algebras, the category of associative algebras, the category of Lie algebras, the category of Poisson algebras, ...) are associated to operads.

The main successful applications of operads in algebra occur in deformation theory: the theory of operads unifies the construction of deformation complexes, gives generalizations of powerful methods of rational homotopy, and brings to light deep connections between the cohomology of algebras, the structure of combinatorial polyhedra, the geometry of moduli spaces of surfaces, and conformal field theory. The new proofs of the existence of deformation-quantizations by Kontsevich and Tamarkin bring together all these developments and lead Kontsevich to the fascinating conjecture that the motivic Galois group operates on the space of deformation-quantizations (see [35]).

The purpose of this monograph is to study not operads themselves, but *modules over operads* as a device to model functors between categories of algebras as effectively as operads model categories of algebras.

Modules over operads occur naturally when one needs to represent universal complexes associated to algebras over operads (see [14, 54]).

Modules over operads have not been studied as extensively as operads yet. However, a generalization of the theory of Hopf algebras to modules over operads has already proved to be useful in various mathematical fields: to organize Hopf invariants in homotopy theory [2]; to study non-commutative generalizations of formal groups [12, 13]; to understand the structure of certain combinatorial Hopf algebras [38, 39]. Besides, the notion of a module over an operad unifies and generalizes classical structures, like Segal's notion of a

Γ-object, which occur in homological algebra and homotopy theory. In [33], Kapranov and Manin give an application of the relationship between modules over operads and functors for the construction of Morita equivalences between categories of algebras.

Our own motivation to write this monograph comes from homotopy theory: we prove, with a view to applications, that functors determined by modules over operads satisfy good homotopy invariance properties.

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Contents

Index and Glossary of Notation

Introduction

Main Ideas and Objectives

The background of the theory of operads is entirely reviewed in the first part of the monograph. The main characters of the story appear in a natural generalization of the symmetric algebra $\mathrm{S}(X)$, the module spanned by tensors $x_1 \otimes \cdots \otimes x_n \in X^{\otimes n}$ divided out by the symmetry relations

$$x_{w(1)} \otimes \cdots \otimes x_{w(n)} \equiv x_1 \otimes \cdots \otimes x_n,$$

where w ranges permutations of $(1, \dots, n)$. Formally, the symmetric algebra is defined by the expansion $\mathrm{S}(X) = \bigoplus_{n=0}^{\infty} (X^{\otimes n})_{\Sigma_n}$, where the notation $(-)_{\Sigma_n}$ refers to a module of coinvariants under the action of the symmetric group of n-letters, denoted by Σ_n. The theory of operads deals with functors $\mathrm{S}(M) : X \mapsto \mathrm{S}(M, X)$ of generalized symmetric tensors

$$\mathrm{S}(M, X) = \bigoplus_{n=0}^{\infty} (M(n) \otimes X^{\otimes n})_{\Sigma_n}$$

with coefficients in objects $M(n)$ equipped with an action of the symmetric groups Σ_n. The structure formed by the coefficient sequence $M = \{M(n)\}_{n \in \mathbb{N}}$ is called a Σ_*-object (or, in English words, a symmetric sequence or a symmetric object). The definition of $\mathrm{S}(M, X)$ makes sense in the setting of a symmetric monoidal category $\mathcal{E}$. The map $\mathrm{S}(M) : X \mapsto \mathrm{S}(M, X)$ defines a functor $\mathrm{S}(M) : \mathcal{E} \to \mathcal{E}$.

In this book we study a generalization of this construction with the aim to model functors on algebras over operads. For an operad $\mathtt{P}$, we use the notation ${}_{\mathtt{P}}\mathcal{E}$ to refer to the category of $\mathtt{P}$-algebras in $\mathcal{E}$. We aim to model functors $F : {}_{\mathtt{R}}\mathcal{E} \to \mathcal{E}$ from a category of algebras over an operad $\mathtt{R}$ to the underlying category $\mathcal{E}$, functors $F : \mathcal{E} \to {}_{\mathtt{P}}\mathcal{E}$ from the underlying category $\mathcal{E}$ to a category of algebras over an operad $\mathtt{P}$, as well as functors $F : {}_{\mathtt{R}}\mathcal{E} \to {}_{\mathtt{P}}\mathcal{E}$ from a category of algebras over an operad $\mathtt{R}$ to another category of algebras over an operad $\mathtt{P}$.

