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PREFACE

The German edition of this book, first published in 1966, has been quite popular; we did not, however, consider publishing an English edition because a number of excellent textbooks in this field already exist. In recent years, however, the wish was frequently expressed that, especially, the description of the relationships between optimization and other subfields of mathematics, which is not to be found in this form in other texts, might be made available to a wider readership; so it was with this in mind that, belatedly, a translation was undertaken after all.

Since the appearance of the German edition, the field of optimization has continued to develop at an unabated rate. A completely current presentation would have required a total reworking of the book; unfortunately, this was not possible. For example, we had to ignore the extensive progress which has been made in the development of numerical methods which do not require convexity assumptions to find local maxima and minima of non-linear optimization problems. These methods are also applicable to boundary value, and other, problems. Many new results, both of a numerical and a theoretical nature, which are especially relevant to applications, are to be found in the areas of optimal contol and integer optimization.

Although these and many other new developments had to be ignored, we hope that the book continues to satisfy the goals set forth in the preface to the German edition.

Finally, we want to take this opportunity to express our gratitude, to Peter R. Wadsack for a careful translation, and to Springer Verlag for kind cooperation.

FROM THE PREFACE TO THE GERMAN EDITION

With this book we would like to provide an introduction to a field which has developed into a great new branch of knowledge in the last thirty years. Indeed, it continues to be the object of intensive mathematical research. This rapid development has been possible because there exists a particularly close contact between theory and application.

Optimization problems have appeared in very different applied fields, including such fields as political economics and management science, for example, where little use was formerly made of mathematical methods. It also has become apparent that questions from very different areas of numerical mathematics may be regarded as examples of optimization. Thus, many types of initial value and boundary value problems of ordinary and partial differential equations, as well as approximation problems, game theoretic questions, and others, reduce to optimization problems. As this field has grown in importance, the number of texts has increased. Thus some justification for yet another text might be required. Now most existing texts deal with some subfield, whether linear or non-linear optimization, game theory, or whatever. So it became our intention to provide a certain overview of the entire field with this book, while emphasizing the connections and interrelations among different fields and subfields, including those previously mentioned. Since it is also our impression that these new fields -- for example, the beautiful general theorems on systems of equations and inequalities -- are not yet generally known, even in mathematical circles, we want to use this book to provide a general, easily comprehensible, and for the practitioner, readily accessible, introduction to this varied field, complete with proofs and unobscured by excessive computational detail. Thus, several deeper concepts, such as the theory of optimal processes (due to Pontrjagin), for one example, or the theory of dynamic optimization (due to Bellman), for another, are not discussed.

The book resulted from a number of courses in the subject given by the authors at the Universität Hamburg. In addition, one of the authors included the theorems of the alternative for systems of equations and inequalities, up to the duality theorem of linear optimization (§5 of this book) in an introductory course on "Analytic Geometry and Algebra"; for these theorems may be presented in a few hours as an immediate sequel to matrix theory and the concept of linear independence of vectors. It seems desirable that the young student become familiar with these things. In some countries they already are covered in high school seminars, for which they are well suited. They contribute to the dissemination of mathematics into other sciences and thus their significance will certainly grow in the future.

TABLE OF CONTENTS

I. LINEAR OPTIMIZATION

§1. Introduction

Using simple applications as examples, we will develop the formulation of the general linear optimization problem in matrix notation.

1.1. The Fundamental Type of Optimization Problem

Example 1. First we discuss a problem in production planning whose mathematical formulation already contains the general form of a linear optimization problem.

A plant may produce q different products. Production consumes resources, specifically, m different types of resources, such as labor, materials, machines, etc., each of limited availability. The production of one unit of the k^{th} product yields a net profit of p_k, $k = 1,\ldots,q$. Thus, if x_1 units of the first product, x_2 units of the second, and generally, x_k units of the k^{th} product are produced, the total profit will be $\sum_{k=1}^{q} p_k x_k$. Our problem is to devise a production plan which maximizes the total profit. In doing so, we must bear in mind that the j^{th} resource is available only up to some maximal finite quantity, b_j, and that the production of one unit of the k^{th} product consumes a quantity a_{jk} of the j^{th} resource. The x_k must be chosen, therefore, to satisfy the inequalities $\sum_{k=1}^{q} a_{jk} x_k \le b_j$, $j = 1,\ldots,m$, and naturally must satisfy also the requirements $x_k \ge 0$.

We can formulate this problem as a linear optimization problem in the following manner.

Let there be given the (always <u>real</u>!) numbers p_k, b_j, a_{jk}, $j = 1,\ldots,m$, $k = 1,\ldots,q$. Find numbers x_k such that

$$Q(x_1,\ldots,x_q) = \sum_{k=1}^{q} p_k x_k = \text{Max!}, \tag{1.1}$$

i.e., is as large as possible, subject to the constraints

$$\sum_{k=1}^{q} a_{jk} x_k \le b_j \quad (j = 1,\ldots,m) \tag{1.2}$$

and the positivity constraints

$$x_k \ge 0 \quad (k = 1,\ldots,q). \tag{1.3}$$

The notation $Q(x_1,\ldots,x_q) = \text{Max!}$, resp. Min!, will be used henceforth. It instructs

1. check whether the function Q possesses a maximum, resp. minimum, subject to the given constraints; and if it does,

2. determine the extreme value and the values of

the variables $x_1,...,x_q$ for which Q attains this extreme.

