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CONTENTS

INTRODUCTION

Let Ω be an open subset of the complex plane. We are interested in two classes of analytic functions on Ω. The first, denoted by $H^\infty(\Omega)$, is the set of bounded analytic functions on Ω; and the second, which we call $A(\Omega)$, is the set of functions in $H^\infty(\Omega)$ possessing continuous extensions to the Riemann sphere S^2. Because of the maximum principle, it is often more convenient to discuss $H^\infty(\Omega)$ and $A(\Omega)$ in terms of the complementary set $E = S^2 \setminus \Omega$ instead of Ω. Rotating S^2 so that $\infty \in \Omega$, we can assume that E is a compact plane set.

Perhaps the best way to describe the problems considered below is to prove two elementary theorems. Let E be a compact plane set and let $\Omega = S^2 \setminus E$ be its complement.

<u>Painlevé's Theorem</u>: Assume that for every $\varepsilon > 0$, the set E can be covered by discs the sum of whose radii does not exceed ε. Then $H^\infty(\Omega)$ consists only of constants.

<u>Proof</u>: For each $\varepsilon > 0$ we cover E by a collection of discs, the sum of whose radii does not exceed ε, and we let Γ_ε be the boundary of the union D_ε of these discs. If $f \in H^\infty(\Omega)$ and $f(\infty) = 0$, then by Cauchy's theorem

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{f(\zeta)d\zeta}{z - \zeta}, \quad z \notin \overline{D}_\varepsilon .$$

Thus $|f(z)| \leq \dfrac{\varepsilon \cdot \|f\|}{2\pi \, \mathrm{dist}(z,\Gamma_\varepsilon)}$. Sending ε to 0 we get $f(z) = 0$ for all $z \in \Omega$.

<u>Theorem</u>: If E has positive area, then $A(\Omega)$ contains non-constant functions.

<u>Proof</u>: Write

$$F(z) = \iint_E \frac{d\xi d\eta}{\zeta - z} \qquad \zeta = \xi + i\eta .$$

Clearly $F(z)$ is analytic on the complement Ω of E, and, being the convolution of the locally integrable function $-1/\zeta$ and the characteristic function of a bounded set, $F(z)$ is continuous on the complex plane. Since $\lim_{z\to\infty} F(z) = 0$, F is in $A(\Omega)$, and F is not constant since $\lim_{z\to\infty} zF(z) = -\text{Area } E$.

That proves the theorem, but let us examine the function F more closely. At infinity $F(z)$ has the expansion

$$F(z) = -\text{Area}(E)/z + a_2/z^2 + \cdots .$$

Define R by $\pi R^2 = \text{Area}(E)$ and let Δ be the disc $\{\zeta : |\zeta - z| \leq R\}$. Then

$$|F(z)| \leq \iint_E \frac{d\xi d\eta}{|\zeta - z|} = \iint_{E\cap\Delta} \frac{d\xi d\eta}{|\zeta - z|} + \iint_{E\setminus\Delta} \frac{d\xi d\eta}{|\zeta - z|} .$$

Since E and Δ have the same area, so do $E\setminus\Delta$ and $\Delta\setminus E$, while the integrand is larger on $\Delta\setminus E$ than it is on $E\setminus\Delta$. Thus

$$\int_{E\setminus\Delta} \frac{d\xi d\eta}{|\zeta - z|} \leq \int_{\Delta\setminus E} \frac{d\xi d\eta}{|\zeta - z|}$$

and so

$$|F(z)| \leq \iint_{\Delta} \frac{d\xi d\eta}{|\zeta - z|} = 2\pi R .$$

This gives us a function $g(z) = F/2\pi R$ in $H^{\infty}(\Omega)$ such that $\|g\| \leq 1$ and

$$g(z) = b_1/z + b_2/z^2 + \cdots$$

where $|b_1| \geq \frac{R}{2} = \frac{1}{2} \sqrt{\frac{\text{Area}(E)}{\pi}}$. In other words, we have estimated the analytic capacity of E (defined in Chapter I below) in terms of the area of E.