To define functors of these types we use left and right modules over operads, the structures formed by Σ_*-objects equipped with left or right operad actions. For a right R-module M and an R-algebra A, we use a coequalizer to make the right R-action on M agrees with the left R-action on A in the object $S(M, A)$. This construction returns an object $S_R(M, A) \in \mathcal{E}$ naturally associated to the pair (M, A) and the map $S_R(M) : A \mapsto S_R(M, A)$ defines a functor $S_R(M) : {}_R\mathcal{E} \to \mathcal{E}$ naturally associated to M. For a left P-module N the map $S(N) : X \mapsto S(N, X)$ defines naturally a functor $S(N) : \mathcal{E} \to {}_P\mathcal{E}$. For a P-R-bimodule N, a right R-module equipped with a left P-action that commutes with the right R-action on N, the map $S_R(N) : A \mapsto S_R(N, A)$ defines naturally a functor $S_R(N) : {}_R\mathcal{E} \to {}_P\mathcal{E}$.

We study the categorical and homotopical properties of functors of these form.

Not all functors are associated to modules over operads, but we check that the categories of modules over operads are equipped with structures that reflect natural operations on functors. As a byproduct, we obtain that usual functors (enveloping operads, enveloping algebras, Kähler differentials, bar constructions, ...), which are composed of tensor products and colimits, can be associated to modules over operads.

In homotopy theory, operads are usually supposed to be cofibrant in the underlying category of Σ_*-objects in order to ensure that the category of algebras over an operad has a well defined model structure. In contrast, the category of right modules over an operad R comes equipped with a natural model structure which is always well defined if the operad R is cofibrant in the underlying symmetric monoidal category. Bimodules over operads form model categories in the same situation provided we restrict ourself to connected Σ_*-objects for which the constant term $N(0)$ vanishes. Thus for modules over operads we have more homotopical structures than at the algebra and functor levels. As a result, certain homotopical constructions, which are difficult to carry out at the functor level, can be realized easily by passing to modules over operads (motivating examples are sketched next). On the other hand, we check that, for functors associated to cofibrant right modules over operads, homotopy equivalences of modules correspond to pointwise equivalences of functors. In the case where algebras over operads form a model category, we can restrict ourself to cofibrant algebras to obtain that any weak-equivalence between right modules over operads induce a pointwise weak-equivalence of functors. These results show that modules over operads give good models for the homotopy of associated functors.

We use that objects equipped with left operad actions are identified with algebras over operads provided we change the underlying symmetric monoidal category of algebras. Suppose that the operad R belongs to a fixed base symmetric monoidal category $\mathcal{C}$. The notion of an R-algebra can be defined in any symmetric monoidal category $\mathcal{E}$ acted on by $\mathcal{C}$, or equivalently equipped with a symmetric monoidal functor $\eta : \mathcal{C} \to \mathcal{E}$.

The category of Σ_*-objects in $\mathcal{C}$ forms an instance of a symmetric monoidal category over $\mathcal{C}$, and so does the category of right $\mathtt{R}$-modules. One observes that a left $\mathtt{P}$-module is equivalent to a $\mathtt{P}$-algebra in the category of Σ_*-objects and a $\mathtt{P}$-$\mathtt{R}$-bimodule is equivalent to a $\mathtt{P}$-algebra in the category of right $\mathtt{R}$-modules, for any operads $\mathtt{P}, \mathtt{R}$ in the base category $\mathcal{C}$.

