Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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K. Kawakubo (Ed.)

Transformation Groups

Proceedings, Osaka 1987

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PREFACE

The International Conference on Transformation Groups was held at Osaka University from December 16 to December 21, 1987. The conference was supported by the expenditure of a government enterprise, the Shoda Fund for International Exchange and Grant-in-Aid for Co-operative Research, The Ministry of Education, Science and Culture.

There were a total of 136 participants from the following countries: Denmark, Finland, Japan, Korea, Libya, Poland, Republic of China, United Kingdom, United States of America, West Germany.

The aim of the conference was to reflect recent advances in the theory of transformation groups and to stimulate discussions for new directions and for future research.

Titles of all lectures given at the conference are listed below.

These Proceedings contain accounts of the lectures presented at the conference as well as articles by those who were invited but could not attend. All papers have been refereed and I take this opportunity to thank the authors and the many referees.

I am extremely grateful to all the speakers and participants who made the conference successful. Also, I would like to express my gratitude to all those people who helped me in preparing the conference. In particular, I would like to thank Professors M. Nakaoka, H. Nagao and S. Murakami for their kind administrative support. Moreover, special thanks are due to my colleagues M. Ochiai, T. Yoshida, K. Yamato, Y. Kamishima, M. Morimoto, M. Sakuma, J. Murakami, T. Kobayashi, I. Nagasaki, Y. Ochi and F. Ushitaki who helped me very much in organizing the conference. It is also my great pleasure to thank Y. Oohori and Y. Nakamura for their invaluable secretarial assistance and excellent typing.

Finally I wish to thank Springer-Verlag for publishing this volume and I hope that this volume will contribute to the further progress of the theory of transformation groups and to the broadening of its scope.

Katsuo Kawakubo

Osaka, January 1989

CHRONOLOGICAL LIST OF TALKS

W. C. Hsiang (Princeton): Transformation groups: past and future

F. Raymond (Ann Arbor): Seifert manifolds modelled on principal bundles

M. Morimoto (Okayama): Bak group and equivariant surgery

M. Nagata (Kyoto): The equivariant homotopy type of the classifying space of normal maps

M. Masuda (Osaka): On equivariant inertia groups

A. Hattori (Tokyo): Actions on symplectic manifolds

S. Jackowski (Warszawa): Homotopy colimits for diagrams of homogeneous spaces

R. Oliver (Aarhus): Finiteness obstructions for homologically nilpotent complexes

I. Madsen (Aarhus): Homotopy theory of equivariant automorphism groups

C. m. Wu (Taipei): The ring structure of $U_*(Z_p)$

A. Yamakawa (Tokyo): Fixed-point free $SU(n)$-actions on acyclic manifolds

T. Asoh (Sendai): Smooth $SL(2,C)$ actions on the 3-sphere

F. Uchida (Yamagata): How to construct analytic actions of non-compact Lie groups on a sphere

D. Y. Suh (Taejon): Smith equivalence of representations

S. Morita (Tokyo): Casson's invariant for homology 3-spheres as a secondary invariant of characteristic classes of surface bundles

A. Assadi (Wisconsin): Some algebraic invariants of finite transformation groups

Y. Kamishima (Sapporo): Conformally flat three-manifolds with compact automorphism groups

T. Yoshida (Okayama): On ideal points of deformation curves of hyperbolic 3-manifolds with one cusp

F. Ushitaki (Osaka):	On extensions of non-linear actions on spheres
M. Kato (Fukuoka):	On abelian complex reflection groups
K. Pawałowski (Poznań):	Fixed point sets of smooth group actions on disks and Euclidean spaces
S. Illman (Helsinki):	Subanalytic sets and equivariant triangulations in the theory of compact Lie transformation groups
K. H. Dovermann (Hawaii):	Isovariant surgery
M. Sakuma (Osaka):	Realization of the symmetry groups of links
E. Ossa (Wuppertal):	Connective K-theory of some classifying spaces
T. Inoue (Osaka):	The equivariant Whitehead torsions of equivariant homotopy equivalences between the unit spheres of representations of finite cyclic groups
I. Nagasaki (Osaka):	Semi-linear actions on spheres
K. B. Lee (Oklahoma):	The Nielsen numbers of homotopically periodic maps of infranil-manifolds
S. H. Weintraub (Bayreuth):	Semi-free group actions on 4-manifolds which are non-trivial on homology
C. B. Thomas (Cambridge):	Free actions by finite p-groups on products of spheres and Yagita's invariant $po(G)$
M. Nakaoka (Osaka):	Borsuk-Ulam type theorems and characteristic polynomials
K. Kawakubo (Osaka):	G-s-cobordism theorems do not hold in general for many compact Lie groups G
K. Komiya (Yamaguchi):	Congruences for the Burnside ring
T. Matumoto (Hiroshima):	Three Riemannian metrics on the moduli space of 1-instantons over S^4 and CP^2
T. Petrie (New Brunswick):	Algebraic transformation groups

