Lecture Notes in Mathematics

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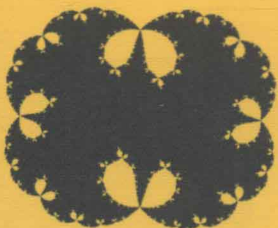
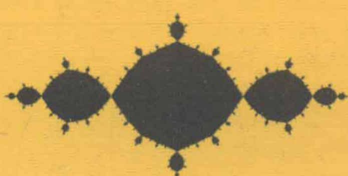
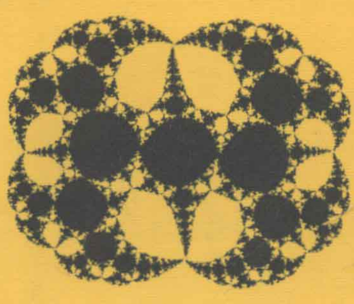
Kevin M. Pilgrim

Combinations of Complex Dynamical Systems







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Preface

The goal of this research monograph is to develop a general combination, decomposition, and structure theory for branched coverings of the two-sphere to itself, regarded as the combinatorial and topological objects which arise in the classification of certain holomorphic dynamical systems on the Riemann sphere. It is intended for researchers interested in the classification of those complex one-dimensional dynamical systems which are in some loose sense *tame*, though precisely what this constitutes we leave open to interpretation. The program is motivated in general by the dictionary between the theories of iterated rational maps and Kleinian groups as holomorphic dynamical systems, and in particular by the structure theory of compact irreducible three-manifolds.

By and large this work involves only topological/combinatorial notions. Apart from motivational discussions, the sole exceptions are (i) the construction of examples which is aided using complex dynamics in §9, and (ii) some familiarity with the Douady-Hubbard proof of Thurston's characterization of rational functions in §§8.3.1 and §10.

The combination and decomposition theory is developed for maps which are not necessarily postcritically finite. However, the proof of the main structure result, the Canonical Decomposition Theorem, depends on Thurston's characterization and is developed only for postcritically finite maps. A survey of known results regarding combinatorics and combination procedures for rational maps is included.

This research was partially supported by NSF grant No. DMS-9996070, the University of Missouri at Rolla, and Indiana University. I thank Albert Goodman for timely advice on group actions which were particularly helpful in proving the results in §7. I thank Curt McMullen for encouraging me to think big. I am especially grateful to Mary Rees and to the referees for valuable comments. Finally, I thank my family for their unwavering support.

Bloomington, Indiana, USA, *Kevin M. Pilgrim*
August, 2003

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Introduction

1.1 Motivation from dynamics–a brief sketch

This work is about the combinatorial aspects of rigidity phenomena in complex dynamics. It is motivated by discoveries of Douady-Hubbard [DH1], Milnor-Thurston [MT], and Sullivan made during the early 1980's (see the preface by Hubbard in [Tan4] for a firsthand account).

In the real quadratic family $f_a(x) = (x^2 + a)/2, a \in \mathbb{R}$, it was proven [MT] that the entropy of f_a as a function of a is continuous, monotone, and increasing as the real parameter varies from $a = 5$ to $a = 8$. A key ingredient of their proof is a complete combinatorial characterization and rigidity result for *critically periodic* maps f_a, i.e. those for which the unique critical point at the origin is periodic. To any map f_a in the family one associates a combinatorial invariant, called its *kneading invariant*. Such an invariant must be *admissible* in order to arise from a map f_a. It was shown that every admissible kneading invariant actually arises from such a map f_a, and that if two critically periodic maps have the same kneading invariant, then they are affine conjugate. In a process called *microimplantation* the dynamics of one map f_a could be "glued" into that of another map f_{a_0} where f_{a_0} is critically periodic to obtain a new map f_{a*a_0} in this family. More precisely: a topological model for the new map is constructed, and its kneading invariant, which depends only on the topological data, is computed. The result turns out to be admissible, hence by the characterization theorem defines uniquely a new map f_{a_0*a}. This construction interprets the cascade of period-doublings as the limit $\lim_{n\to\infty} f_{a_n}$ where $a_{n+1} = a_n * a_0$ and a_0 is chosen so that the critical point is periodic of period two. As an application, it is shown that there exists an uncountable family of maps with distinct kneading invariants but with the same entropy.