Because of these observations, it is natural to assume that operads belong to a base category $\mathcal{C}$ and algebras run over any symmetric monoidal category $\mathcal{E}$ over $\mathcal{C}$. We review constructions of the theory of operads in this relative context. We study more specifically the functoriality of operadic constructions with respect to the underlying symmetric monoidal category. We can deduce properties of functors of the types $\mathrm{S}(N) : \mathcal{E} \to {}_{\mathtt{P}}\mathcal{E}$ and $\mathrm{S}_{\mathtt{R}}(N) : {}_{\mathtt{R}}\mathcal{E} \to {}_{\mathtt{P}}\mathcal{E}$ from this generalization of the theory of algebras over operads after we prove that the map $\mathrm{S}_{\mathtt{R}} : M \mapsto \mathrm{S}_{\mathtt{R}}(M)$ defines a functor of symmetric monoidal categories, like $\mathrm{S} : M \mapsto \mathrm{S}(M)$. For this reason, the book is essentially devoted to the study of the category of right $\mathtt{R}$-modules and to the study of functors $\mathrm{S}_{\mathtt{R}}(M) : {}_{\mathtt{R}}\mathcal{E} \to \mathcal{E}$ associated to right $\mathtt{R}$-modules.

Historical Overview and Prospects

Modules over operads occur naturally once one aims to represent the structure underlying the cotriple construction of Beck [3] and May [47, §9]. As far as we know, a first instance of this idea occurs in Smirnov's papers [56, 57] where an operadic analogue of the cotriple construction is defined. This operadic cotriple construction is studied more thoroughly in Rezk's thesis [54] to define a homology theory for operads.

The operadic bar construction of Getzler-Jones [17] and the Koszul construction of Ginzburg-Kapranov [18] are other constructions of the homology theory of operads. In [14], we prove that the operadic cotriple construction, the operadic bar construction and the Koszul construction are associated to free resolutions in categories of modules over operads, like the bar construction of algebras.

Classical theorems involving modules over algebras can be generalized to the context of operads: in [33], Kapranov and Manin use functors of the form $S_{\mathtt{R}}(N) : {}_{\mathtt{R}}\mathcal{E} \to {}_{\mathtt{P}}\mathcal{E}$ to define Morita equivalences for categories of algebras over operads.

Our personal interest in modules over operads arose from the Lie theory of formal groups over operads. In summary, we use Lie algebras in right modules over operads to represent functors of the form $S_{\mathtt{R}}(\mathtt{G}) : {}_{\mathtt{R}}\mathcal{E} \to {}_{\mathtt{L}}\mathcal{E}$, where $\mathtt{L}$ refers to the operad of Lie algebras. Formal groups over an operad $\mathtt{R}$ are functors on nilpotent objects of the category of R-algebras. For a nilpotent $\mathtt{R}$-algebra A, the object $S_{\mathtt{R}}(\mathtt{G}, A)$ forms a nilpotent Lie algebra and the Campbell-Hausdorff formula provides this object with a natural group structure. Thus the map $A \mapsto S_{\mathtt{R}}(\mathtt{G}, A)$ gives rise to a functor from nilpotent $\mathtt{R}$-algebras to groups.

The Lie theory asserts that all formal groups over operads arise this way (see [11, 12, 13]). The operadic proof of this result relies on a generalization of the classical structure theorems of Hopf algebras to Hopf algebras in right modules over operads. Historically, the classification of formal groups was first obtained by Lazard in [37] in the formalism of "analyzers", an early precursor of the notion of an operad.

Recently, Patras-Schocker [49], Livernet-Patras [39] and Livernet [38] have observed that Hopf algebras in Σ_*-objects occur naturally to understand the structure of certain classical combinatorial Hopf algebras.

Lie algebras in Σ_*-objects were introduced before in homotopy theory by Barratt (see [2], see also [19, 61]) in order to model structures arising from Milnor's decomposition

$$\Omega\Sigma(X_1 \vee X_2) \sim \bigvee_w w(X_1, X_2),$$

where w runs over a Hall basis of the free Lie algebra in 2-generators x_1, x_2 and $w(X_1, X_2)$ refers to a smash product of copies of X_1, X_2 (one per occurrence of the variables x_1, x_2 in w).

In sequels [15, 16] we use modules over operads to define multiplicative structures on the bar complex of algebras. Recall that the bar complex $B(C^*(X))$ of a cochain algebra $A = C^*(X)$ is chain equivalent to the cochain complex of ΩX, the loop space of X (under standard completeness assumptions on X). We obtain that this cochain complex $B(C^*(X))$ comes naturally equipped with the structure of an E_∞-algebra so that $B(C^*(X))$ is equivalent to $C^*(\Omega X)$ as an E_∞-algebra.