TABLE OF CONTENTS

A Personal Perspective of Differentiable Transformation Groups

(Dedicated to Prof. D. Montgomery)

WU-CHUNG HSIANG[1]

Differentiable Transformation Group has been an active subject in topology for a long time and I have been one of the participants since the early 60's. Let me take this opportunity to present some of my personal views. It is impossible to compile a good list of references. There are too many! Since this note is only a personal perspective, I shall skip all the precise references.

I. Early Stage.

A. Hilbert's Fifth Problem. *Let G be a locally compact group effectively acting on a topological manifold (C^0). Hilbert asked whether G is a Lie group.* Since the early 30's, many people had devoted great efforts to solve this problem until Gleason, Montgomery–Zippin proved that G is a Lie group if it also acts transitively on a topological manifold. Many very important developments in various branches of mathematics were created because of this problem. As I see it, differentiable transformation group has its root in Hilbert's Fifth Problem.

B. P. A. Smith Theory. P. A. Smith proved the following theorem in the 40's.

Let $G = Z/p$ (p is a prime) act on a finite complex X such that

$$H^*(X; Z/p) \cong H^*(S^n; Z/p).$$

Then,

$$H^*(X^G; Z/p) \cong H^*(S^l; Z/p)$$

for some l.

This theorem sets the prototype and the direction of future research. If we study an action of G on a space X, we should find a model action of G on a 'good space' Y such that we should compare the topological behaviors of these two actions, the similarities as well as the distinctions. The first level should be the comparison of the cohomology of the fixed point sets of various subgroups. Many problems were posed as the generalizations

[1] Partially supported by an NSF Grant.

of P. A. Smith's theorem. Unfortunately, most of them turned out to be incorrect, e.g. Floyd showed that P. A. Smith's theorem fails to hold for $G = Z/pq$. In any case, most work in the 50's was devoted to study the cohomological structure of the fixed point set.

C. The Principal Orbit Theorem and the Slice Theorem (Montgomery-Samelson-Yang).

Let G be a compact Lie group differentiably acting on a (compact) C^∞ manifold M. It was proved by Montgomery-Samelson-Yang that *there exists a unique absolutely minimal conjugacy class of isotropy subgroups* (P) *such that the set of the orbits of the type* G/P *form an open dense subset of* M.

As we pointed out in the discussion of the P. A. Smith theory, we should have a model of this theorem. Let G act on the Lie algebra of via the adjoint representation. This is the model. In this case, the maximal torus T is P.

Montgomery-Yang also observed the following theorem.

The Slice Theorem. *Let* (G, M) *be given as above and let* $G/H \subset M$ *be an orbit of* G. *Then we may choose a normal tubular neighborhood* $N(G/H)$ *of* G/H *together with a representation of* H *on the normal disc* D^l, $\varphi : H \to O(D^l)$ *such that* $N(G/H)$ *is* G *invariant and it is the total space of the associated bundle*

$$D^l \to G \times_H D^l \to G/H$$

of the prinicpal bundle

$$H \to G \to G/H.$$

These two theorems make the actions of compact Lie groups different from dynamic systems, the actions of $\mathbb{R}$.