Similar combinatorial rigidity phenomena were also observed for maps $f_c(z) = z^2 + c$, $c \in \mathbb{C}$ in the complex setting. For "critically periodic" parameters c for which the critical point at the origin is periodic, the dynamics restricted to the *filled-in Julia set* $K_c = \{z | f^{\circ n}(z) \not\to \infty\}$ looks roughly like a map from a tree to itself (here $f^{\circ n}$ is the n-fold iterate of f). The dynamics of f_c can be faithfully encoded by what became later known as a *Hubbard tree*, a finite planar tree equipped with a self-map, subject to some reasonable admissibility criteria. Alternatively, via what became known as the theory of *invariant laminations*, the dynamics of f can be encoded by a single rational number $\mu = p/q \in (0, 1)$, where the denominator q is odd. As in the setting of interval maps, the manner in which the critically periodic parameters c are deployed in the parameter plane has a rich combinatorial structure. A procedure known as *tuning* generalizes the process of microimplantation. The inverse of tuning became known as *renormalization* and explains the presence of small copies of the Mandelbrot set inside itself.

Among rational maps, Douady and Hubbard noticed from computer experiments that a different combination procedure, now called *mating* , explained the dynamical structure of certain quadratic rational functions in terms of a pair of critically finite polynomials. However, not all such pairs of polynomials were "mateable", i.e. produced a rational map when mated–obstructions could arise.

The combinatorial characterization and rigidity result for critically periodic unimodal interval maps was greatly generalized by Thurston [DH3] to *postcritically finite rational maps*, i.e. those rational maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ acting on the Riemann sphere such that the *postcritical set*

$$P_f = \overline{\cup_{n>0} f^{\circ n}(\{\text{critical points}\})}$$

is finite. This characterization was then applied to completely resolve the question of when two critically finite quadratic polynomials are mateable.

1.2 Thurston's Characterization and Rigidity Theorem. Standard definitions

The following discussion summarizes the main results of [DH3]. Denote by S^2 the Euclidean two-sphere. By a *branched covering* $F : S^2 \to S^2$ we mean a continuous orientation-preserving map of topological degree $d \geq 1$ such that for all $x \in S^2$, there exist local charts about x and $y = F(x)$ sending x and y to $0 \in \mathbb{C}$ such that within these charts, the map is given by $z \mapsto z^{d_x}$, where $d_x \geq 1$ is the *local degree* of F at x. The prototypical example is a rational function $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree at least two. If $d_x \geq 2$ we call x a *critical point*; $d_x - 1$ is its *multiplicity*. The Riemann-Hurwitz formula implies that counted with multiplicity, there are $2d - 2$ such critical points. The *postcritical set* is defined as

$$P_F = \overline{\bigcup_{n>0} F^{\circ n}(\{\text{critical points}\})}$$

and when F is rational the topology and geometry of this set plays a crucial role in the study of complex dynamics in one variable. Note that P_F contains the set of critical values of F, so that in particular $F^{\circ n} : S^2 - F^{-n}(P_F) \to S^2 - P_F$ is an unramified covering for all $n \geq 1$.

The simplest possible behavior of P_F occurs when this set is finite; in this case, F is said to be *postcritically finite* . "Postcritically finite" is sometimes shortened to *critically finite*, and such maps F are called here *Thurston maps.*

Combinatorial equivalence. Two Thurston maps F, G are said to be *combinatorially equivalent* if there exist orientation-preserving homeomorphisms of pairs $h_0, h_1 : (S^2, P_F) \to (S^2, P_G)$ such that $h_0 \circ F = G \circ h_1$ and h_0 is isotopic to h_1 through homeomorphisms agreeing on P_F.