Recall that an E_∞-operad refers to an operad $\mathtt{E}$ equipped with a weak-equivalence $\mathtt{E} \xrightarrow{\sim} \mathtt{C}$, where $\mathtt{C}$ is the operad of associative and commutative algebras. An E_∞-algebra is an algebra over some E_∞-operad. Roughly an E_∞-operad parameterizes operations that make an algebra structure commutative up to a whole set of coherent homotopies.

In the differential graded context, the bar construction $B(A)$ is defined naturally for algebras A over Stasheff's chain operad, the operad defined by the chain complexes of Stasheff's associahedra. We use the letter $\mathtt{K}$ to refer to this operad. We observe that the bar construction is identified with the functor $B(A) = S_{\mathtt{K}}(B_{\mathtt{K}}, A)$ associated to a right $\mathtt{K}$-module $B_{\mathtt{K}}$. We can restrict the bar construction to any category of algebras associated to an operad $\mathtt{R}$ equipped with a morphism $\eta : \mathtt{K} \to \mathtt{R}$. The functor obtained by this restriction process is also associated to a right $\mathtt{R}$-module $B_{\mathtt{R}}$ obtained by an extension of structures from $B_{\mathtt{K}}$. Homotopy equivalent operads $\mathtt{R} \xrightarrow{\sim} \mathtt{S}$ have homotopy equivalent modules $B_{\mathtt{R}} \xrightarrow{\sim} B_{\mathtt{S}}$.

The bar module $B_{\mathtt{C}}$ of the commutative operad $\mathtt{C}$ has a commutative algebra structure that reflects the classical structure of the bar construction of commutative algebras. The existence of an equivalence $B_{\mathtt{E}} \xrightarrow{\sim} B_{\mathtt{C}}$, where $\mathtt{E}$ is any E_∞-operad, allows us to transport the multiplicative structures of the bar module B_{C} to B_{E} and hence to obtain a multiplicative structure on the bar complex of E_∞-algebras. Constructions and theorems of this book are motivated by this application.

Note that modules over operads are applied differently in [25] in the study of structures on the bar construction: according to this article, modules over operads model morphisms between bar complexes of chain algebras.

In [15], we only deal with multiplicative structures on modules over operads and with multiplicative structures on the bar construction, but the bar complex forms naturally a coassociative coalgebra. In a subsequent paper [16], we address coalgebras and bialgebras in right modules over operads in order to extend constructions of [15] to the coalgebra setting and to obtain a bialgebra structure on the bar complex.

For a cochain algebra, the comultiplicative structure of the bar complex $B(C^*(X))$ models the multiplicative structure of the loop space ΩX. Bialgebras in right modules over operads give rise to Lie algebras, like the classical ones. One should expect that Lie algebras arising from the bar module B_{R} are related to Barratt's twisted Lie algebras.

Contents

The sections, paragraphs, and statements marked by the sign '¶' can be skipped in the course of a normal reading. These marks ¶ refer to refinement outlines.

Part I. Categorical and Operadic Background

The purpose of the first part of the book is to clarify the background of our constructions. This part does not contain any original result, but only changes of presentation in our approach of operads. Roughly, we make use of functors of symmetric monoidal categories in standard operadic constructions.

§1. Symmetric Monoidal Categories for Operads

First of all, we give the definition of a symmetric monoidal category $\mathcal{E}$ over a base symmetric monoidal category $\mathcal{C}$. Formally, a symmetric monoidal category over $\mathcal{C}$ is an object under $\mathcal{C}$ in the 2-category formed by symmetric monoidal categories and symmetric monoidal functors. In §1 we give equivalent axioms for this structure in a form suitable for our applications. Besides, we inspect properties of functors and adjunctions between symmetric monoidal categories.