The hypothesis of each sample theorem is measure theoretic (in Painlevé's theorem the measure is one dimensional Hausdorff measure), and each is proved by representing a function as the Cauchy integral of a Borel measure. Our purpose is to survey what can be said concerning two problems:

1°. Representing functions in $H^{\infty}(\Omega)$ as Cauchy-Stieltjes integrals

$$f(z) = \int_{E} \frac{d\mu(\zeta)}{\zeta - z} .$$

2°. Estimating or describing analytic capacity in terms of measures, and applying such estimates to approximation problems.

These notes contain much that is old and a little that is new. Hopefully, they are intelligible to the graduate student who knows elementary real and complex analysis and a little functional analysis, and who is interested in analytic capacity and related fields. As three fine expositions of rational approximation theory [28], [81], [88] are already in print, it seems unnecessary to discuss that theory in any depth. Thus the Melnikov-Vitushkin estimates on line integrals have been ignored, and when we use Vitushkin's approximation techniques in Chapter V we give suitable references but no details.

In some instances a theorem has a person's name attached to it, often simply because that is what the theorem is called. But no doubt some important results have not been ascribed to their originals, and unattributed theorems should not be assumed the author's discovery.

Throughout each chapter there are exercises and problems. Some exercises are very routine, and some problems are old and famous, but the only real distinction is that I think I can do the exercises.

Chapter I is an exposition of the theory of analytic capacity. It begins at the beginning, and thus has some overlap with other sources. Chapter II concerns the Cauchy integral representation. It contains a simple characterization of Cauchy transforms. The relation between bounded analytic functions, Hausdorff measure, and Newtonian potential theory is taken up in Chapter III. In Chapter IV we discuss three examples, and in Chapter V applications are made to approximation theory.

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Certain notation should be mentioned. $\Delta(z,\delta)$ stands for the open disc $\{\zeta : |\zeta - z| < \delta\}$ and $S(z,\delta)$ is the closed square of side

δ and center z. Its sides are parallel to the axes. The symbol μ denotes a finite complex Borel measure; S_μ is its support and $|\mu|$ is its variation measure. Unless otherwise indicated, a.e. refers to area, and $\|f\|$ is the supremum of $|f|$ over its domain. C_0^∞ are the compactly supported infinitely differentiable functions, and $C_0^\infty(D)$ are those with support inside D. A function g is in L^p_{loc} if $|g|^p$ is integrable over every compact set. Finally

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) .$$

CHAPTER I. ANALYTIC CAPACITY

§1. Basic Properties

Let E be a compact plane set and let $\Omega = S^2 \setminus E$. When f is analytic on Ω its derivative at ∞ is computed using the local coordinate $1/z$, so that

$$f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty)) .$$

Expanding f in a Taylor series about ∞,

$$f(z) = a_0 + a_1/z + a_2/z^2 + \cdots$$

we have $f'(\infty) = a_1$. In other words,

$$f'(\infty) = \frac{1}{2\pi i} \int_\Gamma f(\zeta) d\zeta$$

whenever the curve Γ separates E from ∞. Define the <u>analytic capacity</u> and <u>continuous analytic capacity</u> respectively as follows

$$\gamma(E) = \sup\{|f'(\infty)| : f \in H^\infty(\Omega),\ \|f\| \leq 1\}$$

$$\alpha(E) = \sup\{|f'(\infty)| : f \in A(\Omega),\ \|f\| \leq 1\} .$$

<u>Theorem 1.1</u>: $H^\infty(\Omega)$ consists only of the constants if and only if $\gamma(E) = 0$; $A(\Omega)$ consists only of the constants if and only if $\alpha(E) = 0$.

<u>Proof</u>: Clearly, if $\gamma(E) > 0$ then $H^\infty(\Omega)$ contains non-constant functions.

On the other hand, if $H^\infty(\Omega)$ is not trivial there is $f \in H^\infty(\Omega)$ with $f(\infty) = 0$ and $f(z_0) \neq 0$ for some $z_0 \neq \infty$. Then the function $(f(z) - f(z_0))/(z_0 - z)$ is in $H^\infty(\Omega)$ and has derivative $f(z_0)$ at ∞, so that $\gamma(E) > 0$. The same argument shows $A(\Omega)$ is nontrivial if and only if $\alpha(E) > 0$.