In particular, the notation $Q(x_1,...,x_q) = \text{Max}!$ makes no claim about the existence of a maximum. It should be interpreted as merely a statement of the problem.

In the context of linear optimization, we consider problems of the type just described: find the maximum of a function Q (the <u>objective function</u>), which is linear in the variables x_k, where the x_k satisfy a system of linear inequalities and are non-negative. The following variations from this fundamental type also occur.

1. The objective function $Q(x_1,...,x_q)$ of form (1.1) is to be minimized. A switch to $-Q(x_1,...,x_q)$ reduces this case to the one described above.

2. The inequalities read $\geq$ instead of $\leq$. Multiplication by -1 converts such inequalities to form (1.2).

3. In the constraints (1.2) we have $=$ instead of $\geq$. The introduction of <u>slack variables</u>, as we shall see below, reduces form (1.2) to this type, where the contraints form a system of equations.

4. The positivity constraints (1.3) may be omitted (or perhaps are already contained in the constraints (1.2)).

5. Various combinations are also possible. The constraints may consist partially of equations and partially of inequalities. Positivity constraints may be prescribed for some, but not all, of the x_k.

<u>Example 2</u>. With this concrete, if highly idealized, example, we will further clarify, and illustrate graphically,

the formulation of the linear optimization problem.

An agricultural cooperative raises cows and sheep. The cooperative has 50 stalls for cows, and 200 for sheep. It also has 72 acres of pasture. One acre is needed to sustain one cow, while a sheep requires 0.2 acres. To care for the animals, the cooperative can provide up to 10,000 hours of labor per year. A cow requires 150 hours annually, and a sheep, 25. The annual profit that is realized is \$250 per cow and \$45 per sheep.

The cooperative would like to determine the number, x_1, of cows, and x_2, of sheep, which it should keep to maximize the total profit.

Formulated mathematically, this becomes the linear optimization problem

$$\left.\begin{array}{rcl} Q(x_1,x_2) = 250x_1 + 45x_2 & = & \text{Max!} \\ x_1 & \leq & 50 \\ x_2 & \leq & 200 \\ x_1 + 0.2x_2 & \leq & 72 \\ 150x_1 + 25x_2 & \leq & 10000 \\ x_1 \geq 0,\ x_2 & \geq & 0. \end{array}\right\} \qquad (1.4)$$

Figure 1.1 graphically illustrates this problem.

Those points, whose coordinates (x_1,x_2) satisfy all of the inequalities (1.4), are precisely the points of the shaded, six-sided polygon M, boundary points included. Now, $Q(x_1,x_2) = c$ determines a family of parallel lines dependent on the parameter c. Problem (1.4) thus can be formulated as follows. From among all lines of the family

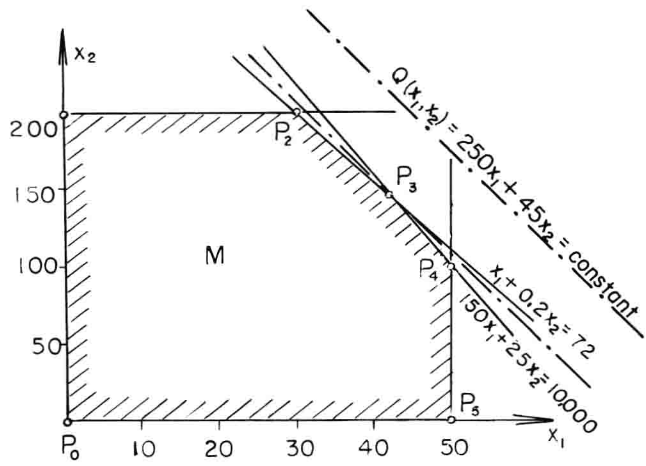

Figure 1.1

which contain points of M, find that line for which c is maximal. Let that line be denoted by $Q(x_1,x_2) = c^*$. Then each point which this line has in common with M, and no other, yields a solution to problem (1.4). It is intuitively clear (and will later be proven in general) that at least one corner point of M must be among these points of intersection of the line $Q(x_1,x_2) = c$ with the polygon M. The case where the intersection contains two corner points occurs only when the constants in $Q(x_1,x_2)$ are adjusted so as to cause a whole side of M to lie in the line $Q(x_1,x_2) = c^*$. In either case, it suffices to compute the values of $Q(x_1,x_2)$ at all corner points of M. The largest value obtained in this way is simultaneously the maximum of $Q(x_1,x_2)$. The coordinates of the corresponding corner point solve the optimization problem. We obtain:

corner	x_1	x_2	$Q(x_1,x_2)$
P_0	0	0	0
P_1	0	200	9000
P_2	32	200	17000
P_3	40	160	17200
P_4	50	100	17000
P_5	50	0	12500

So we see that the maximal profit of $17,200 is attained by keeping 40 cows and 160 sheep.