§2. Symmetric Objects and Functors

We survey categorical properties of the functor $\mathrm{S} : M \mapsto \mathrm{S}(M)$ from Σ_*-objects to functors $F : \mathcal{E} \to \mathcal{E}$. More specifically, we recall the definition of the tensor product of Σ_*-objects, the operation that gives to Σ_*-objects the structure of a symmetric monoidal category over $\mathcal{C}$, and the definition of the composition product of Σ_*-objects, an operation that reflects the composition of functors. For the sake of completeness, we also recall that the functor $\mathrm{S} : M \mapsto \mathrm{S}(M)$ has a right adjoint $\Gamma : G \mapsto \Gamma(G)$. In the case $\mathcal{E} = \mathcal{C} = \Bbbk\,\mathrm{Mod}$, the category of modules over a ring $\Bbbk$, we use this construction to prove that the functor $\mathrm{S} : M \mapsto \mathrm{S}(M)$ is bijective on morphism sets for projective Σ_*-objects or if the ground ring is an infinite field.

§3. Operads and Algebras in Symmetric Monoidal Categories

We recall the definition of an algebra over an operad P. We review the basic examples of the commutative, associative, and Lie operads, which are associated to commutative and associative algebras, associative algebras and Lie algebras respectively.

We assume that the operad P belongs to the base category $\mathcal{C}$ and we define the category ${}_{\mathsf{P}}\mathcal{E}$ of P-algebras in a symmetric monoidal category $\mathcal{E}$ over $\mathcal{C}$. We observe that any functor $\rho : \mathcal{D} \to \mathcal{E}$ of symmetric monoidal categories over $\mathcal{C}$ induces a functor on the categories of P-algebras $\rho : {}_{\mathsf{P}}\mathcal{D} \to {}_{\mathsf{P}}\mathcal{E}$, for any operad P in the base category $\mathcal{C}$. We review the classical constructions of free objects, extension and restriction functors, colimits in categories of algebras over operads, and we check that these constructions are invariant under changes of symmetric monoidal categories. We review the classical definition of endomorphism operads with similar functoriality questions in mind.

At this point, we study the example of algebras in Σ_*-objects and the example of algebras in functors. We make the observation that a left module over an operad P is equivalent to a P-algebra in Σ_*-objects. We also observe that a functor $F : \mathcal{X} \to {}_{\mathsf{P}}\mathcal{E}$, where $\mathcal{X}$ is any source category, is equivalent to a P-algebra in the category of functors of the form $F : \mathcal{X} \to \mathcal{E}$. We use that $\mathrm{S} : M \mapsto \mathrm{S}(M)$ defines a symmetric monoidal functor to obtain the correspondence between left P-modules N and functors $\mathrm{S}(N) : \mathcal{E} \to {}_{\mathsf{P}}\mathcal{E}$.

§4. Miscellaneous Structures Associated to Algebras over Operads

We recall the definition of miscellaneous structures associated to algebras over operads: enveloping operads, which model comma categories of algebras over operads; enveloping algebras, which are associative algebras formed from the structure of enveloping operads; representations, which are nothing but modules over enveloping algebras; and modules of Kähler differentials, which represent the module of derivations of an algebra over an operad. We study applications of these notions to the usual operads: commutative, associative, and Lie. For each example, the operadic definition of an enveloping algebra is equivalent to a standard construction of algebra, and similarly as regards representations and Kähler differentials.

Part II. The Category of Right Modules over Operads and Functors

In the second part of the book, we study categorical structures of right modules over operads and functors. Roughly, we prove that the categories of right modules over operads are equipped with structures that reflect natural operations at the functor level. This part contains overlaps with the literature (with [13] and [54, Chapter 2] in particular). Nevertheless, we prefer to give a comprehensive account of definitions and categorical constructions on right modules over operads.

§5. Definitions and Basic Constructions

First of all, we recall the definition of a right module over an operad $\mathtt{R}$ and of the associated functors $\mathrm{S}_{\mathtt{R}}(M) : A \mapsto \mathrm{S}_{\mathtt{R}}(M, A)$, where the variable A ranges over $\mathtt{R}$-algebras in any symmetric monoidal category $\mathcal{E}$ over the base category $\mathcal{C}$.