If $f \in H^\infty(\Omega)$ and $\|f\| \leq 1$, then

$$g(z) = \frac{f(z) - f(\infty)}{1 - \overline{f(\infty)}f(z)}$$

is in $H^\infty(\Omega)$, $\|g\| \leq 1$, $g(\infty) = 0$ and

$$g'(\infty) = \frac{f'(\infty)}{1 - |f(\infty)|^2} .$$

Thus when computing the extremum $\gamma(E)$, we can restrict our attention to functions vanishing at ∞. Using the standard notation

$$A(E,M) = \{f \in H^\infty(\Omega) : \|f\| \leq M,\ f(\infty) = 0\}$$

we have

$$\gamma(E) = \sup\{|f'(\infty)| : f \in A(E,1)\} .$$

Similarly, letting

$$C(E,M) = A(E,M) \cap A(\Omega) ,$$

we have

$$\alpha(E) = \sup\{|f'(\infty)| : f \in C(E,1)\} .$$

If $f(\infty) = 0$, then $\lim_{z\to\infty} zf(az + b) = af'(\infty)$, and we have the invariance properties

$$\gamma(aE + b) = |a|\gamma(E)$$

$$\alpha(aE + b) = |a|\alpha(E) .$$

It is clear from the definitions that γ and α are monotone: $\gamma(E) \leq \gamma(F)$, $\alpha(E) \leq \alpha(F)$ if $E \subset F$. It is also clear that $\alpha(E) \leq \gamma(E)$. However these two quantities are not commensurate. For, $\gamma(E)$ depends only on the unbounded component of Ω: $\gamma(E) = \gamma(\hat{E})$ where $\hat{E}$ is the union of E and the bounded components of Ω. So if E is the circle $\{z : |z - a| = r\}$, then $\gamma(E) > 0$, but $\alpha(E) = 0$ by Morera's theorem. Another example is obtained by taking E to be an interval on the real axis. Then $\alpha(E) = 0$, again by Morera's theorem, while $\gamma(E) > 0$ because Ω can be mapped conformally onto the unit disc.

When E is connected, so that Ω is simply connected, the class $A(E,1)$ arises in a well known proof of the Riemann mapping theorem [2, p. 222]. Indeed, we have

Theorem 1.2: Assume that E is connected but not a point. Let g be the conformal map of Ω onto the unit disc satisfying $g(\infty) = 0$, $g'(\infty) > 0$. Then $\gamma(E) = g'(\infty)$.

Proof: Since $g \in A(E,1)$, we have $g'(\infty) \leq \gamma(E)$. Let $f \in A(E,1)$. Applying Schwarz's lemma to $F = f \circ g^{-1}$, we have $|F'(0)| \leq 1$. But $F'(0) = f'(\infty)/g'(\infty)$ so that $|f'(\infty)| \leq g'(\infty)$. Therefore $\gamma(E) \leq g'(\infty)$.

Consequently, we see that if E is the disc $\{|z - a| \leq \delta\}$, then $\gamma(E) = \delta$; and the extremal function is $\delta/(z-a)$. And if E is a line segment of length ℓ, then $\gamma(E) = \ell/4$. For this it is enough to take $E = [-2,2]$ and observe that the conformal map $g : S^2 \setminus E \to \Delta(0,1)$ satisfies

$$g^{-1}(w) = w + 1/w \ .$$

We can now estimate analytic capacity in terms of diameters as follows

Corollary 1.3: For any set E

$$\alpha(E) \leq \gamma(E) \leq \text{diam}(E) \ .$$

If E is connected, then

$$\gamma(E) \geq \frac{\text{diam}(E)}{4} \ .$$

Proof: The first assertion follows by monotonicity, because E lies in a disc of radius $\text{diam}(E)$.

To prove the second assertion we can assume that E is not a point.

Let $g(z) = \gamma(E)/z + a_2/z^2 + \cdots$ be the Riemann map of the unbounded component of Ω onto the unit disc. Fix $z_0 \in E$ and write

$$f(w) = \frac{\gamma(E)}{g^{-1}(w) - z_0} \ .$$

Then f is univalent on $|w| < 1$, $f(0) = 0$, and $f'(0) = 1$. By the Koebe-Bieberbach theorem [23, p. 279], the range of f contains $|z| < 1/4$, so that if $z_1 \in E$, we have

$$\frac{\gamma(E)}{|z_1 - z_0|} \geq 1/4 \ .$$

Corollary 1.3 implies that E is totally disconnected when $\gamma(E) = 0$.