D. Borel's Seminar. In 1959, Borel conducted a seminar at The Institute for Advanced Study on transformation groups. This is a systematic study of P. A. Smith theory. One of the important influence of the seminar is that Borel introduced the following construction. *Let* G *be a (compact) group acting on a (finite dimensional) complex* X. We consider

$$X_G = EG \times_G X.$$

In fact, we now define the equivariant cohomology

$$H^*_G(X; \ ) =_{df} H^*(X_G; \ ).$$

Since X_G is the total space of the fibration

$$X \to X_G \to B_G,$$

we may use the spectral sequence to study $H^*_G(X; \ )$. For $F = X^G$, we have the following map of bundles

$$\begin{array}{ccccc} F & \to & F \times BG & \to & BG \\ \downarrow & & \downarrow & & \| \\ X & \to & X_G & \to & BG \end{array}$$

Note that if $G = S^1$, $H^*(BG;\mathbb{Q}) = P[y]$ dim $y = 2$. Borel proved that

$$H^*_{S^1}(X;\mathbb{Q})[y^{-1}] \cong H^*_{S^1}(F;\mathbb{Q})[y^{-1}]$$

and similar results for $G = Z/p$. This gives severe restrictions to the cohomology of the fixed point set. We can generalize this to torus and elementary p-groups by considering various corank 1 subgroups. These facts are the bases of the cohomological studies of the actions of torus and elementary p-groups due to Quillen and Wu-yi Hsiang.

II. Influx of Differential Topology.

In Borel's seminar, it was noted that differentiability was not essential in the previous studies. The influx of differential topology came from several directions.

A. The G-bordism of Conner-Floyd.

Conner-Floyd began their study on G-bordism as they attempted to prove that an odd periodic map on an oriented closed manifold can not have only one fixed point. They used differential topology to study G-actions up to a bordism. Essentially, they were computing the bordism groups of BG.

B. P.A. Smith Theory for Classical Groups.

In the middle 60's, Wu-yi Hsiang and I tried to study differentiable actions of a classical group G on a homotopy sphere Σ^n (or a Euclidean space)

$$\Psi; G \times \Sigma^n \to \Sigma^n.$$

Our main question was *when we can choose a representation*

$$\psi : G \to O(n+1)$$

such that Ψ *'resembles'* ψ. This is the Smith philosophy for 'large' groups. There were earlier examples due to Bredon, but we studied this problem systematically for ten years. We finally understood the picture reasonably well if the principal isotropy subgroup is *not* trivial. Some of the work was done jointly with M. Davis. In our study, differentiability plays an essential rôle.

C. Index Theorem. The G-signature theorem of Atiyah-Singer and the fixed point theorem of Atiyah-Bott have many important applications to topology. In particular, Wall and Petrie applied it to surgery theory which had very important impact on differentiable transformation groups. In fact, there is a PL version of this theory which was used to study the topological conjugacy problem for linear representations of a finite group G. (See J. Shaneson's Warsaw International Congress Report for references.)

D. Explosion. After $G7-G9$, it became clear that we should apply modern differential topology to transformation groups. For a given action

$$\Psi : G \times M \to M,$$

we ought to consider M/G as a stratified space (orbiford) and M as a G-handlebody over M/G.

L. Jones studied the converse of P. A. Smith theory, i.e., building actions of Z/p on discs, etc. R. Oliver studied the non-existence of fixed point for actions of finite groups on a disc, i.e., if we wish to have a model, then we need some restrictions on the group or on the action. M. Davis gave the most satisfactory analysis of the actions of classical groups initiated by Bredon, Wu-yi Hsiang and myself, i.e., large groups may behave better in some circumstances. Finally, Petrie and his school studies G-surgery theory systematically and proved that if we don't put any restriction on G or the action, then the old conjectures were wrong most of the time. For example, after an example due to E. Stein, he showed that there are many differentiable actions on homotopy spheres with one fixed point.

So, this is the qualitative state of differentiable transformation groups.

III. Where Do We Go From Here?

As far as I can see, we shall use differentiable (compact) transformation group theory as a tool. Due to the complications of the examples constructed so far, it is similar to advanced calculus. The theory is very useful but no one can integrate all the improper integrals. We should move on.

A. Algebraic Actions of Classical Groups.

Recently, H. Bass and T. Petrie and their coworkers have started to study algebraic actions on affine spaces. We are still at the infant stage. There is a 'good' étale slice theorem via an analysis on the representation spaces. It seems to me that we understand the object somewhat but we don't understand the morphism. In differential topology, the morphism (i.e., the attaching map) is rather flabby but this is probably not so for algebraic actions. We need some insight.