Orbifolds. The *orbifold* $\mathcal{O}_F$ *associated to* F is the topological orbifold with underlying space S^2 and whose weight $\nu(x)$ at x is the least common multiple of the local degree of F over all iterated preimages of x (infinite weight is interpreted as a puncture). The *Euler characteristic* of $\mathcal{O}_F$

$$\chi(\mathcal{O}_F) = 2 - \sum_{x \in P_F} (1 - 1/\nu(x))$$

is always nonpositive; if it is zero it is called *Euclidean*, or *parabolic*; otherwise it is called *hyperbolic.*

Expanding metrics. For later reference, we discuss expanding metrics. Suppose F is a C^1 Thurston map with orbifold $\mathcal{O}_F$. Let P_F^a denote the punctures of $\mathcal{O}_F$ (i.e. points eventually landing on a periodic critical point under iteration). F is said to be *expanding with respect to a Riemannian metric* $||\cdot||$ on $S^2 - P_F$ if:

1. any compact piecewise smooth curve inside $S^2 - P_F^a$ has finite length,
2. the distance $d(\cdot,\cdot)$ on $S^2 - P_F^a$ determined by lengths of curves computed with respect to $||\cdot||$ is complete,
3. for some constants $C > 0$ and $\lambda > 1$, we have that for any $n > 0$, for any $p \in S^2 - F^{-n}(P_F)$, and any tangent vector $v \in T_p(S^2)$,

$$||Df^n(v)|| > C\lambda^n ||v||.$$

Then we have the useful estimate

$$l(\tilde{\alpha}) < C^{-1}\lambda^{-n} l(\alpha)$$

whenever $\tilde{\alpha}$ is a lift under $f^{\circ n}$ of a curve $\alpha \in S^2 - P_F^a$; here l is length with respect to $||\cdot||$.

Multicurves. Let γ be a simple closed curve in $S^2 - P_F$. By a *multicurve* we mean a collection

$$\Gamma = \{\gamma_1, ..., \gamma_N\}$$

of simple, closed, disjoint, pairwise non-homotopic, non-peripheral curves in $S^2 - P_F$. A curve γ is *peripheral in* $S^2 - P_F$ if some component of its complement contains only one or no points of P_F.

In [DH3] a multicurve Γ is called *F-invariant* (or sometimes, "F-stable") if for any $\gamma \in \Gamma$, each component of $F^{-1}(\gamma)$ is either peripheral with respect to P_F, or is homotopic in $S^2 - P_F$ to an element of Γ. By lifting homotopies, it is easily seen that this property depends only on the set $[\Gamma]$ of homotopy classes of elements of Γ in $S^2 - P_F$. We shall actually require a slightly stronger version of this definition, given in §1.8.3.

Thurston linear map. Let $\mathbb{R}^\Gamma$ be the vector space of formal real linear combinations of elements of Γ. Associated to an F-invariant multicurve Γ is a linear map

$$F_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$$

defined as follows. Let $\gamma_{i,j,\alpha}$ be the components of $F^{-1}(\gamma_j)$ which are homotopic to γ_i in $S^2 - P_F$. Define

$$F_\Gamma(\gamma_j) = \sum_{i,\alpha} \frac{1}{d_{i,j,\alpha}} \gamma_i$$

where $d_{i,j,\alpha}$ is the (positive) degree of the map $F|\gamma_{i,j,\alpha} : \gamma_{i,j,\alpha} \to \gamma_j$. Then F_Γ has spectral radius realized by a real nonnegative eigenvalue $\lambda(F, \Gamma)$, by the Perron-Frobenius theorem.

Thurston's theorem is

Theorem 1.1 (Thurston's characterization and rigidity theorem). *A Thurston map F with hyperbolic orbifold is equivalent to a rational function if and only if for any F-stable multicurve Γ we have $\lambda(f, \Gamma) < 1$. In that case, the rational function is unique up to conjugation by an automorphism of the Riemann sphere.*

Thurston maps with Euclidean orbifold are treated as well. The postcritical set of such a map has either three or four points. In the former case, any such map is equivalent to a rational map unique up to conjugacy. In the latter case, the orbifold has four points of order two, and the map lifts to an endomorphism T_F of a complex torus. Douady and Hubbard show that in this case F is equivalent to a rational map if and only if either (1) the eigenvalues of the induced map on $H_1(T_F)$ are not real, or (2) this induced map is a real multiple of the identity. Here, though, the uniqueness (rigidity) conclusion can fail. For example, in square degrees $d = n^2$, it is possible that T_F is given by $w \mapsto n \cdot w$, so that by varying the shape of the complex torus one obtains a complex one-parameter family of postcritically finite rational maps which are all quasiconformally conjugate. These examples are known as *integral Lattès examples*; see Section 9.3.

