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Edited by A. Dold and B. Eckmann

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Morisuke Hasumi

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PREFACE

The purpose of these notes is to give an account of Hardy classes on infinitely connected open Riemann surfaces and some related topics. Already in this Lecture Notes series we have a beautiful monograph "Hardy Classes on Riemann Surfaces" by Maurice Heins, which appeared in print in 1969. It is therefore natural that our stress should now be placed on some new advances we have seen during subsequent years.

As generally recognized, Hardy classes made their debut in the literature in 1915, when G. H. Hardy discussed the mean growth of functions analytic on the unit disk in his paper [15]. The theory of these very useful classes of functions was laid its foundation in the work of Hardy himself, J. E. Littlewood, F. and M. Riesz, G. Szegö among others. And we now have a large and still growing amount of literature in this area. Speaking roughly, Hardy classes have been studied most intensively in the case of the unit disk from the cradle for its importance as well as simplicity. The case of finitely connected surfaces, planar or not, has drawn much attention and enjoyed considerable progress in recent years.

Opposed to this, our knowledge seems to be relatively small in the case of infinitely connected surfaces. The classical theory of Hardy classes deals mostly with the unit disk and does not have much direct bearing on our present problem. From 1950's downwards, functional-analytic methods have found their successful applications in the field of complex function theory including Hardy classes and the abstract Hardy class theory thus created has grown to form the core of the newly-born theory of function algebras, as evidenced from Gamelin's book [10] for example. Nevertheless, the case of infinitely connected surfaces, as I understand it, still lies beyond the reach of the new theory and needs an independent study as in the case of polydisks and balls. The structure of general Riemann surfaces is not yet very well known and so we should begin with this basic question: "For which class of infinitely connected open Riemann surfaces can one get a fruitful extension of the classical theory of Hardy classes?" In the present notes we will try to give an answer, partial at least, to this question.

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Our idea for attacking the problem is very simple and says that any nice surface should carry an ample family of holomorphic functions. But what do we mean by this? The candidate we wish to put forward here as most promising is the class of Riemann surfaces of Parreau-Widom type (abbreviated to "PWS" in the following). The definition will be given in Chapter V. This class was first introduced by M. Parreau [52] in 1958 and also, perhaps independently, by H. Widom [70] in 1971 from very different motives. They used different definitions, which later turned out to be essentially the same. The main reason why we are interested in this class of surfaces is expressed in the following fundamental result of Widom:

(A) An open Riemann surface R is of Parreau-Widom type if and only if every unitary flat complex line bundle over R has nonconstant bounded holomorphic sections.

Moreover, surfaces of this kind inherit many other nice properties of the unit disk or finitely connected surfaces. We list most relevant results in the following, where R denotes a PWS.

(B) Every positive harmonic function on R has a limit along almost every Green line issuing from any fixed origin in R.

(C) The Dirichlet problem on the space of Green lines on R for any bounded measurable boundary function has a unique solution, which converges to the given boundary function along almost every Green line.

(D) Almost every Green line ℓ on R issuing from any fixed origin $\mathcal{O}$ converges to a point, say b_ℓ, in the Martin boundary Δ of R and the correspondence $\ell \to b_\ell$ is measure-preserving with respect to the Green measure on the one hand, and the harmonic measure on Δ for the point $\mathcal{O}$ on the other. Furthermore, the usual solution of the Dirichlet problem for R with any bounded measurable boundary function on Δ converges to the data along almost every Green line.

The statement (D) means in particular that the Brelot-Choquet problem (see Brelot [5]) has a completely affirmative answer for any PWS. It seems indeed that PWS's form the first general class of surfaces of infinite genus for which this problem has a positive solution. Moreover, the statement (D) can be refined so as to have the following:

(E) Stolz regions with vertex at almost every point in Δ can be defined and the boundary behavior of analytic maps from R can be analysed in detail as in the case of the unit disk.