The simplest semi-simple group is $SL(2,\mathbb{R})$. Inside this group, we have two important subgroups

$$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}.$$

Can we use some techniques from dynamic systems to study the restricted actions of these subgroups (from the given action of $SL(2, \mathbb{R})$)?

B. Elliptic Cohomology and S^1 Actions on Loop Spaces.

Landweber and Stong are making calculations of elliptic cohomology from bordism point of view. So far, it seems to be an ad hoc approach. Segal makes some philosophical suggestions. Due to its relation with S^1 actions on loop spaces, it should be a good playground for people interested in (compact) group actions.

Smooth SL(2,C) actions on the 3-sphere

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1. Introduction

The special linear group $G=SL(2,C)$ contains $K=SU(2)$ as a maximal compact subgroup, and there are two equivariant diffeomorphism classes of smooth K actions on S^3 represented by linear actions.

In this note we study the equivariant homeomorphism classes of smooth G actions on S^3.

In case of transitive actions ; we have

Theorem 1.1. *There are real analytic G actions ϕ_r on S^3 for $r \in R$ (see (3.1)), which are not equivariantly homeomorphic to each other, and any transitive G actions on S^3 is equivariantly diffeomorphic to some ϕ_r.*

In case of non-transitive actions ; the classification of G actions on S^3 can be reduced to that of triads of subsets A and B_j $(j=1,2)$ of $S^1(\subset C)$ satisfying

(A1) $A(\neq \emptyset)$ is a finite union of closed intervals, $A \cap J(A) = \emptyset$ and the components of A alternate with those of $J(A)$, where J is the reflection on S^1 in the real line. (A2) B_j $(j=1,2)$ are open in S^1 and $B_1 \cup B_2 \subset A - \partial A$.

Such triads (A, B_j) and (A', B_j') are called A-equivalent if there is an orientation preserving homeomorphism Φ of S^1 onto itself such that $\Phi J = J \Phi$ and

(1) $\Phi(A) = A'$, $\Phi(B_j) = B_j'$ or (2) $\Phi(A) = J(A')$, $\Phi(B_j) = J(B_{3-j}')$ $(j=1,2)$.

We see the following theorem.

Theorem 1.2. *There is a one-to-one correspondence between the equivariant homeomorphism classes of non-transitive smooth G actions on S^3 and the A-equivalence classes of triads with* (A1-2).

As the corollary to these theorems, we have

Corollary 1.3. (i) *There are infinitely many (non-equivalent) smooth G actions on S^3 which are not equivariantly homeomorphic to any real analytic one.*
(ii) *Any real analytic G action on S^3 has a finite (odd) number of orbits, and non-transitive real analytic ones are determined by the number of their orbit ; among them the unique linear action has five orbits.*

Recently F. Uchida shows that our method is useful to study SO(p,q) actions on S^{p+q-1}.

2. Subalgebras of sl(2,C) and subgroups of SL(2,C)

The Lie algebra of $G=SL(2,C)$ is

$$\mathbf{g} = \mathrm{sl}(2,C) = \{X \in M(2,C)\,;\, \mathrm{Trace}\, X = 0\}$$

with the bracket operation $[X,Y] = XY - YX$. In this section we prepare some results on subalgebras of **g** and subgroups of G.

Choose a R-basis of **g**

$$K_1 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ K_2 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ K_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ H_j = iK_j \quad (j = 1,2,3).$$

Then these satisfy the following relations:

(2.1) $-[K_a, K_b] = K_c = [H_a, H_b],\ -[K_a, H_b] = H_c = [K_b, H_a]$ and $[K_a, H_a] = 0$
for $(a, b, c) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$.

We say that subalgebras **u** and **u'** of **g** are conjugate in G if $\mathrm{Ad}(g)\mathbf{u}' = \mathbf{u}$ for some $g \in G$, where $\mathrm{Ad} : G \to GL(\mathbf{g})$ is the adjoint representation of G.