Idea of the proof. The idea of the proof is the following. Associated to F is a Teichmüller space $\mathcal{T}_F$ modelled on (S^2, P_F), and an analytic self-map $\sigma_F : \mathcal{T}_F \to \mathcal{T}_F$. The existence of a rational map combinatorially equivalent to

F is equivalent to the existence of a fixed point of σ_F. The map σ_F is distance-nonincreasing for the Teichmüller metric, and if the associated orbifold is hyperbolic, σ_F^2 decreases distances, though not necessarily uniformly. To find a fixed point, one chooses arbitrarily $\tau_0 \in \mathcal{T}_F$ and considers the sequence $\tau_i = \sigma_F^{\circ i}(\tau_0)$. If $\{\tau_i\}$ fails to converge, then the length of the shortest geodesic on τ_i, in its natural hyperbolic metric, must become arbitrarily small. In this case, for some i sufficiently large, the family of geodesics on τ_i which are both sufficiently short and sufficiently shorter than any other geodesics on τ_i form an invariant multicurve whose leading eigenvalue cannot be less than one, i.e. is a *Thurston obstruction* .

For a nonperipheral simple closed curve $\gamma \subset S^2 - P_F$ let $l_\tau(\gamma)$ denote the hyperbolic length of the unique geodesic on the marked Riemann surface given by τ which is homotopic to γ. In [DH3] the authors show by example that it is possible for curves of two different obstructions to intersect, thus preventing their lengths from becoming simultaneously small. Hence, if Γ is an obstruction and $\gamma \in \Gamma$, then it is not necessarily true that $\inf_i\{l_{\tau_i}(\gamma)\} = 0$. Moreover, their proof does not explicitly show that if F is obstructed, then $\inf_i\{l_{\tau_i}(\gamma)\} = 0$ for some fixed curve γ. Thus it is conceivable that, for each i, there is a curve γ_i such that

$$\inf_i\{l_{\tau_i}(\gamma_i)\} = 0$$

while for fixed i

$$\inf_j\{l_{\tau_j}(\gamma_i)\} > 0.$$

In [Pil2] this possibility was ruled out:

Theorem 1.2 (Canonical obstruction). *Let F be a Thurston map with hyperbolic orbifold, and let Γ_c denote the set of all homotopy classes of non-peripheral, simple closed curves γ in $S^2 - P_F$ such that $l_{\tau_i}(\gamma) \to 0$ as $i \to \infty$. Then Γ_c is independent of τ_i. Moreover:*

1. *If Γ_c is empty, then F is combinatorially equivalent to a rational map.*
2. *Otherwise, Γ_c is an F-stable multicurve for which $\lambda(F, \Gamma_c) \geq 1$, and hence is a canonically defined Thurston obstruction to the existence of a rational map combinatorially equivalent to F.*

The proof also showed, with the same hypotheses,

Theorem 1.3 (Curves degenerate or stay bounded). *Let γ be a nonperipheral, simple closed curve in $S^2 - P_F$.*

1. *If $\gamma \in \Gamma_c$, then $l_{\tau_i}(\gamma) \to 0$ as $i \to \infty$.*
2. *If $\gamma \notin \Gamma_c$, then $l_{\tau_i}(\gamma) \geq E$ for all i, where E is a positive constant depending on τ_0 but not on γ.*

1.3 Examples

Formal mating. Formal mating is a combination process which takes as input two monic complex polynomials f, g of the same degree d and returns as output a branched covering $F = F_{f,g}$ of the two-sphere. Let f, g be two monic complex polynomials of degree $d \geq 2$. Compactify the complex plane $\mathbb{C}$ to $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty \cdot \exp(2\pi i t), t \in \mathbb{R}/\mathbb{Z}\}$ by adding the circle at infinity, thus making $\tilde{\mathbb{C}}$ homeomorphic to a closed disk. Extend f continuously to $\tilde{f} : \tilde{\mathbb{C}}_f \to \tilde{\mathbb{C}}_f$ by setting $f(\infty \cdot \exp(2\pi i t)) = \infty \cdot \exp(2\pi i d t)$ and do the same for g.