The estimate on analytic capacity given in Corollary 1.3 is sharp in the case of a line segment. In the introduction we gave another estimate:

$$\gamma(E) \geq \alpha(E) \geq \frac{1}{2} \sqrt{\frac{\text{Area}(E)}{\pi}} \ .$$

In Chapter III this will be improved by a factor of 2, so that it is sharp in the case of a disc.

When E is not compact define

$$\gamma(E) = \sup\{\gamma(K) : K \text{ compact, } K \subset E\}$$

$$\alpha(E) = \sup\{\alpha(K) : K \text{ compact, } K \subset E\} \ .$$

It is then clear that $\gamma(U) = \alpha(U)$ for all open sets U. A normal family argument shows

$$\gamma(E) = \inf\{\alpha(U) : U \text{ open, } U \supset E\}$$

when E is compact.

The condition $\gamma(E) = 0$ is necessary and sufficient for the set E to be removable for bounded analytic functions.

<u>Theorem 1.4</u>: Let E be a relatively closed subset of an open set U and assume $\gamma(E) = 0$. If $f \in H^\infty(U\setminus E)$, then f has a unique extension in $H^\infty(U)$.

Proof: The uniqueness is trivial because E is nowhere dense. Let $z_0 \in E$. Since by 1.3 E is totally disconnected, there is an analytic simple closed curve Γ in $U \setminus E$ which encloses z_0. Let D be the domain bounded by Γ. Using the Cauchy integral we can write $f = f_1 + f_2$ in a neighborhood of Γ, where $f_1 \in H^\infty(D)$, and $f_2 \in H^\infty(S^2 \setminus (E \cap D))$. Since $\gamma(E \cap D) = 0$, f_2 is constant, and f extends analtyically to D.

The same result is true if $f \in A(U \setminus E)$ and γ is replaced by α. However the above argument only works if E is a compact subset of U. The simplest proof of the full result uses Vitushkin's localization operator ([28] II, 1.7) and would be a digression at this point.

Theorem 1.5: If E is a relatively closed subset of an open set U and $\alpha(E) = 0$, then every $f \in A(U \setminus E)$ has a unique extension in $A(U)$.

The Semi-additivity Problem: Show there exists a constant C such that

$$\gamma(E_1 \cup E_2) \leq C(\gamma(E_1) + \gamma(E_2))$$

for some reasonable class of sets (like the Borel sets). Equivalent formulations of this quite important conjecture are given in [18] and [81]. It is not known whether there is a constant C such that

$$\gamma(E_1 \cup E_2) \leq C\gamma(E_1)$$

for all compact E_2 such that $\gamma(E_2) = 0$. The same problems for the continuous analytic capacity α are also open and from the point of view of rational approximation theory the important question is whether or not

$$\alpha(E_1 \cup E_2) \leq C\alpha(E_1)$$

when E_2 is compact and $\alpha(E_2) = 0$.

<u>Exercise 1.6</u>: Prove that if $\{E_n\}$ is a decreasing sequence of compact sets and $E = \bigcap E_n$, then $\gamma(E) = \lim \gamma(E_n)$. Prove that if $\{F_n\}$ is a sequence of compact sets such that $\gamma(F_n) = 0$, then $\gamma(\bigcup F_n) = 0$. Determine if the above two assertions hold with γ replaced by α.

<u>Exercise 1.7</u>: In §6 it is proved that any subset of $[0,1]$ with positive inner Lebesgue measure has positive analytic capacity. Use this fact and the usual construction of a non-measurable set to exhibit subsets $\{E_n\}$ of $[0,1]$ such that $\gamma(E_n) = 0$ for all n but $\bigcup E_n = [0,1]$. Use the same ideas to construct two sets E_1 and E_2 such that $E_1 \cup E_2 = [0,1]$, but $\gamma(E_1) = \gamma(E_2) = 0$.

§2. Schwarz's Lemma

As is suggested by the proof of Theorem 1.2, there is a close connection between analytic capacity and the Schwarz lemma. Indeed we have the following inequalities which, though elementary, are the reasons that the extremal quantity $\alpha(E)$ is so important in approximation theory.

<u>Theorem 2.1</u>: Let $f \in A(E,1)$. Then for $z_0 \in \Omega$ we have

$$|f(z_0)| \leq \frac{\gamma(E)}{\operatorname{dist}(z_0,E)} .$$