Consider the following subalgebras

(2.2) $\mathbf{k} = \langle K_1, K_2, K_3\rangle,\ \mathbf{u}_r = \langle K_1, rK_2 - H_3, rK_3 + H_2\rangle\ (r \in R)$ and
$\mathbf{v}_\varepsilon = \langle K_1, H_1, K_2 - \varepsilon H_3, K_3 + \varepsilon H_2\rangle\ (\varepsilon = \pm 1)$,
where $\langle\ \rangle$ denotes a R-vector space spanned by the elements in the angle bracket.

Proposition 2.3. *Let* **u** *be a proper subalgebra of* **g** *with* $\dim \mathbf{u} \geqq 3$. *If* $K_1 \in \mathbf{u}$, *then* **u** *is* $\mathbf{k}, \mathbf{u}_r\ (r \in R)$ *or* $\mathbf{v}_\varepsilon\ (\varepsilon = \pm 1)$. *Here* $\mathbf{u}_r$ *is conjugate to* $\mathbf{k}$, $\mathbf{u}_0$ *and* $\mathbf{u}_1$ *if* $|r|>1$, $|r|<1$ *and* $|r| = 1$, *respectively, and further* $\mathbf{v}_1$ *is conjugate to* $\mathbf{v}_{-1}$.

Proof. If $K_1 \in \mathbf{u}$, then $\mathbf{u} \cap \mathbf{k} = \langle K_1\rangle$ or $\mathbf{k}$. The relation (2.1) implies that $\mathbf{u} = \mathbf{u}_r$ or $\mathbf{v}_\varepsilon$ (resp. $\mathbf{k}$) in case $\mathbf{u} \cap \mathbf{k} = \langle K_1\rangle$ (resp. $\mathbf{u} \cap \mathbf{k} = \mathbf{k}$). *q.e.d.*

Corollary 2.4. *Any proper subalgebra* **u** *of* **g** *with* $\dim \mathbf{u} \geqq 3$ *is conjugate to one of*

$$\mathbf{k} = \langle K_1, K_2, K_3\rangle,\ \mathbf{u}_0 = \langle K_1, H_2, H_3\rangle,\ \mathbf{u}_1 = \langle K_1, K_2 - H_3, K_3 + H_2\rangle$$
$$\mathbf{v}_1 = \langle K_1, H_1, K_2 - H_3, K_3 + H_2\rangle \text{ and } \mathbf{w}_r = \langle rK_1 + H_1, K_2 - H_3, K_3 + H_2\rangle (r \in R),$$

and these subalgebras are not conjugate to each other.

Proof. The relation (2.1) implies that $\mathbf{u}$ is conjugate to $\mathbf{w}_r$ if $\mathbf{u} \cap \mathbf{k} = \{0\}$. Thus the first half holds by the above proposition. The second half follows by using $\det \mathbf{u} = \{\det X \,;\, X \in \mathbf{u}\}$ and the Killing form of $\mathbf{g}$, which are Ad(G)-invariant. *q.e.d.*

Now consider the following connected closed subgroups of G.

(2.5) $$K = SU(2),\ U_r = \{\begin{pmatrix} z & (r-1)w \\ -(r+1)\overline{w} & \overline{z} \end{pmatrix};\, z, w \in C, |z|^2 + (r^2-1)|w|^2 = 1\}$$

$$\text{and } W_r = \{\begin{pmatrix} \exp(ri-1)x & 0 \\ z & \exp(1-ri)x \end{pmatrix};\, x \in R, z \in C\}\ (r \in R).$$

Lemma 2.6. (i) *The subalgebras* $\mathbf{k}$, $\mathbf{u}_r$ *and* $\mathbf{w}_r$ *are the Lie algebras of* K, U_r *and* W_r, *respectively.*

(ii) The coset space G/U_r is homeomorphic to

$$R^3 \text{ if } |r|>1 \text{ and } S^2\times R \text{ if } |r|\leqq 1.$$

Proof. (i) is clear. (ii) $G/K \approx R^3$ holds, because K is a maximal compact subgroup of G. There is a transitive G action on $S^2\times R$ with an isotropy subgroup U_r ($|r|\leqq 1$), and hence $G/U_r \approx S^2\times R$ ($|r|\leqq 1$). *q.e.d.*

Lemma 2.7. $G = KLU_r = KLK$ *(r* $\in$ *R) for* $L = \{\mathrm{diag}\,(x, 1/x) \in M(2,C)\,;\, x>0\}$.