Let $S^2_{f,g}$ denote the quotient space $\tilde{\mathbb{C}}_f \cup \tilde{\mathbb{C}}_g / \sim$, where $\infty \cdot \exp(2\pi i t) \sim \infty \cdot \exp(-2\pi i t)$. The *formal mating* $F_{f,g} : S^2_{f,g} \to S^2_{f,g}$ is defined as the map induced by $\tilde{f}$ on $\tilde{\mathbb{C}}_f$ and by $\tilde{g}$ on $\tilde{\mathbb{C}}_g$.

1.3.1 A realizable mating

Let $f(z) = z^2 - 1$ and $g(z) = z^2 + c$ where c is the unique complex parameter for which the critical point at the origin is periodic of period three and $\mathrm{Im}(c) > 0$. Then the formal mating F of f and g is combinatorially equivalent to a rational map; see Figure 1.1.

1.3.2 An obstructed mating

Let $f(z) = g(z) = z^2 - 1$ and denote by F the formal mating of f and g. Since the origin is periodic of period two under $z^2 - 1$, the postcritical set of F has four points and the orbifold $\mathcal{O}_F$ is the four-times punctured sphere. F is not combinatorially equivalent to a rational map. To see this, let $\gamma \in S^2 = S^2_{f,g}$ be the simple closed curve formed by two copies, one in each of $\tilde{\mathbb{C}}_f, \tilde{\mathbb{C}}_g$, of $\{\infty \cdot \exp(2\pi i 1/3)\} \cup R_{1/3} \cup \{\alpha\} \cup R_{2/3} \cup \{\infty \cdot \exp(2\pi i 2/3)\}$ where α is the common landing point of $R_{1/3}, R_{2/3}$. (see §1.5.1 for relevant definitions, or just look at Figure 1.2 below.)

Since $z^2 - 1$ interchanges $R_{1/3}$ and $R_{2/3}$, F sends γ to itself by an orientation-reversing homeomorphism. Hence $\Gamma = \{\gamma\}$ is an invariant multicurve for which the Thurston matrix is simply (1), and so Γ is a Thurston obstruction. Note, however, that Γ is also an obstruction to the existence of a branched covering G combinatorially equivalent to F which is expanding with respect to some metric, since lifts of γ must be shrunk by a definite factor.

Informally, one could decompose this example as follows (see Figure 1.3).

Let $\mathcal{S}_0(y)$ denote the component of $S^2 - \{\gamma\}$ containing the two critical points, and let $\mathcal{S}_0(x)$ denote the component of $S^2 - \{\gamma\}$ containing the two critical values. Regard $\mathcal{S}_0(x)$ as a subset of one copy of the sphere $\mathcal{S}_x = S^2 \times \{x\}$, and $\mathcal{S}_0(y)$ as a subset of a different copy of the sphere $\mathcal{S}_y = S^2 \times \{y\}$. Let $\mathcal{S} = \mathcal{S}_x \sqcup \mathcal{S}_y = S^2 \times \{x, y\}$. Let $\mathcal{S}_1(x) = \mathcal{S}_0(x) \subset \mathcal{S}_x$ and let $\mathcal{S}_1(y) \subset \mathcal{S}_y$ be the unique component of $S^2 - F^{-1}(\gamma)$ contained in $\mathcal{S}_0(y)$. The original map F determines branched covering maps $\mathcal{S}_1(x) \to \mathcal{S}_0(y)$ and $\mathcal{S}_1(y) \to \mathcal{S}_0(x)$.