It is possible to generalize the Cauchy-Read theorem to PWS's. The theorem consists of two statements, whose refinements are called, respectively, the direct Cauchy theorem--(DCT) for short--and the inverse Cauchy theorem. Concerning these, we first have:

(F) The inverse Cauchy theorem holds for any PWS.

The converse of this statement is also true. Namely, we have:

(G) If R is a hyperbolic Riemann surface and if the set $H^\infty(R)$ of bounded holomorphic functions on R separates the points of R, then the inverse Cauchy theorem holds if and only if R is a PWS.

Thus the inverse Cauchy theorem almost characterizes surfaces of Parreau-Widom type. On the other hand, the direct Cauchy theorem--an utmost refinement of the Cauchy integral formula--is not always valid. As we shall see, there exist PWS's of infinite genus for which the direct Cauchy theorem holds, while there exist planar PWS's for which the direct Cauchy theorem fails.

The corona problem for PWS's can be studied with some interesting results. First we have:

(H) Every PWS R can be embedded homeomorphically as an open subset in the maximal ideal space of the Banach algebra $H^\infty(R)$.

This, however, is all one can say about an arbitrary PWS. The corona problem for general PWS's has a negative answer, contrary to the expectation one might have. A curious fact in this connection is that a PWS can satisfy the corona theorem without satisfying the direct Cauchy theorem. These two things look like independent. Finally, we state another fact:

(I) If a PWS R is regular in the sense of potential theory, then $H^\infty(R)$ is dense in the space of all holomorphic functions on R in the topology of uniform convergence on compact subsets of R.

We have been talking about relevant properties of PWS's to be discussed in these notes. Apart from PWS's, this book contains a detailed account of the classification problem of plane regions in terms of Hardy classes. As it turns out, almost all that happen in the category of Riemann surfaces can happen in the category of plane regions. Intuitively, PWS's are surfaces which are good in some sense or other. So our last result shows that plane regions can be as ill-behaved as one can imagine.

In writing these notes I have been led by the feeling that the surfaces of Parreau-Widom type probably form the widest family of well-behaved surfaces as far as the theory of Hardy classes is concerned. It is hoped that our description in the following pages will justify this feeling somehow or other. One thing I wish to note is that our study is not a mere adaptation of the existing knowledge of Riemann surfaces. It also aims at finding some new facts which are neither too general nor too special. The present notes are not complete in any sense but just reflect the author's personal interest in the field.

At all event I hope that our effort would help not only extend the theory of Hardy classes but also increase our knowledge of Riemann surfaces in general.

The prerequisites for reading these notes are the fundamentals of advanced complex function theory and some knowledge of functional analysis. As for the function theory, we assume that the reader has some acquaintance with the facts to be found in Chapters I, II, III, V of Ahlfors and Sario [AS] and also in the first four chapters of the book [CC] by Constantinescu and Cornea. As for the functional analysis, Chapter 1 of Hoffman [34] may be useful, if not sufficient.

We now comment on the contents of the present notes. The function-theoretic prerequisites are sketched in Chapter I without proof. In order to deal reasonably with Hardy classes on multiply-connected open surfaces, we rely on two concepts: multiplicative analytic functions and Martin compactification. These are explained in Chapters II and III, respectively. Chapter IV contains preliminary observations on Hardy classes, where the boundary behavior is our principal concern. The main body of this book begins in Chapter V. There, the definition of surfaces of Parreau-Widom type is given after Widom. After showing, by means of regularization, that this definition is essentially equivalent to Parreau's, we present a detailed proof of Widom's fundamental theorem mentioned in (A) above. In Chapter VI we discuss the Dirichlet problem on the space of Green lines (see (B), (C)) and then solve the Brelot-Choquet problem for PWS's (see (D)). Two types of Cauchy theorems--direct and inverse--on PWS's form the main theme of Chapter VII. There, the statement (F) is established by using Green lines and, as an application, it is shown that H^{∞} is a maximal weak-star closed subalgebra of L^{∞} on the Martin boundary. Next, the direct Cauchy theorem (DCT) is precisely stated. We prove a weaker version of (DCT), which is valid for any hyperbolic Riemann surface, together with some application. On the other hand, (DCT) itself fails sometimes. But this cannot be seen until we know something about invariant subspaces, which are studied in Chapter VIII. In Chapter VIII we classify closed H^{∞}-submodules of L^p--(shift-)invariant subspaces of L^p--on the Martin boundary of a PWS. Corresponding to the known results for the case of the unit disk, we consider two principal types of invariant subspaces, which are called doubly invariant and simply invariant, respectively. As for doubly invariant subspaces the situation is rather simple for any PWS. But the so-called Beurling type theorem for simply invariant subspaces is not always valid for PWS's. It is proved in fact that the