3. Transitive actions

In this section we state an immediate consequence of the previous section, and prove Theorem 1.1.

For each $r \in R$ the analytic $G = SL(2,C)$ action on $S^3 = C^2 - \{0\}/R^+$, defined by

(3.1) $\phi_r(X, [P]) = [\exp(ir\log(\|XP\|/\|P\|))XP]\ \ (X \in G, P \in C^2 - \{0\})$,

is transitive and its isotropy subgroup at $[{}^t(0,1)] \in S^3$ is W_r of (2.5).

Proof of Theorem 1.1. The equivariant homeomorphism classes of transitive G actions on S^3 are classified by the conjugacy classes of connected subgroups U of G with $G/U \approx S^3$. Therefore the theorem follows from Corollary 2.4 and Lemma 2.6. *q.e.d.*

4. Non-transitive actions

To begin with we prepare some results on smooth actions. Let $\phi : G \times M \to M$ be a smooth G action on M, and denote by $\mathbf{g}$ and $\mathbf{X}(M)$ the Lie algebras of G and smooth vector fields on M, respectively.

(4.1)([3 ; Ch.II, Th.II]) The map $\phi^+ : \mathbf{g} \to \mathbf{X}(M)$, given by

$$\phi^+(X)_p h = \lim_{t\to 0} \frac{h(\phi(\exp(-tX), p)) - h(p)}{t} \quad (X \in \mathbf{g})$$

for any smooth function h around $p \in M$, is a Lie algebra homomorphism.

The isotropy subalgebra $\mathbf{g}_p = \{X \in \mathbf{g} \,;\, \phi^+(X)_p = 0\}$ at $p \in M$ is the Lie algebra of the isotropy subgroup G_p of G at p.

In case that $\mathbf{g}$ is simple and ϕ is non-trivial, ϕ^+ is monomorphic, and hence $\mathbf{g}$ is regarded as a subalgebra of $\mathbf{X}(M)$ by identifying $X = \phi^+(X)$.

From now on we shall classify non-transitive smooth $G = SL(2,C)$ actions on S^3, and set

$$G = SL(2,C),\ K = SU(2),\ T = \{\mathrm{diag}(z, z) \in K \,;\, |z| = 1\} (\subset K),$$
$$\mathbf{g} = \mathbf{sl}(2,C),\ \mathbf{k} = \mathbf{su}(2) = \langle K_1, K_2, K_3 \rangle \ \text{ and } \ S^3 = H/R^+$$
$$\text{for } H = \{P \in M(2,C) \,;\, 0 \neq P = P^*\}.$$

The following (4.2) is known (cf.[1 ; Th.1.3]).

(4.2) Any non-transitive (and non-trivial) smooth K action on S^3 is equivariantly diffeomorphic to the linear action ;

$$\psi_0 : K \times S^3 \to S^3,\ \psi_0(X, [P]) = [XPX^*] \ (X \in K, P \in H).$$

The fixed point sets of ψ_0 under T and K are

(4.3) $(C \supset) S^1 = F(T, S^3) \supset F(K, S^3) = \{\pm 1\}$

by the diffeomorphism $S^1 \ni x + iy \to \mathrm{diag}(x+y,\ x-y) \in F(T,\ S^3)$ $(x,\ y \in R)$. Thus the reflection J of S^1 is given by

(4.4) $$J(z) = \psi_0(j, z) \ \text{ for } z \in S^1 \ \text{ and } \ j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in K.$$

To classify non-transitive G actions on S^3, we may assume by (4.2)

(4.5) $\phi : G \times S^3 \to S^3$ is a smooth G-action on S^3 such that its restricted K action coincides with ψ_0, i.e. $\phi | K \times S^3 = \psi_0$.

Lemma 4.6. (i) *The map* $\ell : R \times F(T, S^3) \to F(T, S^3)$,

(4.7) $\ell(t, z) = \phi(\exp(-tH_1), z)$ $(t \in R, z \in F(T, S^3))$ *for* $H_1 \in \mathbf{g}$,