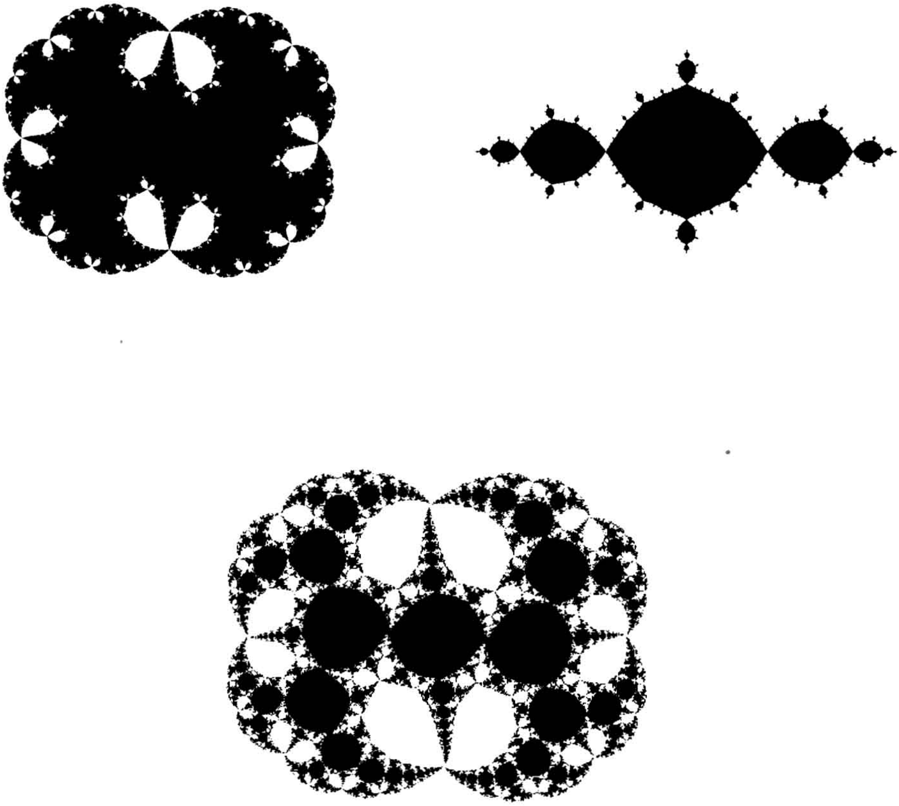

Fig. 1.1. A realizable mating. The filled-in Julia set of $f(z) = z^2 - 1$ is shown at top right in black. The complement of the filled-in Julia set of $g(z)$ is shown in black at top left in a chart near infinity. The Julia set of the mating of f and g is the boundary between the black and white region in the figure at the bottom.

To complete the decomposition, we must extend over the unshaded regions–the complement of $\mathcal{S}_1(x), \mathcal{S}_1(y)$. Note that the boundary components of $\mathcal{S}_1(y), \mathcal{S}_1(x)$ map by degree one onto their images. We must make a choice of such an extension. To keep things as simple as possible, we extend by a homeomorphism. The result is a continuous branched covering map

$$\mathcal{F} : \mathcal{S} \to \mathcal{S}$$

which interchanges the two spheres $\mathcal{S}_x, \mathcal{S}_y$. The "postcritical set", defined in the obvious way, still consists of four points: two period 2 critical points in the sphere $\mathcal{S}_y$, and two period 2 critical values in the sphere $\mathcal{S}_x$.