Beurling type theorem is valid if and only if (DCT) is. In this connection, examples in Chapter X may be interesting. We give there three types of construction: the first defines PWS's of infinite genus (of Myrberg type) for which (DCT) holds; the second yields a family of planar PWS's for which (DCT) fails but the corona theorem holds; and the third gives PWS's for which the corona theorem is false. In the same chapter we also prove the statements (H) and (I). In Chapter IX we first prove the statement (G), which characterizes PWS's among hyperbolic Riemann surfaces, and then collect a couple of conditions on PWS's equivalent to (DCT). Finally in Chapter XI we solve Heins' problem concerning classification of plane regions by using Hardy classes. The statement (E) will not be proved but just sketched in Chapter VI. There are three appendices and a list of references, which is by no means exhaustive.

My interest in the subject treated here was first aroused while I was visiting the University of California at Berkeley in 1962-64. For this I am indebted to Professors E. Bishop, H. Helson and J. L. Kelley. It was C. Neville's thesis [45], that led me in 1972 to a serious study of Hardy classes on infinitely connected Riemann surfaces. I would thus like to thank Professor L. A. Rubel, who showed me the thesis right after its completion. I also owe thanks to Professor L. Carleson, who invited me to the Mittag-Leffler Institute during its Beurling Year in 1976-77, when I was able to deepen my knowledge of Hardy classes. The primitive version of these notes was written then. New discoveries have made the notes expand subsequently. Most of the main chapters were rewritten for a series of lectures I gave on the present topics at Tokyo Metropolitan University during the week of July 5, 1982. Particular thanks are due to Professors M. Sakai and S. Yamashita, who organized the lectures and made some valuable suggestions. I have much benefited by very helpful remarks from Professors Z. Kuramochi and H. Widom and Dr. M. Hayashi, to whom I wish to express my appreciation.

I would like to dedicate this book to Professor Zirô Takeda, who is my first teacher on the research level and whose inspiration and encouragement have remained with me as fresh as ever.

Mito, Ibaraki
July, 1983

Morisuke Hasumi

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CHAPTER I. THEORY OF RIEMANN SURFACES: A QUICK REVIEW

Some basic results in the theory of Riemann surfaces are collected here for our later reference. They are stated without proof but most of them can be found together with complete proofs either in Ahlfors and Sario, Riemann Surfaces or in Constantinescu and Cornea, Ideale Ränder Riemannscher Flächen, referred to as [AS] or [CC] below.

§1. TOPOLOGY OF RIEMANN SURFACES

We refer to [AS] for most basic definitions concerning Riemann surfaces, which will not be given here, e.g. conformal structure, local variable, parametric disk, intersection number, etc. In what follows, R denotes a Riemann surface. Unless otherwise stated, all Riemann surfaces are assumed to be connected.

1. Exhaustion

1A. Every connected open set in R is called a (sub-)region (or domain) in R. Every region in R is supposed to have the conformal structure induced from that of R. A region D in R is called a regular region in R if it is relatively compact, the boundary ∂D of D in R consists of a finite number of nonintersecting analytic curves, and $R \setminus D$ has no compact components.