1.2. The Fundamental Type in Matrix Notation

The fundamental type of linear optimization problem, described by (1.1), (1.2), and (1.3), will now be reformulated in a more concise notation by the introduction of vectors and matrices. The p_k's, b_j's, and x_k's are collected in (column) vectors,

$$\underset{\sim}{p} = \begin{pmatrix} p_1 \\ p_2 \\ \cdots \\ p_q \end{pmatrix}, \quad \underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix}, \quad \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_q \end{pmatrix} \tag{1.5}$$

the a_{jk}'s into the matrix

$$\underset{\sim}{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mq} \end{pmatrix}. \tag{1.6}$$

The transpose matrix of $\underset{\sim}{A}$ will be denoted by $\underset{\sim}{A}'$:

$$\underset{\sim}{A}' = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1q} & a_{2q} & \cdots & a_{mq} \end{pmatrix},$$

and correspondingly, a column vector, say $\underset{\sim}{p}$, is transposed to the row vector

$$\underset{\sim}{p}' = (p_1, p_2, \ldots, p_q).$$

The linear optimization problem now reads as follows.

Let $\underset{\sim}{p}$ and $\underset{\sim}{b}$ be given real vectors, as in (1.5), and let $\underset{\sim}{A}$ be a given real matrix, as in (1.6). Find the real vector $\underset{\sim}{x}$ for which

$$Q(\underset{\sim}{x}) = \underset{\sim}{p}'\underset{\sim}{x} = \text{Max!} \tag{1.1a}$$

subject to the <u>constraints</u>

$$\underset{\sim}{A}\underset{\sim}{x} \leq \underset{\sim}{b} \tag{1.2a}$$

and the <u>positivity</u> <u>constraints</u>

$$\underset{\sim}{x} \geq \underset{\sim}{0}. \tag{1.3a}$$

Here $\underset{\sim}{0}$ is the zero vector. The relation $\geq$ or $\leq$ between vectors means that the corresponding relation holds for each component. By introducing a dummy vector, $\underset{\sim}{y} = \underset{\sim}{b} - \underset{\sim}{A}\underset{\sim}{x}$, the inequalities (1.2a) may be transformed into equations. Instead of (1.2a) we have the equations

$$\underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{y} = \underset{\sim}{b} \tag{1.2b}$$

and to (1.3a) we add the further positivity constraints

$$\underset{\sim}{y} = \begin{pmatrix} y_1 \\ \cdots \\ y_m \end{pmatrix} \geq \underset{\sim}{0} \tag{1.3b}$$

A vector with non-negative components which is used in this manner to transform inequalities into equations is called a slack variable vector, and its components are called slack variables. (But notice that this process does not reduce the total number of inequalities, as new constraints have been added.)

Now set

$$\left.\begin{array}{l} \hat{\underset{\sim}{x}} = \begin{pmatrix} \underset{\sim}{x} \\ \underset{\sim}{y} \end{pmatrix} = \begin{pmatrix} x_1 \\ \cdots \\ x_q \\ y_1 \\ \cdots \\ y_m \end{pmatrix}, \quad \tilde{\underset{\sim}{p}} = \begin{pmatrix} \underset{\sim}{p} \\ \underset{\sim}{0} \end{pmatrix} = \begin{pmatrix} p_1 \\ \cdots \\ p_q \\ 0 \\ \cdots \\ 0 \end{pmatrix} \left.\vphantom{\begin{matrix}0\\ \cdots \\ 0\end{matrix}}\right\} m \text{ components}, \qquad \tilde{\underset{\sim}{b}} = \underset{\sim}{b}, \\ \tilde{\underset{\sim}{A}} = (\underset{\sim}{A}, \underset{\sim}{E}_m) = \begin{pmatrix} a_{11} & \cdots & a_{1q} & 1 & 0 & \cdots & 0 \\ a_{21} & \cdots & a_{2q} & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mq} & 0 & 0 & \cdots & 1 \end{pmatrix}, \end{array}\right\} \tag{1.7}$$

$$n = q + m \tag{1.8}$$

where $\underset{\sim}{E}_m$ is the m-dimensional identity matrix. Then (1.1a), (1.2a), and (1.3a) become equivalent to

$$Q(\tilde{\underset{\sim}{x}}) = \tilde{\underset{\sim}{p}}'\tilde{\underset{\sim}{x}} = \text{Max!}, \tag{1.1c}$$

$$\tilde{\underset{\sim}{A}}\tilde{\underset{\sim}{x}} = \tilde{\underset{\sim}{b}}, \tag{1.2c}$$