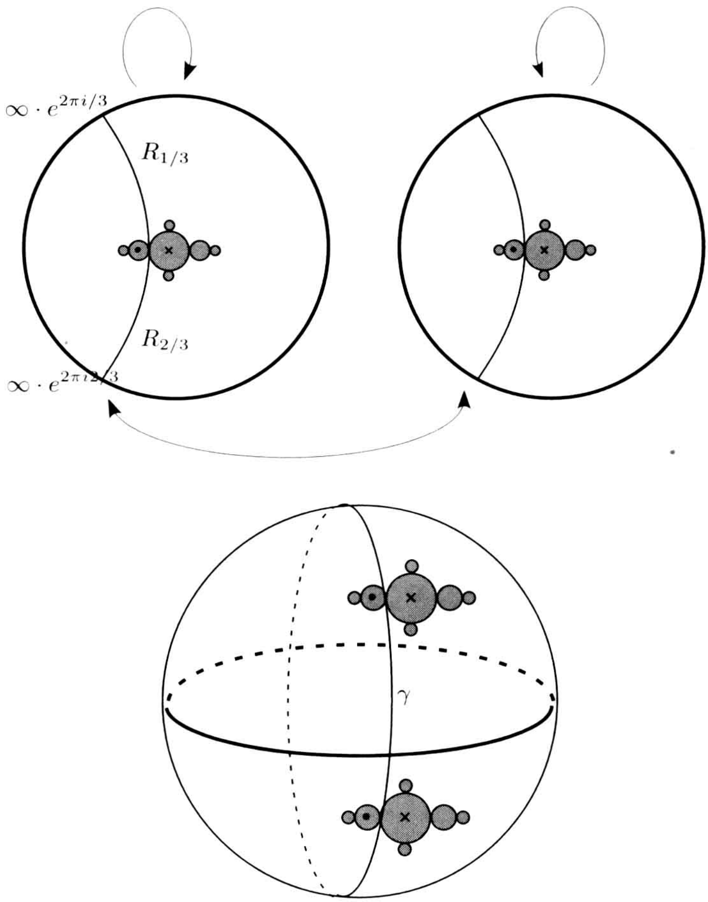

Fig. 1.2. An obstructed mating. Postcritical points are indicated by solid dots and critical points by crosses. The two overlapping crosses and dots correspond to the two period 2 critical points.

Identify $S^2 \times \{x, y\}$ with $\widehat{\mathbb{C}} \times \{x, y\}$ via a homeomorphism so that the postcritical set of $\mathcal{F}$ is $\{0, \infty\} \times \{x, y\}$. With a suitable generalization of the notion of combinatorial equivalence to maps defined on unions of spheres (see §4.2), $\mathcal{F}$ is combinatorially equivalent to the map which sends $(z, y) \to (z^2, x)$ and $(z, x) \to (z, y)$.

1.3.3 An obstructed expanding Thurston map

Here is a general construction. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ be a matrix with integral coefficients. The linear map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by A preserves the lattice $\mathbb{Z}^2$ and thus descends to an endomorphism $T_A : T^2 \to T^2$ of the torus

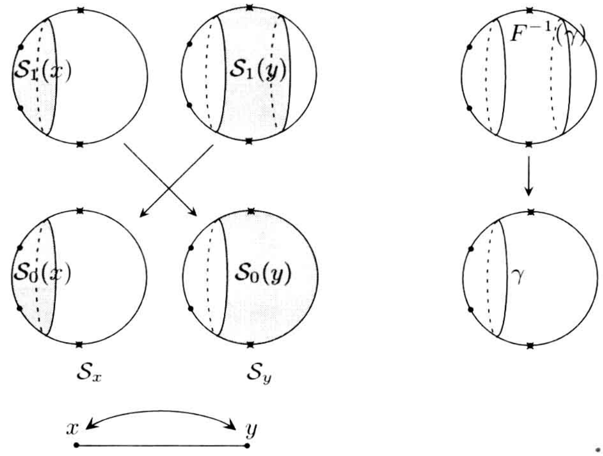

Fig. 1.3. Decomposition of the obstructed mating by cutting along the obstruction. Postcritical points are indicated by solid dots and critical points by crosses. The two overlapping crosses and dots correspond to the two period 2 critical points.

$T^2 = \mathbb{R}^2/\mathbb{Z}^2$. This endomorphism commutes with the involution $\iota : (x, y) \to (-x, -y)$. The quotient space $T^2/(v \sim \iota(v))$ is topologically a sphere S^2 and so T_A descends to a map $F_A : S^2 \to S^2$. The set of critical values of F_A is the image on the sphere of the set of points of order at most two on the torus. Since the endomorphism on the torus must preserve this set of four points, F_A is postcritically finite.

If e.g. $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ then F_A is expanding with respect to the orbifold metric inherited from the Euclidean metric on the torus. Let γ be the curve which is the image of the line $x = 1/4$. Then $\Gamma = \{\gamma\}$ is a multicurve whose Thurston matrix is $(1/2 + 1/2 + 1/2) = (3/2)$ and is therefore an obstruction; see Figure 1.4 where the metric sphere is represented as a "rectangular pillowcase" i.e. the union of two rectangles along their common boundary.