Theorem. If R is an open Riemann surface, then there exists an increasing sequence $\{R_n\}_{n=1}^{\infty}$ of regular regions in R such that $Cl(R_n)$ is included in R_{n+1} for each $n = 1, 2, \ldots$ and $R = \cup_{n=1}^{\infty} R_n$. ([AS], Ch. II, 12D)

Any sequence of regular regions R_n in R having this property is called a regular exhaustion of R.

1B. Existence of a regular exhaustion shows that every Riemann surface R admits a locally finite covering consisting of parametric disks and hence that R can be regarded as a polyhedron, i.e. a triangulated surface, which we denote by $K = K(R)$ ([AS], Ch. I, 46A).

Theorem. $K = K(R)$ is an orientable polyhedron. It is a finite polyhedron if and only if R is a compact or compact bordered surface.

1C. Let K be an orientable polyhedron. A finite subcomplex P of K is called a canonical subcomplex if (i) P is a polyhedron and (ii) every component of $K \setminus P$ is infinite and has a single contour. An increasing sequence $\{P_n\}_{n=1}^{\infty}$ of canonical subcomplexes of K is called a canonical exhaustion of K if K is the union of P_n and if every border simplex of P_{n+1} does not belong to P_n.

Theorem. Let R be an open Riemann surface. Then the polyhedron $K = K(R)$ has a subdivision which permits a canonical exhaustion. ([AS], Ch. I, 29A)

By a canonical (sub-)region in R we mean a regular region D such that every component of $R \setminus D$ has a single contour.

Corollary. Every open Riemann surface R admits a regular exhaustion $\{R_n\}_{n=1}^{\infty}$ consisting of canonical subregions.

2. The Homology Groups

2A. Let R be a Riemann surface and $H_1(R)$ the 1-dimensional singular homology group of R ([AS], Ch. I, 33B). By 1C the surface R can be regarded as an orientable polyhedron, which we denote by $K = K(R)$. Let $H_1(K)$ be the 1-dimensional homology group of this polyhedron K ([AS], Ch. I, 23D). Every oriented 1-simplex in K is regarded as a singular 1-simplex on the surface R. This gives rise to an isomorphism of $H_1(K)$ onto $H_1(R)$, which is called the canonical isomorphism ([AS], Ch. I, 34A).

2B. Thus the properties of the group $H_1(R)$ can be obtained by looking into $H_1(K)$. A finite or infinite sequence of cycles in K, labeled in pairs by a_i, b_i, is called a canonical sequence if $a_i \times a_j = b_i \times b_j = 0$, $a_i \times b_i = 1$ and $a_i \times b_j = 0$ for $i \neq j$, denoting by $a \times b$ the intersection number of 1-chains a and b ([AS], Ch. I, 31A).

If R is a compact surface, then K is a finite orientable polyhedron and there exists a canonical sequence a_i, b_i, $i = 1,\dots, g$, which forms a basis for $H_1(K)$. Hence, $\dim H_1(R) = \dim H_1(K) = 2g$.

The number g is called the genus of the surface R.

If $\overline{R}$ is a compact bordered surface with q contours $c_0, \ldots, c_{q-1}$, $q \geqq 1$, then there exists in K a canonical sequence a_i, b_i, $i = 1, \ldots, g$, which, together with all but one contours, forms a basis of $H_1(K)$. Hence, $\dim H_1(\overline{R}) = \dim H_1(K) = 2g + q - 1$ ([AS], Ch. I, 31D). The number g is again the genus of $\overline{R}$.

2C. Finally let R be an open surface, so that K is an orientable open polyhedron. Let D be a regular subregion of R. Since the closure $Cl(D)$ of D is regarded as a compact bordered surface, it defines a finite polyhedron $K(Cl(D))$. By applying a suitable subdivision to K if necessary we may assume that $K(Cl(D))$ is a subcomplex of K. So we have a natural homomorphism of $H_1(K(Cl(D)))$ into $H_1(K)$. Since no components of $R \setminus D$ are compact, we see that the homomorphism is in fact an isomorphism and therefore that $H_1(K(Cl(D)))$ is identified with a subgroup of $H_1(K)$. Turning to the region D itself, we have

Theorem. Let D be a regular subregion of R. Then the group $H_1(D)$ is regarded as a subgroup of $H_1(R)$ by identifying every 1-chain in D with one in R.