In the book, we use the notation $\mathcal{M}$ for the category of Σ_*-objects in the base category $\mathcal{C}$ and the notation $\mathcal{F}$ for the category of functors $F : \mathcal{E} \to \mathcal{E}$, where $\mathcal{E}$ is any given symmetric monoidal category over $\mathcal{C}$. The map $\mathrm{S} : M \mapsto \mathrm{S}(M)$ defines a functor $\mathrm{S} : \mathcal{M} \to \mathcal{F}$. In a similar way, we use the notation $\mathcal{M}_{\mathtt{R}}$ for the category of right $\mathtt{R}$-modules and the notation $\mathcal{F}_{\mathtt{R}}$ for the category of functors $F : {}_{\mathtt{R}}\mathcal{E} \to \mathcal{E}$. The map $\mathrm{S}_{\mathtt{R}} : M \mapsto \mathrm{S}_{\mathtt{R}}(M)$ defines a functor $\mathrm{S}_{\mathtt{R}} : \mathcal{M}_{\mathtt{R}} \to \mathcal{F}_{\mathtt{R}}$.

§6. Tensor Products

For any source category $\mathcal{A}$, the category of functors $F : \mathcal{A} \to \mathcal{E}$ has a natural tensor product inherited pointwise from the symmetric monoidal category $\mathcal{E}$. In the first part of this introduction, we mention that the category of Σ_*-objects $\mathcal{M}$ comes also equipped with a natural tensor product, as well as the category of right $\mathtt{R}$-modules $\mathcal{M}$. The definition of the tensor product of Σ_*-objects is recalled in §2. The tensor product of right $\mathtt{R}$-modules is derived from the tensor product of Σ_*-objects and this definition is recalled in §6.

Classical theorems assert that the functor $\mathrm{S} : M \mapsto \mathrm{S}(M)$ satisfies $\mathrm{S}(M \otimes N) \simeq \mathrm{S}(M) \otimes \mathrm{S}(N)$. In §6, we check that the functor $\mathrm{S}_{\mathtt{R}} : M \mapsto \mathrm{S}_{\mathtt{R}}(M)$ satisfies similarly $\mathrm{S}_{\mathtt{R}}(M \otimes N) \simeq \mathrm{S}_{\mathtt{R}}(M) \otimes \mathrm{S}_{\mathtt{R}}(N)$. Formally, we prove that the map $\mathrm{S}_{\mathtt{R}} : M \mapsto \mathrm{S}_{\mathtt{R}}(M)$ defines a functor of symmetric monoidal categories over $\mathcal{C}$:

$$(\mathcal{M}_{\mathtt{R}}, \otimes, 1) \xrightarrow{\mathrm{S}_{\mathtt{R}}} (\mathcal{F}_{\mathtt{R}}, \otimes, 1).$$

§7. Universal Constructions on Right Modules over Operads

An operad morphism $\psi : \mathtt{R} \to \mathtt{S}$ gives rise to extension and restriction functors $\psi_! : {}_{\mathtt{R}}\mathcal{E} \rightleftarrows {}_{\mathtt{S}}\mathcal{E} : \psi^!$. The composition of functors $F : {}_{\mathtt{R}}\mathcal{E} \to \mathcal{E}$ with the restriction functor $\psi^! : {}_{\mathtt{S}}\mathcal{E} \to {}_{\mathtt{R}}\mathcal{E}$ defines an extension functor on functor categories: $\psi_! : \mathcal{F}_{\mathtt{R}} \to \mathcal{F}_{\mathtt{S}}$. In the converse direction, the composition of functors $G : {}_{\mathtt{S}}\mathcal{E} \to \mathcal{E}$ with the extension functor $\psi_! : {}_{\mathtt{R}}\mathcal{E} \to {}_{\mathtt{S}}\mathcal{E}$ defines a restriction functor $\psi^! : \mathcal{F}_{\mathtt{S}} \to \mathcal{F}_{\mathtt{R}}$. These extension and restriction functors $\psi_! : \mathcal{F}_{\mathtt{R}} \rightleftarrows \mathcal{F}_{\mathtt{S}} : \psi^!$ form a pair of adjoint functors.

At the level of module categories, we also have extension and restriction functors $\psi_! : \mathcal{M}_{\mathtt{R}} \rightleftarrows \mathcal{M}_{\mathtt{S}} : \psi^!$ that generalize the classical extension and restriction functors of modules over associative algebras. We prove that these operations on modules correspond to the extension and restriction of functors. Explicitly, we have natural functor isomorphisms $\psi_! \, \mathrm{S}_{\mathtt{R}}(M) \simeq \mathrm{S}_{\mathtt{S}}(\psi_! M)$ and $\psi^! \, \mathrm{S}_{\mathtt{R}}(M) \simeq \mathrm{S}_{\mathtt{S}}(\psi^! M)$. Besides, we check the coherence of these isomorphisms with respect to adjunction relations and tensor structures.