Using a similar decomposition process as in the previous example, we may produce a map $\mathcal{F} : S^2 \times \{x, y\} \to S^2 \times \{x, y\}$, this time sending each component to itself by a degree two branched covering.

Note that since the components of $F^{-1}(\gamma)$ map by degree two, the extension over the complements of $\mathcal{S}_1(x), \mathcal{S}_1(y)$ is now more complicated. Again, to keep things as simple as possible, we extend so that these complementary components, which are disks, map onto their images (again disks) by a quadratic branched covering which is ramified at a single point (say at z_x, z_y) which we arrange to be fixed points of $\mathcal{F}$.

It turns out that the resulting map $\mathcal{F}$ is combinatorially equivalent to the map of $\widehat{\mathbb{C}} \times \{x, y\}$ to itself given by $(z, x) \mapsto (z^2 - 2, x)$ and $(z, y) \mapsto (z^2 - 2, y)$ (the points z_x, z_y are identified with the point $\infty \in \widehat{\mathbb{C}}$).

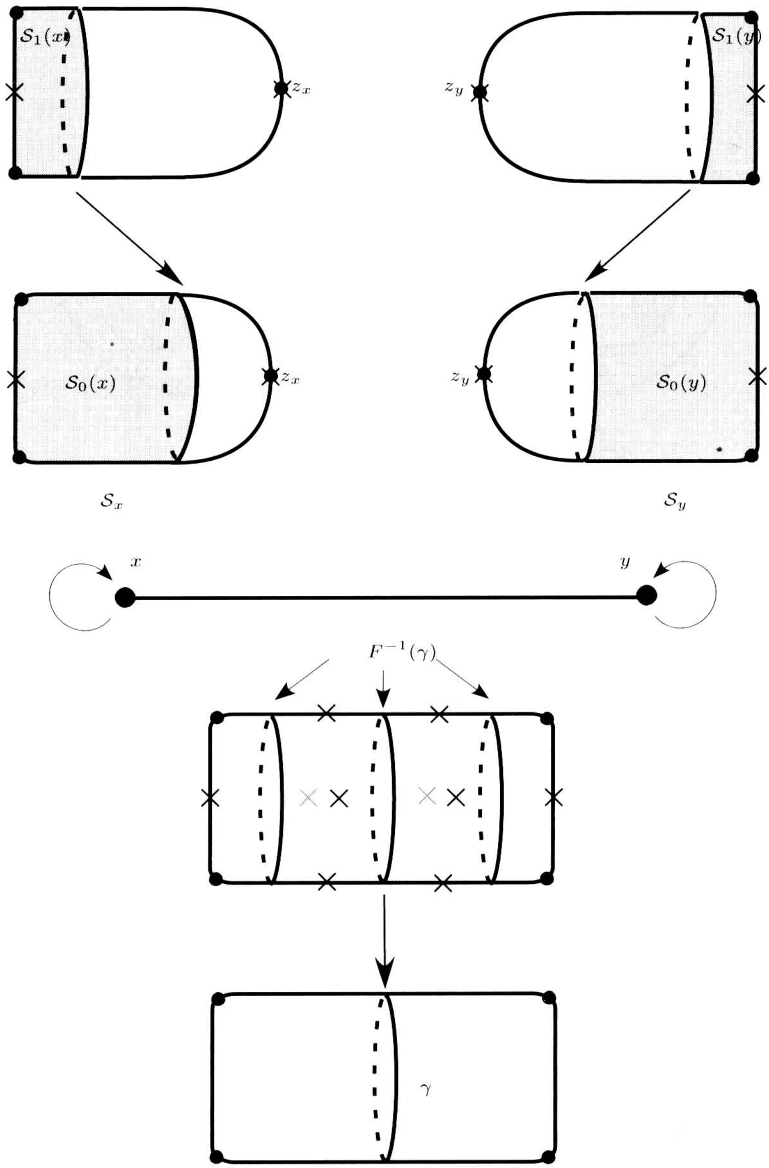

Fig. 1.4. An obstructed expanding map.