In case D is a canonical subregion of R, it is seen moreover that $H_1(D)$ is a direct summand of the free abelian group $H_1(R)$. See [AS], Ch. I, §§31-32 for a detailed discussion.

3. The Fundamental Group

3A. Let $\mathcal{O}$ be a point in R, which is held fixed. The fundamental group $F_{\mathcal{O}}(R)$, referred to the "origin" $\mathcal{O}$, is defined to be the multiplicative group of homotopy classes of closed curves issuing from $\mathcal{O}$ ([AS], Ch. I, 9D). Every closed curve γ from $\mathcal{O}$ can be considered as a singular 1-simplex. We thus get a natural homomorphism of $F_{\mathcal{O}}(R)$ onto $H_1(R)$, which takes the homotopy class of γ to its homology class.

Theorem. Under the natural homomorphism the homology group $H_1(R)$ is isomorphic with the quotient group $F_{\mathcal{O}}(R)/[F_{\mathcal{O}}(R)]$ of $F_{\mathcal{O}}(R)$ modulo the commutator subgroup $[F_{\mathcal{O}}(R)]$ of $F_{\mathcal{O}}(R)$. ([AS], Ch. I, 33D)

3B. By a character of an abstract group G we mean any homomorphism of G into the circle group (= the multiplicative group of complex numbers of modulus one). The set of all characters of G forms a group with respect to the pointwise multiplication of characters. The group thus obtained is called the character group of G and is denoted by G^*. The preceding theorem shows that $H_1(R)^*$ is isomorphic with $\mathcal{F}_0(R)^*$. Combined with the last remark in 2C we have

Theorem. Let D be a canonical subregion of an open Riemann surface R such that $0 \in D$. Then $\mathcal{F}_0(D)$ is a subgroup of $\mathcal{F}_0(R)$ and every character of $\mathcal{F}_0(D)$ is the restriction of some character of $\mathcal{F}_0(R)$.

§2. CLASSICAL POTENTIAL THEORY

4. Superharmonic Functions

4A. Let D be a region in a Riemann surface R. An extended real-valued function s on D is called superharmonic if (i) $-\infty < s(z) \leqq +\infty$ on D and $s \not\equiv +\infty$; (ii) s is lower semicontinuous; and (iii) for every a in D there exists a parametric disk V with center a such that $\mathrm{Cl}(V) \subseteqq D$ and

$$s(a) \geqq \frac{1}{2\pi}\int_0^{2\pi} s(re^{it})dt$$

for $0 < r \leqq 1$, where V is identified with the open unit disk. A function s on D is subharmonic if $-s$ is superharmonic. A function on D is harmonic if it is both superharmonic and subharmonic.

4B. A collection $\mathcal{S}$ of superharmonic functions on D is called a Perron family if (i) for every pair of elements s_1, s_2 in $\mathcal{S}$ there exists an s_3 in $\mathcal{S}$ with $\min\{s_1(z), s_2(z)\} \geqq s_3(z)$ for all z in D; (ii) for every s in $\mathcal{S}$ and every parametric disk V with $\mathrm{Cl}(V) \subseteqq D$, there exists an s' in $\mathcal{S}$ such that

$$s'(z) \leqq \frac{1}{2\pi}\int_0^{2\pi} s(e^{it})\mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) dt$$

for all z in V; and (iii) $\mathcal{S}$ has a subharmonic minorant, i.e. there exists a subharmonic function s'' on D such that $s \geqq s''$ for all s in $\mathcal{S}$. A collection $\mathcal{S}$ of subharmonic functions on D is a Perron family if $-\mathcal{S} = \{-s\colon s \in \mathcal{S}\}$ is a Perron family of superharmonic functions. Perron families are very useful in constructing harmonic functions. The reason is given by the following

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