In the particular case of the unit morphism of an operad, we obtain that the composite of a functor $\mathrm{S}_{\mathtt{R}}(M) : {}_{\mathtt{R}}\mathcal{E} \to \mathcal{E}$ with the free $\mathtt{R}$-algebra functor is identified with the functor $\mathrm{S}(M) : \mathcal{E} \to \mathcal{E}$ associated to the underlying Σ_*-object of M. In the converse direction, for a Σ_*-object L, we obtain that the composite of the functor $\mathrm{S}(L) : \mathcal{E} \to \mathcal{E}$ with the forgetful functor $U : {}_{\mathtt{R}}\mathcal{E} \to \mathcal{E}$ is identified with the functor associated to a free right $\mathtt{R}$-module associated to L.

§8. Adjunction and Embedding Properties

The functor $\mathrm{S}_{\mathtt{R}} : \mathcal{M}_{\mathtt{R}} \to \mathcal{F}_{\mathtt{R}}$ has a right adjoint $\Gamma_{\mathtt{R}} : \mathcal{F}_{\mathtt{R}} \to \mathcal{M}_{\mathtt{R}}$, like the functor $\mathrm{S} : \mathcal{M} \to \mathcal{F}$. In the case $\mathcal{E} = \mathcal{C} = \Bbbk\,\mathrm{Mod}$, the category of modules over a ring $\Bbbk$, we use this result to prove that the functor $\mathrm{S}_{\mathtt{R}} : M \mapsto \mathrm{S}_{\mathtt{R}}(M)$ is bijective on morphism sets in the same situations as the functor $\mathrm{S} : M \mapsto \mathrm{S}(M)$ on Σ_*-objects.

§9. Algebras in Right Modules over Operads

We study the structure of algebras in a category of right modules over an operad. We observe at this point that a bimodule over operads $\mathtt{P}, \mathtt{R}$ is equivalent to a P-algebra in right R-modules. We use that $S_R : M \mapsto S_R(M)$ defines a symmetric monoidal functor to obtain the correspondence between P-R-bimodules N and functors $S(N) : {}_R\mathcal{E} \to {}_P\mathcal{E}$, as in the case of left P-modules. We review applications of the general theory of §3 to this correspondence between P-algebra structures.

§10. Miscellaneous Examples

To give an illustration of our constructions, we prove that enveloping operads, enveloping algebras and modules of Kähler differentials, whose definitions are recalled in §4, are instances of functors associated to modules over operads. Besides we examine the structure of these modules for classical operads, namely the operad of associative algebras A, the operad of Lie algebras L, and the operad of associative and commutative algebras C.

New examples of functors associated to right modules over operads can be derived by using the categorical operations of §6, §7 and §9.

Part III. Homotopical Background

The purpose of the third part of the book is to survey applications of homotopical algebra to the theory of operads. We review carefully axioms of model categories to define (semi-)model structures on categories of algebras over operads. We study with more details the (semi-)model categories of algebras in differential graded modules.

This part does not contain any original result, like our exposition of the background of operads, but only changes of presentations. More specifically, we observe that crucial verifications in the homotopy theory of algebras over operads are consequences of general results about the homotopy of functors associated to modules over operads. In this sense, this part gives first motivations to set a homotopy theory for modules over operads, the purpose of the fourth part of the book.

§11. Symmetric Monoidal Model Categories For Operads

First of all, we review the axioms of model categories and the construction of new model structures by adjunction from a given model category. The model categories introduced in this book are defined by this adjunction process.

The notion of a symmetric monoidal model category gives the background for the homotopy theory of operads. We give axioms for a symmetric monoidal model category $\mathcal{E}$ over a base symmetric monoidal model category $\mathcal{C}$.

We prove that the category of Σ_*-objects inherits a model structure and forms a symmetric monoidal model category over the base category. Then we